Introduction to Cryptography
CS 355

Lecture 24

Diffie-Hellman and Discrete Log

Lecture Outline

- The Discrete Log problem
- The Diffie-Hellman protocol

Discrete Logarithm Problem (DLP)

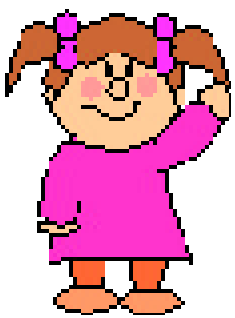
- Given a multiplicative group (G, *), an element g in G having order n and an element y in the subgroup generated by g, denoted <g>
- Find the unique integer x such that

$$g^x \bmod n = y$$

- i.e., x is the discrete logarithm $\log_g y$
- For example, given the group Z_p*, where p is a 1024-bit prime, let g be an element having order q, where q is a 160-bit prime
  - q | (p-1)
  - e.g., Z_7*={3,2,6,4,5,1}, we choose the subgroup {2,4,1}

The Diffie-Hellman Protocol

- Key agreement protocol, both A and B contribute to the key
- Setup: p prime and g generator of Z_p^*, p and g public.



g^a mod p $\longrightarrow$

$\longleftarrow$ g^b mod p

Pick random, secret a
Compute and send g^a mod p

$K = (g^b \bmod n)^a = g^{ab} \bmod p$

Pick random, secret b
Compute and send g^b mod p

$K = (g^a \bmod n)^b = g^{ab} \bmod p$

Diffie-Hellman Key Establishment

- A and B wishes to establish a shared secret key so that no eavesdropper can compute the key:
- A and B shares public parameters a group Z_p and a generator g
  - A randomly chooses x and send g^x mod p to B
  - B randomly chooses y and send g^y mod p to A
  - Both A and B can compute g^{xy} mod p
  - It is (believed to be) infeasible for an eavesdropper to compute g^{xy} mod p
  - A and B can establish a shared secret without sharing any secret to start with

CDH and DDH

- Security of the Diffie-Hellman key establishment protocol based on the CDH problem
- Computational Diffie-Hellman (CDH)
  – Given a multiplicative group (G, *), an element $g \in G$ having order q, given g^x and g^y, find g^{xy}
- Decision Diffie-Hellman (DDH)
  – Given a multiplicative group (G, *), an element $g \in G$ having order q, given g^x, g^y, and g^z, determine if $g^{xy} \equiv g^z \bmod n$
- Discrete Log is at least as hard as CDH, which is at least as hard as DDH.

Choices of Parameters

- Why use an element of order q, instead of just using a generator for $Z_p{}^*$?
- Answer:
  - it is often beneficial to have order being a prime
    - e.g., given e, one can find d s.t. $g^{ed}=g$
  - Balance security and size
    - p needs to be large enough for discrete log to be hard, thus 1024 bits
    - we want the group to be relative small, so that an index to an element in the group is short (e.g., 160 bits)
      - it needs to be large enough to prevent exhaustive search

Algorithms for The Discrete Log Problem

- There are generic algorithms that work for every cyclic group
  - Pollard Rho
  - Pohlig-Hellman
- There are algorithms that work just for some groups such as Z_p^*
  - e.g., the index calculus algorithms
  - these algorithms are much more efficient
  - therefore, 1024 bits are needed for adequate level of security

Bit Security in Discrete Log

- Even though it is difficult to find $\log_g x$, it is possible to determine some bits in $\log_g x$
  - e.g., let g be the generator of Z_p^*, consider the least significant bit (LSB) of $\log_g x$
    - recall that $\log_g x$ is even iff. x is quadratic residue in Z_p^*
- However, finding some bits (aka. hard-core bits) is as hard as computing discrete log
  - in Z_p^*, when $p-1=2^s t$, where t is odd, computing the s least significant bits are easy, computing the s+1 LSB is difficult

One Way Functions

- A function f(x) is a one-way function if
  - given a, it is easy to compute f(a).
  - yet given b, it is difficult to find a such that f(a)=b.
- Examples of one-way functions
  - Modular exponentiation $f(x) = g^x \bmod p$
  - Multiplication $f(x,y) = x \cdot y$
- One way functions are the foundations for modern cryptography, yet we do not know whether they exist or not.
  - existence of one-way functions imply $P \neq NP$

Coming Attractions …

- ElGamal Encryption