Introduction to Cryptography
CS 355

Lecture 19

RSA

Review: Number Theory

Definition An integer n > 1 is called a prime number if its positive divisors are 1 and n.
Definition Any integer number n > 1 that is not prime is called a composite number.

Theorem (Fundamental Theorem of Arithmetic)

$$n = p_1^{e_1} p_2^{e2} \ldots p_k^{ek}$$

Definition The greatest common divisor of a and b, denoted by gcd(a, b), is the largest number that divides both a and b.
Definition Two integers a > 0 and b > 0 are relatively prime if gcd(a, b) = 1.

Review: Euler Phi Function

Definition: A reduced set of residues (RSR) modulo m is a set of integers R each relatively prime to m, so that every integer relatively prime to m is congruent to exactly one integer in R.

Definition: Given n, $Z_n^* = \{a \mid 0<a<n \text{ and } \gcd(a,n)=1\}$ is the standard RSR modulo n.

Definition

Given an integer n, $\Phi(n) = |Z_n^*|$ is the size of RSR modulo n.

Theorem: If $\gcd(m,n) = 1$, $\Phi(mn) = \Phi(m)\,\Phi(n)$

Fact: $\Phi(p)=p-1$ for prime p

Review: Euler's Theorem

Euler's Theorem

Given integer n > 1, such that gcd(a, n) = 1 then

$$a^{\Phi(n)} \equiv 1 \pmod{n}$$

Corollary: Given integer n > 1, such that gcd(a, n) = 1 then $a^{\Phi(n)-1}$ mod n is a multiplicative inverse of a mod n.

Corollary: Given integer n > 1, x, y, and a positive integers with gcd(a, n) = 1. If $x \equiv y \pmod{\Phi(n)}$, then

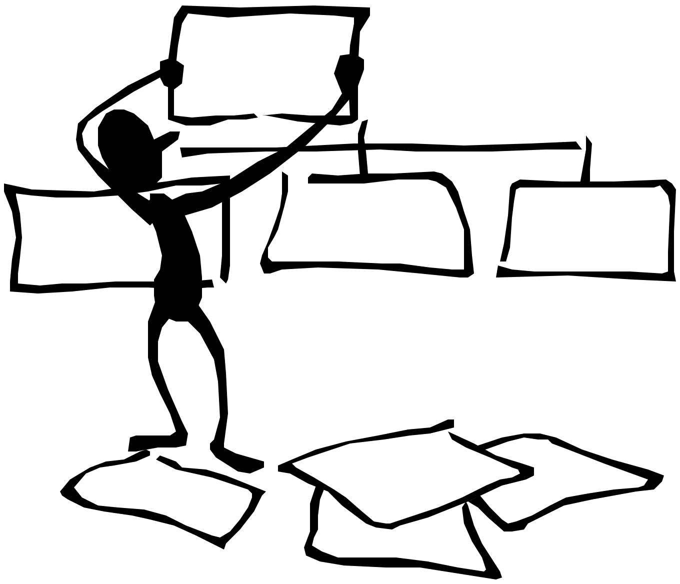
$$a^x \equiv a^y \pmod{n}.$$

Corollary (Fermat's "Little" Theorem):

$$a^{p-1} \equiv 1 \pmod{n}$$

Lecture Outline

- Why public key cryptography?
- Overview of Public Key Cryptography
- RSA
  – square & multiply algorithm
  – RSA implementation
- Pohlig-Hellman



Limitation of Secret Key (Symmetric) Cryptography

- Sender and receiver must share the same key
  - needs secure channel for key distribution
  - impossible for two parties having no prior relationship

Public Key Cryptography Overview

- Proposed in Diffie and Hellman (1976) “New Directions in Cryptography”
  - public-key encryption schemes
  - public key distribution systems
    - Diffie-Hellman key agreement protocol
  - digital signature
- Public-key encryption was proposed in 1970 by James Ellis
  - in a classified paper made public in 1997 by the British Governmental Communications Headquarters
- Diffie-Hellman key agreement and concept of digital signature are still due to Diffie & Hellman

Public Key Encryption

- Public-key encryption
  - each party has a PAIR (K, K^{-1}) of keys: K is the **public** key and K^{-1} is the **secret** key, such that
    $\mathbf{D}_{K^{-1}}[\mathbf{E}_K[M]] = M$
  - Knowing the public-key and the cipher, it is computationally infeasible to compute the private key
  - Public-key crypto system is thus known to be *asymmetric* crypto systems
  - The public-key K may be made publicly available, e.g., in a publicly available directory
  - Many can encrypt, only one can decrypt

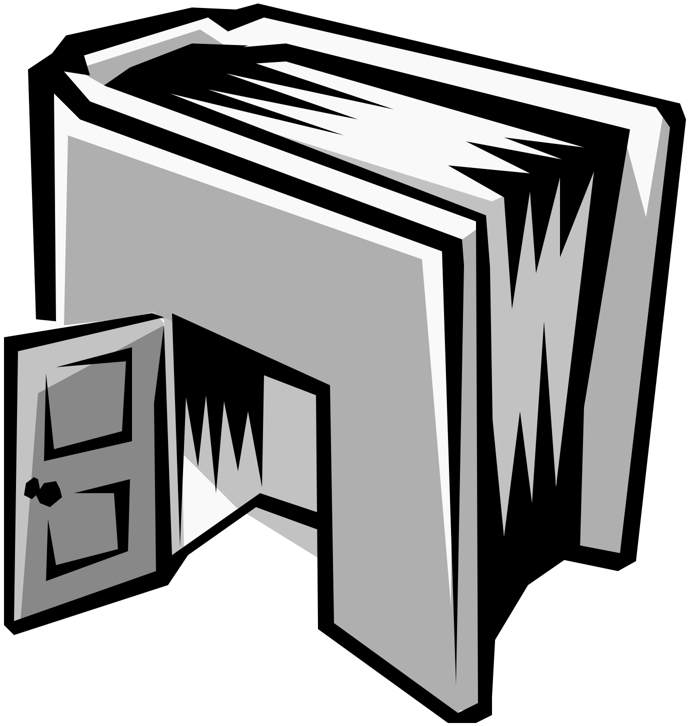
Public-Key Encryption Needs One-way Trapdoor Functions

- Given a public-key crypto system,
  - Alice has public key K
  - $\mathbf{E}_K$ must be a one-way function, knowing y= $\mathbf{E}_K[x]$, it should be difficult to find x
  - However, $\mathbf{E}_K$ must not be one-way from Alice's perspective. The function $\mathbf{E}_K$ must have a trapdoor such that knowledge of the trapdoor enables one to invert it

Trapdoor One-way Functions

Definition:

A function f: $\{0,1\}^* \rightarrow \{0,1\}^*$ is a trapdoor one-way function iff f(x) is a one-way function; however, given some extra information it becomes feasible to compute f^{-1}: given y, find x s.t. $y = f(x)$



RSA Algorithm

- Invented in **1978** by Ron **R**ivest, Adi **S**hamir and Leonard **A**dleman
  - Published as R L Rivest, A Shamir, L Adleman, "*On Digital Signatures and Public Key Cryptosystems*", Communications of the ACM, vol 21 no 2, pp120-126, Feb 1978
- Security relies on the difficulty of factoring large composite numbers
- Essentially the same algorithm was discovered in 1973 by Clifford Cocks, who works for the British intelligence

Z_{pq}*

- Let p and q be two large primes
- Denote their product n=pq.
- $Z_n^* = Z_{pq}^*$ contains all integers in the range [1,pq-1] that are relatively prime to both p and q
- The size of Z_n^* is

  $$\Phi(pq) = (p-1)(q-1)=n-(p+q)+1$$
- For every $x \in Z_{pq}^*$, $x^{(p-1)(q-1)} \equiv 1$

Exponentiation in Z_{pq}*

- Motivation: We want to use exponentiation for encryption
- Let e be an integer, $1<e<(p-1)(q-1)$
- When is the function $f(x)=x^e$, a one-to-one function in Z_{pq}*?
- If x^e is one-to-one, then it is a permutation in Z_{pq}*.

Exponentiation in Z_{pq}*

- Claim: If e is relatively prime to (p-1)(q-1) then $f(x)=x^e$ is a one-to-one function in Z_{pq}*

- Proof by constructing the inverse function of f. As gcd(e,(p-1)(q-1))=1, then there exists d and k s.t. ed=1+k(p-1)(q-1)
- Let $y=x^e$, then $y^d=(x^e)^d=x^{1+k(p-1)(q-1)}=x \pmod{pq}$, i.e., $g(y)=y^d$ is the inverse of $f(x)=x^e$.

RSA Public Key Crypto System

Key generation:

Select 2 large prime numbers of about the same size, p and q

Compute $n = pq$, and $\Phi(n) = (q-1)(p-1)$

Select a random integer e, $1 < e < \Phi(n)$, s.t. $\gcd(e, \Phi(n)) = 1$

Compute d, $1 < d < \Phi(n)$ s.t. $ed \equiv 1 \bmod \Phi(n)$

Public key: (e, n)

Secret key: d

RSA Description (cont.)

Encryption

Given a message M, $0 < M < n$ $M \in Z_n - \{0\}$

use public key (e, n)

compute $C = M^e \bmod n$ $C \in Z_n - \{0\}$

Decryption

Given a ciphertext C, use private key (d)

Compute $C^d \bmod n = (M^e \bmod n)^d \bmod n = M^{ed} \bmod n = M$

RSA Example

- $p = 11$, $q = 7$, $n = 77$, $\Phi(n) = 60$
- $d = 13$, $e = 37$ ($ed = 481$; $ed \bmod 60 = 1$)

- Let $M = 15$. Then $C \equiv M^e \bmod n$
  - $C \equiv 15^{37} \pmod{77} = 71$

- $M \equiv C^d \bmod n$
  - $M \equiv 71^{13} \pmod{77} = 15$

Coming Attractions …

- Fast exponentiation algorithm
- Pohlig-Hellman Exponentiation Cipher
- Distribution of Prime Numbers