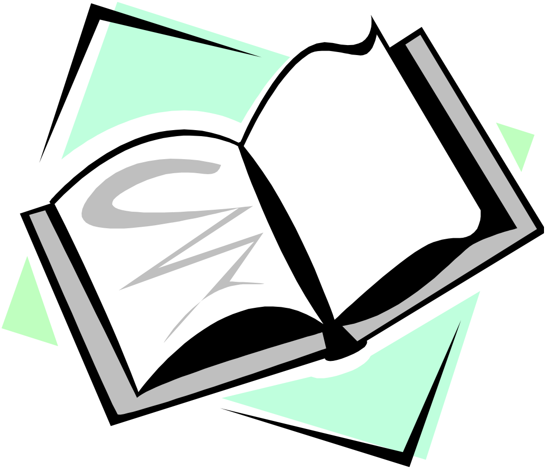
Cryptography
CS 555

Topic 17: Textbook RSA encryption

Outline and Readings

- Outline
  - One-way functions
  - RSA
- Readings:
  - Katz and Lindell: Chapter 6.0, 6.1.1, 6.1.2, 7.2



Towards One-Way Function

- We know how to use Pseudo-Random Generator (PRG) and Pseudo-Random Function (PRF) to construct encryption schemes and MAC.
- We know what algorithms that are used in practice in instantiate PRG and PRF.
  - But we cannot prove that they are PRG or PRF; we can only assume that they are
- Can we prove that some constructions are PRG or PRF based on something else?

One-Way Function

- A function f is one-way if
  - It is easy to compute
  - It is hard to invert, that is, given y=f(x), where x is randomly chosen, it is difficult to find x' such that f(x')=y
- A one-way permutation is length-preserving (input and output have the same size) and one-to-one.
- Candidates for one-way functions
  - Multiplication: f(x,y) = xy

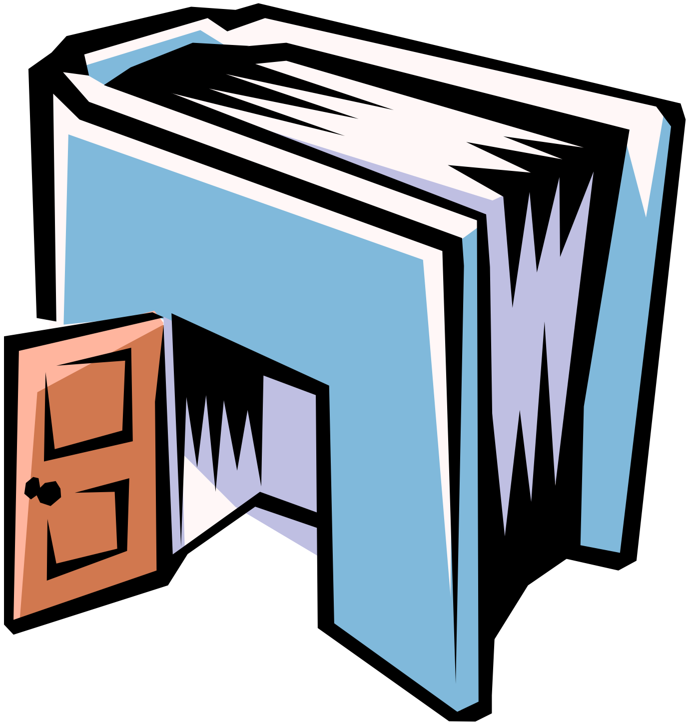
Relationship of One-Way Functions and Cryptography

- Secure encryption and MAC schemes imply/require the existence of one-way functions
- Given a one-way function, one can construct PRG, PRF, PRP
  - Thus one can construct secure encryption and MAC schemes
  - Details are more suitable for 655
- One-way functions are foundation of modern cryptography theory

Trapdoor One-way Functions

Definition:

A function f: $\{0,1\}^* \rightarrow \{0,1\}^*$ is a trapdoor one-way function iff f(x) is a one-way function; however, given some extra information it becomes feasible to compute f^{-1}: given y, find x s.t. y = f(x)



Public-Key Encryption Needs One-way Trapdoor Functions

- Given a public-key crypto system,
  - Alice has public key K
  - $\mathbf{E}_K$ must be a one-way function, knowing y= $\mathbf{E}_K[x]$, it should be difficult to find x
  - However, $\mathbf{E}_K$ must not be one-way from Alice's perspective. The function $\mathbf{E}_K$ must have a trapdoor such that knowledge of the trapdoor enables one to invert it

RSA Algorithm

- Invented in **1978** by Ron **R**ivest, Adi **S**hamir and Leonard **A**dleman
  - Published as R L Rivest, A Shamir, L Adleman, "*On Digital Signatures and Public Key Cryptosystems*", Communications of the ACM, vol 21 no 2, pp120-126, Feb 1978
- Security relies on the difficulty of factoring large composite numbers
- Essentially the same algorithm was discovered in 1973 by Clifford Cocks, who works for the British intelligence

Z_{pq}*

- Let p and q be two large primes
- Denote their product n=pq.
- $Z_n^* = Z_{pq}^*$ contains all integers in the range [1,pq-1] that are relatively prime to both p and q
- The size of Z_n^* is

  $\Phi(pq) = (p-1)(q-1)=n-(p+q)+1$
- For every $x \in Z_{pq}^*$, $x^{(p-1)(q-1)} \equiv 1$

Exponentiation in Z_{pq}^*

- Motivation: We want to use exponentiation for encryption

- Let e be an integer, $1<e<(p-1)(q-1)$

- When is the function $f(x)=x^e$, a one-to-one function in Z_{pq}^*?
- If x^e is one-to-one, then it is a permutation in Z_{pq}^*.

Review: Euler's Theorem

Euler's Theorem

Given integer $n > 1$, such that $\gcd(a, n) = 1$ then

$$a^{\Phi(n)} \equiv 1 \pmod{n}$$

Corollary: Given integer $n > 1$, such that $\gcd(a, n) = 1$ then $a^{\Phi(n)-1} \bmod n$ is a multiplicative inverse of $a \bmod n$.

Corollary: Given integer $n > 1$, x, y, and a positive integers with $\gcd(a, n) = 1$. If $x \equiv y \pmod{\Phi(n)}$, then

$$a^x \equiv a^y \pmod{n}.$$

Corollary (Fermat's "Little" Theorem):

$$a^{p-1} \equiv 1 \pmod{p}$$

Exponentiation in Z_{pq}*

- Claim: If e is relatively prime to (p-1)(q-1) then $f(x)=x^e$ is a one-to-one function in Z_{pq}*

- Proof by constructing the inverse function of f. As gcd(e,(p-1)(q-1))=1, then there exists d and k s.t. ed=1+k(p-1)(q-1)
- Let $y=x^e$, then $y^d=(x^e)^d=x^{1+k(p-1)(q-1)}=x \pmod{pq}$, i.e., $g(y)=y^d$ is the inverse of $f(x)=x^e$.

RSA Public Key Crypto System

Key generation:

Select 2 large prime numbers of about the same size, p and q

Compute $n = pq$, and $\Phi(n) = (q-1)(p-1)$

Select a random integer e, $1 < e < \Phi(n)$, s.t. $\gcd(e, \Phi(n)) = 1$

Compute d, $1 < d < \Phi(n)$ s.t. $ed \equiv 1 \bmod \Phi(n)$

Public key: (e, n)

Private key: d

RSA Description (cont.)

Encryption

Given a message M, $0 < M < n$ $\quad M \in Z_n - \{0\}$

use public key (e, n)

compute $C = M^e \bmod n$ $\quad C \in Z_n - \{0\}$

Decryption

Given a ciphertext C, use private key (d)

Compute $C^d \bmod n = (M^e \bmod n)^d \bmod n = M^{ed} \bmod n = M$

RSA Example

- $p = 11$, $q = 7$, $n = 77$, $\Phi(n) = 60$
- $d = 13$, $e = 37$ ($ed = 481$; $ed \bmod 60 = 1$)

- Let $M = 15$. Then $C \equiv M^e \bmod n$
  - $C \equiv 15^{37} \pmod{77} = 71$

- $M \equiv C^d \bmod n$
  - $M \equiv 71^{13} \pmod{77} = 15$

Coming Attractions …

- RSA Security
- Prime number generation

- Reading: Katz & Lindell: 7.2