Security Analytics

Topic 4: Review of Probability and Statistics

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Based on slides by Prof. Jenifer Neville and Chris Clifton

Readings

- Chapter 4 Data Analysis and Uncertainty
  - Sections 4.1 to 4.3
- Handout: “Background on Probability and Statistics”
- Stefan Axelsson. The Base-Rate Fallacy and the Difficulty of Intrusion Detection. In ACM TISSEC 2000.

Quiz

1. A standard normal distribution has:
   (a) mean equal to the variance
   (b) mean equal 1 and variance equal 1
   (c) mean equal 0 and variance equal 1
   (d) mean equal 0 and standard deviation equal 0
   (e) none of these
2. **True or False**: P(A and B) = P(A|B)P(B|A).
3. **True or False**: If P(A|B) = P(A) then A and B are independent.
4. A card is drawn at random from a deck of playing cards. If it is red, the player wins 1 dollar; if it is black, the player loses 2 dollars. Find the expected value of the game.
5. An urn contains eight balls, two of which are red and six white. Two balls are drawn at random. What is the probability that at least one of the balls drawn is red?

Quiz

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4. A card is drawn at random from a deck of playing cards. If it is red, the player wins 1 dollar; if it is black, the player loses 2 dollars. Find the expected value of the game.
   E = P(red)*1 + P(black)*(-2) = 1/2 - 1 = -1/2
5. An urn contains eight balls, two of which are red and six white. Two balls are drawn at random. What is the probability that at least one of the balls drawn is red?
   1-P(two white) = 1 - (6/8)^2 = 1 - 9/16 = 7/16

Probability

- Probability theory *(some disagreement)*
  - Concerned with interpretation of probability
  - 17th century: Pascal and Fermat develop probability theory to analyze games of chance
- Probability calculus *(universal agreement)*
  - Concerned with manipulation of mathematical representations
  - 1933: Kolmogorov states axioms of modern probability

Probability basics

- Basic notion: **Random variable (RV)**
  - A variable that can take one of a set of possible values
  - *X* refers to random variable; *x* refers to a value of that random variable
- Types of random variables
  - Discrete RV has a finite set of possible values; Continuous RV can take any value within an interval
  - **Boolean**: e.g., Warning (is there a storm warning? = <yes, no>)
  - **Discrete**: e.g., Weather is one of <sunny,rainy,cloudy,snow>
  - **Continuous**: e.g., Temperature

Probability basics

- **Sample space (S)**
  - Set of all possible outcomes of an experiment
- **Event**
  - Any subset of *outcomes* contained in the sample space S
  - When events ***A*** and ***B*** have no outcomes in common they are said to be *mutually exclusive*

Examples

Random variable(s)	**Sample space**
One coin toss	H, T
Two coin tosses	HH, HT, TH, TT
Select one card	2♥, 2♠, ..., A♣ (52)
Play a chess game	Win, Lose, Draw
Inspect a part	Defective, OK
Cavity and toothache	TT, TF, FT, FF

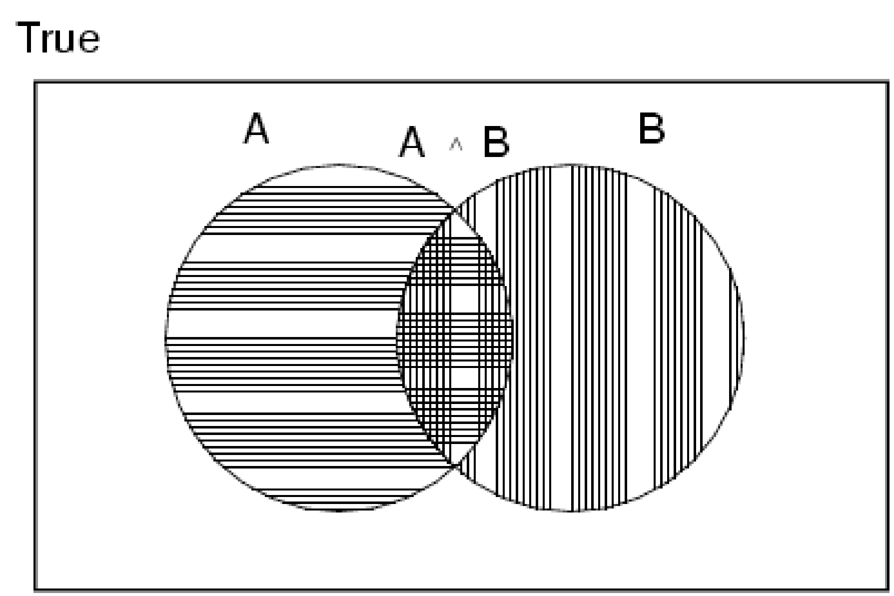
Axioms of probability

- For a sample space S with possible events, a function that associates real values with each event A is called a ***probability function*** if the following properties are satisfied:
  1. $0 \leq P(A) \leq 1$ for every A
  2. $P(S) = 1$
  3. $P(A_1 \lor A_2 \ldots \lor A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

     if $A_1, A_2, \ldots, A_n$ *are pairwise mutually exclusive events*

Implications of axioms

- For any events ***A, B***
  - P(A) = 1 - P($\neg A$)
  - P(true) = 1   and   P(false) = 0
  - If A and B are mutually exclusive then P($A \wedge B$) = 0
  - P($A \vee B$) = P(A) + P(B) - P($A \wedge B$)



Permutations and combinations

- An **ordered** sequence of k objects taken from a set of n distinct objects without replacement, is called a **permutation** of size k
  - The number of permutations of size k that can be constructed from the n objects is:

$$P_{k,n} = \frac{n!}{(n-k)!}$$

- An **unordered** sequence of k objects taken from a set of n distinct objects without replacement, is called a **combination** of size k
  - The number of combinations of size k that can be constructed from the n objects is:

$$C_{k,n} = \frac{P_{k,n}}{k!} = \frac{n!}{k!(n-k)!}$$

Example

- An urn contains ten balls, six of which are red and four of which are white.
  Five balls are drawn at random (without replacement).
  What is the probability of drawing three red and two white balls?

$$\frac{C_{3,6} \cdot C_{2,4}}{C_{5,10}} = \frac{\binom{6}{3}\binom{4}{2}}{\binom{10}{5}} = \frac{6!}{3!3!}\frac{4!}{2!2!}\frac{5!5!}{10!}$$

- An urn contains five balls, numbered from 1 to 5. Three balls are drawn at random. What is the probability that we draw the sequence 3, 4, 1?

$$\frac{1}{P_{3,5}} = \frac{(5-3)!}{5!}$$

Probability distribution

- **Probability distribution** *(i.e., probability mass function or probability density function)* specifies the probability of observing every possible value of a random variable
- Discrete (*probability mass function)*
  - Denotes probability that *X* will take on a particular value:

$$P(X = x)$$

- Continuous (*probability density function*)
  - Probability of any particular point is 0, have to consider probability within an interval:

$$P(a < X < b) = \int_a^b p(x)dx$$

Joint probability

- **Joint probability distribution** for a set of random variables gives the probability of every atomic event on those random variables

  E.g., P(Weather, Warning) = a 4 × 2 matrix of values:

	sunny	rainy	cloudy	snow
warning $= Y$	0.005	0.08	0.02	0.02
warning $= N$	0.415	0.12	0.31	0.03

- Every question about events can be answered by the joint distribution

Conditional probability

- **Conditional** (or posterior) probability:
  - e.g., P(warning=Y | snow=T) = 0.4
  - Complete conditional distributions specify conditional probability for all possible combinations of a set of RVs:
    P(warning | snow) =
    { P(warning = Y | snow = T),
    P(warning = N | snow = T), }
    { P(warning = Y | snow = F),
    P(warning = N | snow = F) }
- If we know more, then we can update the probability by conditioning on more evidence
  - e.g., if Windy is also given then P(warning | snow, windy) = 0.5

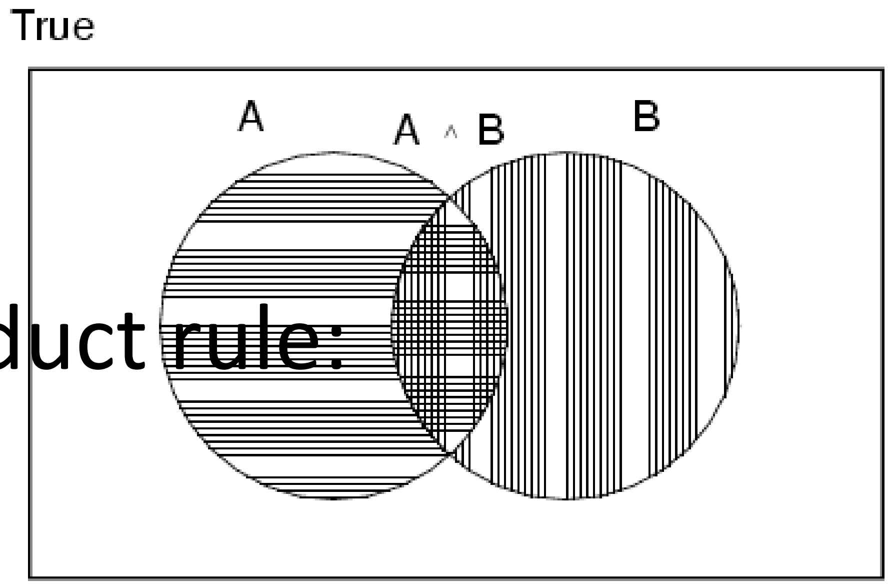
Conditional probability

- Definition of conditional probability:

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \quad \text{if } P(B) > 0$$

- **Product rule** gives an alternative formulation:

$$\begin{aligned} P(A \wedge B) &= P(A|B)P(B) \\ &= P(B|A)P(A) \end{aligned}$$



- **Bayes rule** uses the product rule:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Example

- Conditional probability:

$$P(A|B) = \frac{P(A \wedge B)}{P(B)} \quad \text{if } P(B) > 0$$

- Example: What is P(sunny | warning = Y)?

	sunny	rainy	cloudy	snow
warning $= Y$	0.005	0.08	0.02	0.02
warning $= N$	0.415	0.12	0.31	0.03

Conditional probability

- **Chain rule** is derived by successive application of product rule:

$$
\begin{aligned}
P(X_1, \dots, X_n) &= P(X_n | X_1, \dots, X_{n-1}) P(X_1, \dots, X_{n-1}) \\
&= P(X_n | X_1, \dots, X_{n-1}) P(X_{n-1} | X_1, \dots, X_{n-2}) P(X_1, \dots, X_{n-2}) \\
&= \dots \\
&= \prod_{i=1}^{n} P(X_i | X_1, \dots, X_{i-1})
\end{aligned}
$$

Marginal probability

- **Marginal** (or unconditional) probability corresponds to belief that event will occur regardless of conditioning events
- Marginalization: $$P(A) = \sum_{b \in B} P(A, b)$$ $$= \sum_{b \in B} P(A|b)P(b)$$
- Example: What is P(cloudy)?

	sunny	rainy	cloudy	snow
warning $= Y$	0.005	0.08	0.02	0.02
warning $= N$	0.415	0.12	0.31	0.03

Independence

- **A and B are independent iff:**
  - $P(A|B) = P(A)$ or $P(B|A) = P(B)$ or $P(A, B) = P(A)\,P(B)$
  - *Knowing B tells you nothing about A*
- **Examples**
  - Coin flip 1 and coin flip 2?
  - Weather and storm warning?
  - Weather and coin flip=H?
  - Weather and election?

Conditional independence

- Two variables *A* and *B* are **conditionally** independent given *Z*
  iff for all values of *A, B, Z:*
  $$P(A, B \mid Z) = P(A \mid Z)\, P(B \mid Z)$$

- ***Note***: *independence does not imply conditional independence or vice versa*

Example 1

- **Conditional independence does not imply independence**
- Gender and lung cancer are not independent
  $$P(C \mid G) \neq P(C)$$
- Gender and lung cancer are conditionally independent given smoking
  $$P(C \mid G, S) = P(C \mid S)$$
- Why? Because gender indicates likelihood of smoking, and smoking causes cancer

Example 2

- **Independence does not imply conditional independence**
- Sprinkler-on and raining are independent
  $$P(S \mid R) = P(S)$$
- Sprinkler-on and raining are not conditionally independent given
  grass is wet
  $$P(S \mid R, W) \neq P(S \mid R)$$
- Why? Because once we know the grass is wet, if it's not raining, then the explanation for the grass being wet has to be the sprinkler

Example

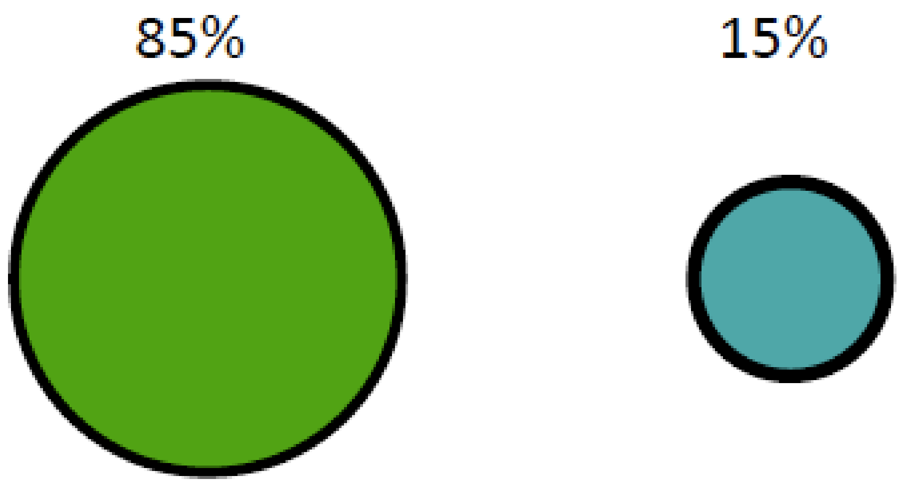
- You flip a fair coin twice
  1. The first flip is heads
  2. The second flip is tails
  3. The two flips are not the same
- Are (1) and (2): *independent*? *Conditionally independent*? *Neither*?

The Base Rate Fallacy

- Taxi-cab problem *(Tversky & Kahneman '72)*
  - 85% of the cabs are Green
  - 15% of the cabs are Blue
  - An accident eyewitness reports a Blue cab
  - But she is wrong 20% of the time.
- What is the probability that the cab is Blue?
  - Participants tend to overestimate probability, most answer 80%
  - They ignore baseline prior probability of blue cabs.

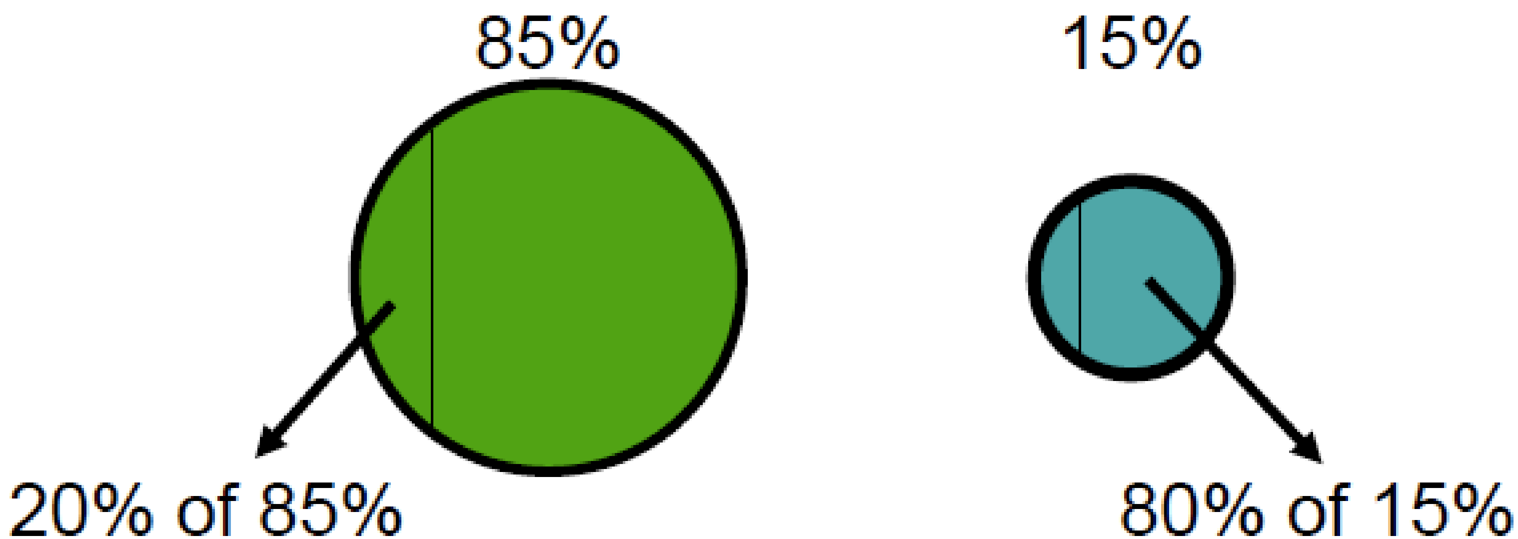
A priori (beforehand)

$$P(green) = 0.85$$
$$P(blue) = 0.15$$



$$P(seeBlue|blue) = 0.80$$
$$P(seeBlue|green) = 0.20$$

After accident (only cars reported as being blue)



Base Rate Fallacy

- How to compute probability

$$
\begin{aligned}
P(blue|seeBlue) &= \frac{P(blue \wedge seeBlue)}{P(seeBlue)} \\
&= \frac{P(seeBlue|blue)P(blue)}{P(seeBlue)} \\
&= \frac{P(seeBlue|blue)P(blue)}{P(seeBlue|blue)P(blue) + P(seeBlue|green)P(green)} \\
&= \frac{0.80 \cdot 0.15}{(0.80 \cdot 0.15) + (0.20 \cdot 0.85)} \\
&= 0.41
\end{aligned}
$$

Most people answered 80%

Medical Test

- In the 1980's in the US, a HIV test was used that had the following properties:
  There were 4% false positives
  There were 100% true positives
- About 0.4% of the male population was HIV positive
- If a man tested HIV positive, what is the probability he is actually HIV positive?

Representation

- P(positive | no HIV) = .04 (4% false positives)
- P(positive | HIV) = 1 (100% true positives)
- P(HIV) = .004 (0.4% HIV positive rate)

- want: P(HIV | positive) = ?

	HIV	no HIV
Positive	P(positive | HIV)P(HIV)	P(positive | noHIV)P(noHIV)
Negative	P(negative | HIV)P(HIV)	P(negative | noHIV)P(noHIV)

Representation

- P(positive | no HIV) = .04 (4% false positives)
- P(positive | HIV) = 1 (100% true positives)
- P(HIV) = .004 (0.4% HIV positive rate)

- want: P(HIV | positive) = ?

	HIV	no HIV
Positive	P(positive | HIV)P(HIV)= (1)(.004) = .004	P(positive | noHIV)P(noHIV)= (.04)(.996) = .03984
Negative	P(negative | HIV)P(HIV)= (0)(.004) = 0	P(negative | noHIV)P(noHIV)= (.96)(.996) = .95616

Solution

- P(HIV | positive) = .004 / (.004 + .03984)
  = .091

	HIV	no HIV
Positive	P(positive | HIV)P(HIV)= (1)(.004) = .004	P(positive | noHIV)P(noHIV)= (.04)(.996) = .03984
Negative	P(negative | HIV)P(HIV)= (0)(.004) = 0	P(negative | noHIV)P(noHIV)= (.96)(.996) = .95616

Base Rate Fallacy in Intrusion Detection

- Assumptions in the hypothesized system:
  - Few tens of workstations running UNIX
  - Few servers running UNIX
  - Couple of dozen users
  - Capable of generating 1,000,000 audit records per day (with C2 compliant logging)
  - Single site security officer (SSO)
  - 10 audit records affected in the average intrusion
  - 2 intrusions per day => 20 records per 1,000,000 account to actual intrusions

Stefan Axelsson. The Base-Rate Fallacy and the Difficulty of Intrusion Detection. ACM TISSEC. 2000.

Base Rate Fallacy in Intrusion Detection (Continued)

Calculation of Bayesian detection rates

- I: Intrusive behavior
- A: Presence of an intrusion alarm
- With the assumptions, we have:
  - $P(I) = 2 \cdot 10^{-5}$; $P(\neg I) = 1 – P(I) = 0.99998$
  - Detection rate or True positive rate: $P(A|I)$
  - False alarm rate: $P(A|\neg I)$
  - False negative rate: $P(\neg A|I) = 1 - P(A|I)$
  - True negative rate: $P(\neg A|\neg I) = 1 - P(A|\neg I)$
- Maximize
  - $P(I|A)$: Bayesian detection rate
  - $P(\neg I|\neg A)$

Base Rate Fallacy in Intrusion Detection (Continued)

- For $P(A|I)=1$, $P(A|\neg I)=1\cdot 10^{-5}$, we get $P(I|A)$ as 0.66
- For $P(A|I)=0.7$, $P(A|\neg I)=1\cdot 10^{-5}$, we get $P(I|A)$ as 0.58
- Even for large detection rate, viz. $P(A|I)$, Bayesian detection rate is dominated by the factor of false alarm rate, viz. factor of $P(A|\neg I)$
- $P(I|A)$ close to 50% will induce SSO to ignore all (or most) of the alarms generated

Base Rate Fallacy in Intrusion Detection (Lessons)

- Intrusion detection is difficult in real world
- The “effectiveness” of an intrusion detection system depends not just on its ability to detect intrusive behavior but on its ability to suppress false alarms
- Comparison shows anomaly-based detection methods have larger false alarm rates than signature-based detection, but signature-based detection methods cannot provide protection against novel intrusions

Expectation

- Denotes the expected value or mean value of a random variable X
- Discrete

$$E[X] = \sum_x x \cdot p(x)$$

- Continuous

$$E[X] = \int_x x \cdot p(x) dx$$

- Expectation of a function

$$E[h(X)] = \sum_x h(x) \cdot p(x)$$

$$E[aX + b] = a \cdot E[X] + b$$

$$E[X + Y] = E[X] + E[Y]$$

Example

- Let X be a random variable that represents the number of heads which appear when a fair coin is tossed three times.
- X = {0, 1, 2, 3}
- P(X=0) = 1/8; P(X=1) = 3/8; P(X=2) = 3/8; P(X=3) = 1/8
- What is the expected value of X, E[X]?

$$\begin{aligned} E[X] &= (0 \cdot \frac{1}{8}) + (1 \cdot \frac{3}{8}) + (2 \cdot \frac{3}{8}) + (3 \cdot \frac{1}{8}) \\ &= \frac{3}{2} \end{aligned}$$

Variance

- Denotes the expectation of the squared deviation of X from its mean
- Variance

$$\begin{aligned} Var(X) &= E[(x - E[X])^2] \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

- Standard deviation

$$\sigma = \sqrt{Var(X)}$$

- Variance of a function

$$Var(aX + b) = a^2 \cdot Var(X)$$

$$Var(h(X)) = \sum_x (h(x) - E[h(x)])^2 \cdot p(x)$$

Example

- Let X be a random variable that represents the number of heads which appear when a fair coin is tossed three times.
- X = {0, 1, 2, 3}

$$E[X] = (0 \cdot \frac{1}{8}) + (1 \cdot \frac{3}{8}) + (2 \cdot \frac{3}{8}) + (3 \cdot \frac{1}{8})$$
$$= \frac{3}{2}$$

- What is the variance of X, Var(X)?

$$\begin{aligned} Var(X) &= \left(\left[0 - \frac{3}{2}\right]^2 \cdot \frac{1}{8}\right) + \left(\left[1 - \frac{3}{2}\right]^2 \cdot \frac{3}{8}\right) + \left(\left[2 - \frac{3}{2}\right]^2 \cdot \frac{3}{8}\right) + \left(\left[3 - \frac{3}{2}\right]^2 \cdot \frac{1}{8}\right) \\ &= \left(\frac{9}{4} \cdot \frac{1}{8}\right) + \left(\frac{1}{4} \cdot \frac{3}{8}\right) + \left(\frac{1}{4} \cdot \frac{3}{8}\right) + \left(\frac{9}{4} \cdot \frac{1}{8}\right) \\ &= \frac{3}{4} \end{aligned}$$

Common distributions

- Bernoulli
- Binomial
- Multinomial
- Poisson
- Normal

Bernoulli

- Binary variable (0/1) that takes the value of 1 with probability p
  - E.g., Outcome of a fair coin toss is Bernoulli with *p=0.5*

$$P(x) = p^x(1-p)^{1-x}$$

$$E[X] = 1(p) + 0(1-p) = p$$

$$\begin{aligned} Var(X) &= E[X]^2 - (E[X])^2 \\ &= 1^2(p) + 0^2(1-p) - p^2 \\ &= p(1-p) \end{aligned}$$

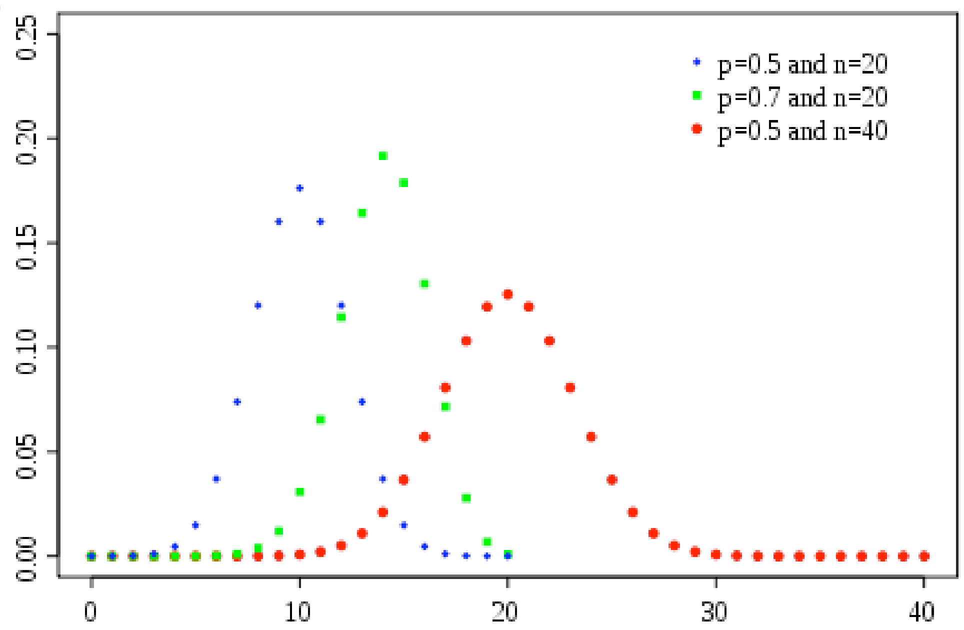
Binomial

- Describes the number of successful outcomes in *n* independent Bernoulli(p) trials
  - E.g., Number of heads in a sequence of 10 tosses of a fair coin is Binomial with *n=10* and *p=0.5*

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E[X] = np$$

$$Var[X] = np(1-p)$$



Multinomial

- Generalization of binomial to *k* possible outcomes; outcome *i* has probability p_i of occurring
  - E.g., Number of {outs, singles, doubles, triples, homeruns} in a sequence of 10 times at bat is Multinomial
- Let X_i denote the number of times the *i*-th outcome occurs in *n* trials:

$$P(x_1, ...x_k) = \binom{n}{x_1, ...x_k} p_1^{x_1} p_2^{x_2} ... p_k^{x_k}$$

$$E[X_i] = np_i$$

$$Var(X_i) = np_i(1 - p_i)$$

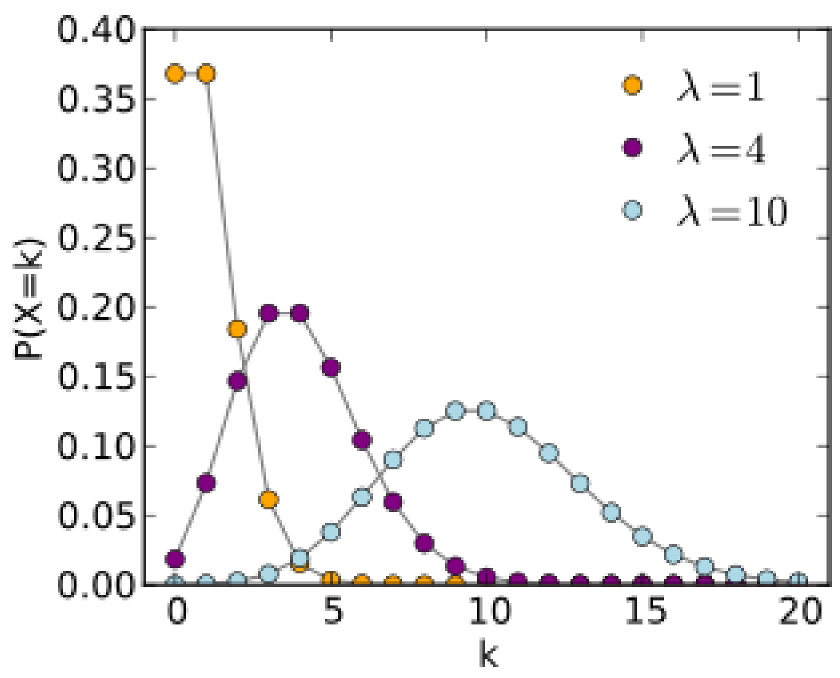
Poisson

- Describes the number of successful outcomes occurring in a fixed interval of time (or space) if the "successes" occur *independently* with a known average rate
  - E.g., Number of emergency calls to a service center per hour, when the average rate per hour is *λ=10*

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$E[X] = \lambda$$

$$Var[X] = \lambda$$



Normal (Gaussian)

- Important distribution gives well-known bell shape

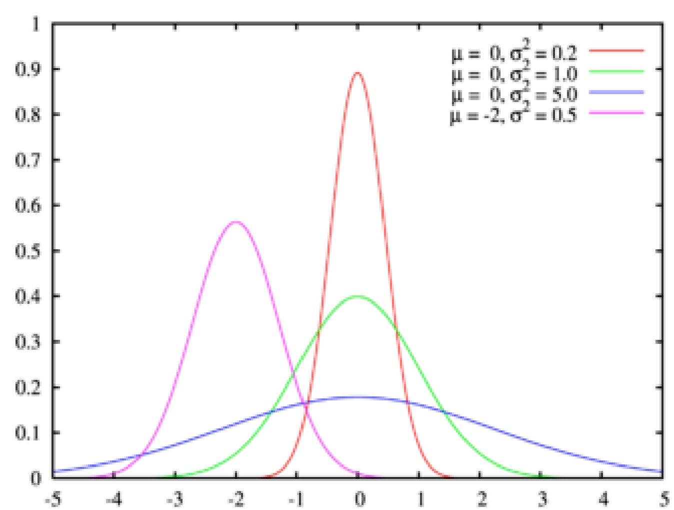
$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$E[X] = \mu$$

$$Var(X) = \sigma^2$$

- **Central limit theorem:**

Distribution of the mean of n samples becomes normally distributed as n ↑, regardless of the distribution of the underlying population



Multivariate RV

- A multivariate random variable $\boldsymbol{X}$ is a set $X_1, X_2, \ldots, X_p$ of random variables
- **Joint** density function: $P(\mathbf{x})=P(x_1, x_2, \ldots, x_p)$
- **Marginal** density function: the density of any subset of the complete set of variables, e.g.,:

$$P(x_1) = \sum_{x_2, x_3} p(x_1, x_2, x_3)$$

Conditional density function: the density of a subset conditioned on particular values of the others, e.g.,:

$$P(x_1 | x_2, x_3) = \frac{p(x_1, x_2, x_3)}{p(x_2, x_3)}$$

Frequentist view of Probability

- Dominant perspective for last century
- Probability is an **objective** concept
  - Defined as the frequency of an event occurring under repeated trials in "same" situation
  - E.g., number of heads in repeated coin tosses
- Restricts application of probability to repeatable events

Bayesian view

- Increasing importance over last decade
  - Due to increase in computational power that facilitates previously intractable calculations
- Probability is a **subjective** concept
  - Defined as individual degree-of-belief that event will occur
  - E.g., belief that we will have another snow storm tomorrow
- Begin with prior belief estimates and update those by conditioning on observed data

Calculating probabilities: Bayesian

- *Begin with prior belief estimates*: P(A)
  - E.g., After the Seahawks won their conference, Vegas casinos believed the Seahawks were likely to win the Superbowl over the Patriots:
    $P(S \text{ wins})=0.525$, $P(P \text{ wins})=0.475$
- Observe data
  - But then Vegas observed a heavy majority of the betters (80%) chose the Patriots, which is unlikely given their current belief
- *Update belief by conditioning on observed data*
  $P(A|data) = P(data|A)\ P(A) / P(data)$
  - So they updated their belief to increase the the Patriots's chance of a win:
    $P(S \text{ wins} \mid \text{betting}) = P(\text{betting} \mid S \text{ wins})\ P(S \text{ wins}) / P(\text{betting}) = 0.50$
- Even when the same data is observed, if people have different priors, they can end up with different posterior probability estimates P(A|data)

Bayesian vs. frequentist

- Bayesian central tenet:
  - Explicitly model all forms of uncertainty
  - E.g., Parameters, model structure, predictions
- Frequentist often model same uncertainty but in less-principled manner, e.g.,:
  - Parameters set by cross-validation
  - Model structure averaged in ensembles
  - Smoothing of predicted probabilities
- Although interpretation of probability is different, underlying calculus is the same