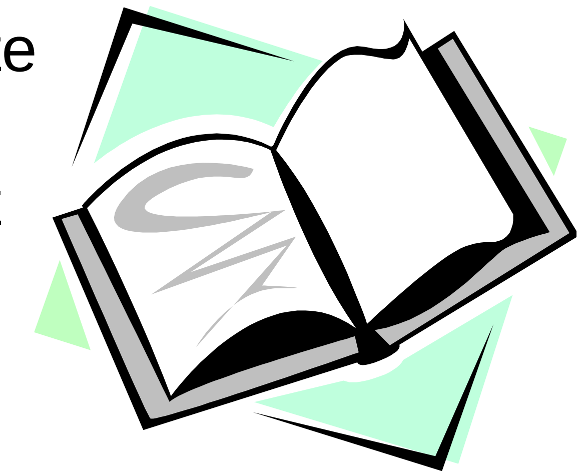
Cryptography
CS 555

Topic 6: Number Theory Basics

Outline and Readings

- Outline
  - Divisibility, Prime and composite numbers, The Fundamental theorem of arithmetic, Greatest Common Divisor, Modular operation, Congruence relation
  - The Extended Euclidian Algorithm
  - Solving Linear Congruence



- Readings:
  - Katz and Lindell: 7.1.1, 7.1.2

Divisibility

Definition

Given integers a and b, with $a \neq 0$, a divides b (denoted $a|b$) if $\exists$ integer k, s.t. $b = ak$.

a is called a **divisor** of b, and b a **multiple** of a.

Proposition:

(1) If $a \neq 0$, then $a|0$ and $a|a$. Also, $1|b$ for every b

(2) If $a|b$ and $b|c$, then $a | c$.

(3) If $a|b$ and $a|c$, then $a | (sb + tc)$ for all integers s and t.

Divisibility (cont.)

Theorem (Division algorithm)
Given integers a, b such that $a>0$, $a<b$ then there exist two unique integers q and r, $0 \leq r < a$ s.t. $b = aq + r$.

Proof:
Uniqueness of q and r:
assume $\exists$ q' and r' s.t $b = aq' + r'$, $0 \leq r' < a$, q' integer
then $aq + r = aq' + r' \Rightarrow a(q-q') = r'-r \Rightarrow q-q' = (r'-r)/a$
as $0 \leq r,r' < a \Rightarrow -a < (r'-r) < a \Rightarrow -1 < (r'-r)/a < 1$
So $-1 < q-q' < 1$, but q-q' is integer, therefore
$q = q'$ and $r = r'$

Prime and Composite Numbers

Definition

An integer $n > 1$ is called a prime number if its positive divisors are 1 and n.

Definition

Any integer number $n > 1$ that is not prime, is called a composite number.

Example

Prime numbers: 2, 3, 5, 7, 11, 13, 17 …

Composite numbers: 4, 6, 25, 900, 17778, …

Decomposition in Product of Primes

Theorem (Fundamental Theorem of Arithmetic)
Any integer number n > 1 can be written as a product of prime numbers (>1), and the product is unique if the numbers are written in increasing order.

$$n = p_1^{e_1} p_2^{e2} \ldots p_k^{ek}$$

Example: $84 = 2^2 \bullet 3 \bullet 7$

Classroom Discussion Question (Not a Quiz)

- Are the total number of prime numbers finite or infinite?

Greatest Common Divisor (GCD)

Definition

Given integers $a > 0$ and $b > 0$, we define $\gcd(a, b) = c$, the greatest common divisor (GCD), as the greatest number that divides both a and b.

Example

$\gcd(256, 100)=4$

Definition

Two integers $a > 0$ and $b > 0$ are relatively prime if $\gcd(a, b) = 1$.

Example

25 and 128 are relatively prime.

GCD as a Linear Combination

Theorem
Given integers a, b > 0 and a > b, then d = gcd(a,b) is the least positive integer that can be represented as ax + by, x, y integer numbers.

Proof: Let t be the smallest positive integer s.t. t = ax + by.
We have d | a and d | b $\Rightarrow$ d | ax + by, so d | t, so d $\leq$ t.
We now show t ≤ d.
First t | a; otherwise, a = tu + r, 0 < r < t;
r = a - ut = a - u(ax+by) = a(1-ux) + b(-uy), so we found another linear combination and r < t. Contradiction.
Similarly t | b, so t is a common divisor of a and b, thus

t ≤ gcd (a, b) = d. So t = d.

Example

gcd(100, 36) = 4 **=** 4 $\times$ 100 – 11 $\times$ 36 = 400 - 396

GCD and Multiplication

Theorem

Given integers a, b, m >1. If
gcd(a, m) = gcd(b, m) = 1, then gcd(ab, m) = 1

Proof idea:

$ax + ym = 1 = bz + tm$

Find u and v such that $(ab)u + mv = 1$

GCD and Division

Theorem
Given integers $a>0$, b, q, r, such that $b = aq + r$,
then $\gcd(b, a) = \gcd(a, r)$.

Proof:
Let $\gcd(b, a) = d$ and $\gcd(a, r) = e$, this means

$d \mid b$ and $d \mid a$, so $d \mid b - aq$, so $d \mid r$
Since $\gcd(a, r) = e$, we obtain $d \le e$.

$e \mid a$ and $e \mid r$, so $e \mid aq + r$, so $e \mid b$,
Since $\gcd(b, a) = d$, we obtain $e \le d$.

Therefore $d = e$

Finding GCD

Using the Theorem: Given integers a>0, b, q, r, such that b = aq + r, then gcd(b, a) = gcd(a, r).

Euclidian Algorithm

Find gcd (b, a)

while $a \neq 0$ *do*

$r \leftarrow b \bmod a$

$b \leftarrow a$

$a \leftarrow r$

return b

QuickTime™ and a TIFF (Uncompressed) decompressor are needed to see this picture.

Euclidian Algorithm Example

Find gcd(143, 110)

$$143 = 1 \times 110 + 33$$
$$110 = 3 \times 33 + 11$$
$$33 = 3 \times 11 + 0$$

gcd (143, 110) = 11

Modulo Operation

Definition:

$$a \bmod n = r \Leftrightarrow \exists q, \text{s.t. } a = q \times n + r$$

$$\text{where } 0 \le r \le n-1$$

Example:

7 mod 3 = 1

-7 mod 3 = 2

Congruence Relation

Definition: Let a, b, n be integers with n>0, we say that $a \equiv b \pmod{n}$,

if $a - b$ is a multiple of n.

Properties: $a \equiv b \pmod{n}$

if and only if $n \mid (a - b)$
if and only if $n \mid (b - a)$
if and only if $a = b+k\cdot n$ for some integer k
if and only if $b = a+k\cdot n$ for some integer k

E.g., $32 \equiv 7 \pmod{5}$, $-12 \equiv 37 \pmod{7}$,
$17 \equiv 17 \pmod{13}$

Properties of the Congruence Relation

Proposition: Let a, b, c, n be integers with $n>0$

1. $a \equiv 0 \pmod{n}$ if and only if $n \mid a$
2. $a \equiv a \pmod{n}$
3. $a \equiv b \pmod{n}$ if and only if $b \equiv a \pmod{n}$
4. if $a \equiv b$ and $b \equiv c \pmod{n}$, then $a \equiv c \pmod{n}$

Corollary: Congruence modulo n is an equivalence relation.

Every integer is congruent to exactly one number in $\{0, 1, 2, \ldots, n-1\}$ modulo n

Equivalence Relation

Definition

A binary relation R over a set Y is a subset of $Y \times Y$.

We denote a relation $(a,b) \in R$ as aRb.

•example of relations over integers?

Definition

A relation is an equivalence relation on a set Y, if R is

Reflexive: aRa for all $a \in R$

Symmetric: for all $a, b \in R$, $aRb \Rightarrow bRa$.

Transitive: for all $a,b,c \in R$, aRb and $bRc \Rightarrow aRc$

Example

"=" is an equivalence relation on the set of integers

More Properties of the Congruence Relation

Proposition: Let a, b, c, n be integers with $n>0$

If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then:

$a + c \equiv b + d \pmod{n}$,

$a - c \equiv b - d \pmod{n}$,

$a \cdot c \equiv b \cdot d \pmod{n}$

E.g., $5 \equiv 12 \pmod{7}$ and $3 \equiv -4 \pmod{7}$, then, …

Multiplicative Inverse

Definition: Given integers n>0, a, b, we say that b is a **multiplicative inverse of** a **modulo** n if

$$ab \equiv 1 \pmod{n}.$$

Proposition: Given integers n>0 and a, then a has a multiplicative inverse modulo n if and if only if a and n are relatively prime.

Towards Extended Euclidian Algorithm

- **Theorem:** Given integers a, b > 0, then d = gcd(a,b) is the least positive integer that can be represented as ax + by, x, y integer numbers.
- How to find such x and y?

The Extended Euclidian Algorithm

First computes	Then computes	And
$b = q_1 a + r_1$	$x_0 = 0$	$y_0 = 1$
$a = q_2 r_1 + r_2$	$x_1 = 1$	$y_1 = 0$
$r_1 = q_3 r_2 + r_3$	$x_2 = -q_1 x_1 + x_0$	$y_2 = -q_1 y_1 + y_0$
…	…	…
$r_{k-3} = q_{k-1} r_{k-2} + r_{k-1}$	$x_k = -q_{k-1} x_{k-1} + x_{k-2}$	$y_k = -q_{k-1} y_{k-1} + y_{k-2}$
$r_{k-2} = q_k r_{k-1}$		

We have $ax_k + by_k = r_{k-1} = \gcd(a,b)$

Extended Euclidian Algorithm

```
Extended_Euclidian (a,b)
    x=1;  y=0;  d=a;  r=0;  s=1;  t=b;
    while (t>0) {
                                q = ⌊d/t⌋;
          u=x-qr;  v=y-qs;  w=d-qt;
          x=r;     y=s;     d=t;
          r=u;     s=v;     t=w;
    }
    return (d, x, y)
end
```

Invariants:

$ax + by = d$

$ar + bs = t$

Another Way

Find gcd(143, 111)

$143 = 1 \times 111 + 32$

$111 = 3 \times 32 + 15$

$32 = 2 \times 15 + 2$

$15 = 7 \times 2 + 1$

gcd (143, 111) = 1

$32 = 143 - 1 \times 111$

$15 = 111 - 3 \times 32$

$= 4 \times 111 - 3 \times 143$

$2 = 32 - 2 \times 15$

$= 7 \times 143 - 9 \times 111$

$1 = 15 - 7 \times 2$

$= 67 \times 111 - 52 \times 143$

Linear Equation Modulo n

If gcd(a, n) = 1, the equation

$$ax \equiv 1 \bmod n$$

has a unique solution, $0 < x < n$. This solution is often represented as $a^{-1} \bmod n$

Proof: if $ax_1 \equiv 1 \pmod n$ and $ax_2 \equiv 1 \pmod n$, then $a(x_1-x_2) \equiv 0 \pmod n$, then $n \mid a(x_1-x_2)$, then $n \mid (x_1-x_2)$, then $x_1-x_2=0$

How to compute $a^{-1} \bmod n$?

Examples

Example 1:

- Observe that $3 \cdot 5 \equiv 1 \pmod 7$.
- Let us try to solve $3 \cdot x+4 \equiv 3 \pmod 7$.
- Subtracts 4 from both side, $3 \cdot x \equiv -1 \pmod 7$.
- We know that $-1 \equiv 6 \pmod 7$.
- Thus $3 \cdot x \equiv 6 \pmod 7$.
- Multiply both side by 5, $3 \cdot 5 \cdot x \equiv 5 \cdot 6 \pmod 7$.
- Thus, $x \equiv 1 \cdot x \equiv 3 \cdot 5 \cdot x \equiv 5 \cdot 6 \equiv 30 \equiv 2 \pmod 7$.
- Thus, any x that satisfies $3 \cdot x+4 \equiv 3 \pmod 7$ must satisfy $x \equiv 2 \pmod 7$ and vice versa.

Question: To solve that $2x \equiv 2 \pmod 4$.
Is the solution $x \equiv 1 \pmod 4$?

Linear Equation Modulo (cont.)

To solve the equation

$$ax \equiv b \bmod n$$

When gcd(a,n)=1, compute $x = a^{-1} b \bmod n$.

When gcd(a,n) = d >1, do the following

- If d does not divide b, there is no solution.
- Assume d|b. Solve the new congruence, get x_0

$$(a/d)x \equiv b/d \ (\bmod\ n/d)$$

- The solutions of the original congruence are x_0, $x_0+(n/d)$, $x_0+2(n/d)$, …, $x_0+(d-1)(n/d)$ (mod n).

Solving Linear Congruences

Theorem:

- Let a, n, z, z' be integers with n>0. If gcd(a,n)=1, then $az \equiv az'$ (mod n) if and only if $z \equiv z'$ (mod n).
- More generally, if d:=gcd(a,n), then $az \equiv az'$ (mod n) if and only if $z \equiv z'$ (mod n/d).

Example:

- $5 \cdot 2 \equiv 5 \cdot -4 \pmod{6}$
- $3 \cdot 5 \equiv 3 \cdot 3 \pmod{6}$

Coming Attractions …

- More on secure encryption
- Reading: Katz & Lindell: 3.4, 3.5, 3.6