Information Security CS 526

Topic 3

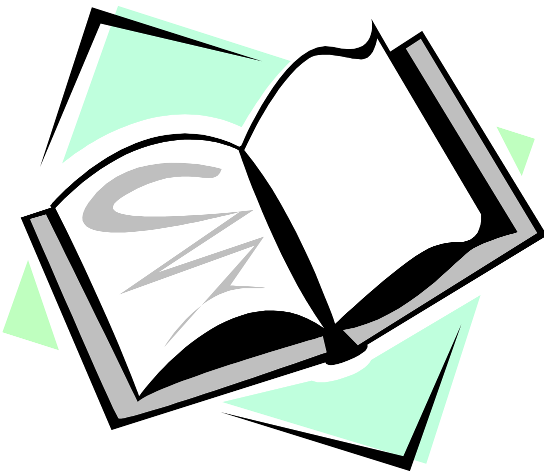
Cryptography: One-time Pad, Information Theoretic Security, and Stream CIphers

Announcements

- HW1 is out, due on Sep 5
  - Start early, late policy is 3 total late days
- TA Office hour set
  - Mohammed Almeshekah: Office: LWSN2161; Office hour: Monday 10am to 11am, Thursday 9am to 10am
  - Weining Yang: Office: LWSN 2161; Office hour: Wednesday 10am to 11am, Friday 1pm to 2pm
- Planned project 1: web security, to be assigned on Sep 19
- Planned project 2: implementation of file encryption

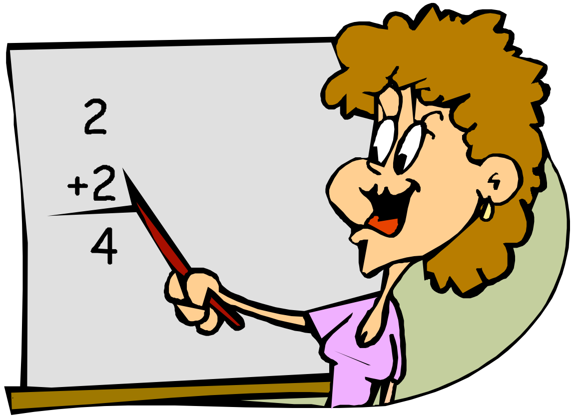
Readings for This Lecture

- Required reading from wikipedia
  - One-Time Pad
  - Information theoretic security
  - Stream cipher
  - Pseudorandom number generator



- Stream ciphers on Dan Boneh's Cryptography I course on Coursea

Begin Math



Random Variable

Definition

A **discrete random variable, X,** consists of a finite set $\mathcal{X}$, and a probability distribution defined on $\mathcal{X}$. The probability that the random variable **X** takes on the value x is denoted **Pr**[**X** =x]; sometimes, we will abbreviate this to **Pr**[x] if the random variable **X** is fixed. It must be that

$$0 \le \Pr[x] \quad \text{for all } x \in \mathcal{X}$$

$$\sum_{x \in \mathcal{X}} \Pr[x] = 1$$

Example of Random Variables

- Let random variable $\mathbf{D_1}$ denote the outcome of throwing one die (with numbers 0 to 5 on the 6 sides) randomly, then $\mathcal{D}=\{0,1,2,3,4,5\}$ and $\mathbf{Pr}[\mathbf{D_1}=i] = 1/6$ for $0 \leq i \leq 5$
- Let random variable $\mathbf{D_2}$ denote the outcome of throwing a second such die randomly
- Let random variable $\mathbf{S_1}$ denote the sum of the two dice, then $\mathcal{S} =\{0,1,2,\ldots,10\}$, and

  $\mathbf{Pr}[\mathbf{S_1}=0] = \mathbf{Pr}[\mathbf{S_1}=10] = 1/36$

  $\mathbf{Pr}[\mathbf{S_1}=1] = \mathbf{Pr}[\mathbf{S_1}=9] = 2/36 = 1/18$

  …
- Let random variable $\mathbf{S_2}$ denote the sum of the two dice modulo 6, what is the distribution of $\mathbf{S_2}$?

Relationships between Two Random Variables

Definitions

Assume **X** and **Y** are two random variables,
then we define:

- joint probability: **Pr**[x, y] is the probability that **X** takes value x and **Y** takes value y.
- conditional probability: **Pr**[x|y] is the probability that **X** takes value x given that **Y** takes value y.

  Pr[x|y] = **Pr**[x, y] / **Pr**[y]
- independent random variables: **X** and **Y** are said to be independent if **Pr**[x,y] = **Pr**[x]P[y], for all $x \in \mathcal{X}$ and all $y \in \mathcal{Y}$.

Examples

- Joint probability of $\mathbf{D_1}$ and $\mathbf{D_2}$
  for $0 \le i,j \le 5$, $\mathbf{Pr}[\mathbf{D_1}=i, \mathbf{D_2}=j] = ?$
- What is the conditional probability $\mathbf{Pr}[\mathbf{D_1}=i \mid \mathbf{D_2}=j]$ for $0 \le i, j \le 5$?
- Are $\mathbf{D_1}$ and $\mathbf{D_2}$ independent?

- Suppose $\mathbf{D_1}$ is plaintext and $\mathbf{D_2}$ is key, and $\mathbf{S_1}$ and $\mathbf{S_2}$ are ciphertexts of two different ciphers, which cipher would you use?

Examples to think after class

- What is the joint probability of $\mathbf{D_1}$ and $\mathbf{S_1}$?
- What is the joint probability of $\mathbf{D_2}$ and $\mathbf{S_2}$?

- What is the conditional probability $\mathbf{Pr}[\mathbf{S_1}=s \mid \mathbf{D_1}=i]$ for $0 \le i \le 5$ and $0 \le s \le 10$?
- What is the conditional probability $\mathbf{Pr}[\mathbf{D_1}=i \mid \mathbf{S_2}=s]$ for $0 \le i \le 5$ and $0 \le s \le 5$?

- Are $\mathbf{D_1}$ and $\mathbf{S_1}$ independent?
- Are $\mathbf{D_1}$ and $\mathbf{S_2}$ independent?

Bayes' Theorem

If P[y] > 0 then

$$P[x \mid y] = \frac{P[x]P[y \mid x]}{P[y]}$$

$$P[y] = \sum_{x \in X} P[x, y] = \sum_{x \in X} P[x]p[y \mid x]$$

Corollary

X and Y are independent random variables iff P[x|y] = P[x], for all x $\in$ X and all y $\in$ Y.

End Math

One-Time Pad

- Fix the vulnerability of the Vigenere cipher by using very long keys

- Key is a random string that is at least as long as the plaintext
- Encryption is similar to shift cipher
- Invented by Vernam in the 1920s

One-Time Pad

Let $Z_m = \{0,1,\ldots,m-1\}$ be the alphabet.

Plaintext space = Ciphtertext space = Key space = $(Z_m)^n$

The key is chosen uniformly randomly

Plaintext $X = (x_1\ x_2\ \ldots\ x_n)$

Key $K = (k_1\ k_2\ \ldots\ k_n)$

Ciphertext $Y = (y_1\ y_2\ \ldots\ y_n)$

$e_k(X) = (x_1+k_1\ \ x_2+k_2\ \ldots\ x_n+k_n) \bmod m$

$d_k(Y) = (y_1-k_1\ \ y_2-k_2\ \ldots\ y_n-k_n) \bmod m$

The Binary Version of One-Time Pad

Plaintext space = Ciphtertext space =

Keyspace = $\{0,1\}^n$

Key is chosen randomly

For example:

- Plaintext is 11011011
- Key is 01101001
- Then ciphertext is 10110010

Bit Operators

- Bit AND

  $0 \wedge 0 = 0$ $0 \wedge 1 = 0$ $1 \wedge 0 = 0$ $1 \wedge 1 = 1$

- Bit OR

  $0 \vee 0 = 0$ $0 \vee 1 = 1$ $1 \vee 0 = 1$ $1 \vee 1 = 1$

- Addition mod 2 (also known as Bit XOR)

  $0 \oplus 0 = 0$ $0 \oplus 1 = 1$ $1 \oplus 0 = 1$ $1 \oplus 1 = 0$

- Can we use operators other than Bit XOR for binary version of One-Time Pad?

How Good is One-Time Pad?

- Intuitively, it is secure …
  - The key is random, so the ciphertext is completely random
- How to formalize the confidentiality requirement?
  - Want to say "certain thing" is not learnable by the adversary (who sees the ciphertext). But what is the "certain thing"?
- Which (if any) of the following is the correct answer?
  - The key.
  - The plaintext.
  - Any bit of the plaintext.
  - Any information about the plaintext.
    - E.g., the first bit is 1, the parity is 0, or that the plaintext is not "aaaa", and so on

Shannon (Information-Theoretic) Security = Perfect Secrecy

- Basic Idea: Ciphertext should reveal no "information" about Plaintext

Definition. An encryption over a message space $\mathcal{M}$ is perfectly secure if

$\forall$ probability distribution over $\mathcal{M}$

$\forall$ message $m \in \mathcal{M}$

$\forall$ ciphertext $c \in \mathcal{C}$ for which $\Pr[C=c] > 0$

We have

$$\Pr[\mathbf{PT}=m \mid \mathbf{CT}=c] = \Pr[\mathbf{PT} = m].$$

Explanation of the Definition

– Pr [**PT** = m] is what the adversary believes the probability that the plaintext is m, before seeing the ciphertext

– Pr [**PT** = m | **CT**=c] is what the adversary believes after seeing that the ciphertext is c

– Pr [**PT**=m | **CT**=c] = Pr [**PT** = m] means that after knowing that the ciphertext is C_0, the adversary's belief does not change.

Equivalent Definition of Perfect Secrecy

Definition. An encryption scheme over a message space $\mathcal{M}$ is perfectly secure if $\forall$ probability distribution over $\mathcal{M}$, the random variables **PT** and **CT** are independent. That is,

$\forall$ message $m \in \mathcal{M}$

$\forall$ ciphertext $c \in \mathcal{C}$

$\Pr[\mathbf{PT}{=}m \wedge \mathbf{CT}{=}c] = \Pr[\mathbf{PT} = m] \Pr[\mathbf{CT} = c]$

Note that this is equivalent to: When $\Pr[\mathbf{CT} = c] \neq 0$, we have

$\Pr[\mathbf{PT} = m] = \Pr[\mathbf{PT}{=}m \wedge \mathbf{CT}{=}c] / \Pr[\mathbf{CT} = c] = \Pr[\mathbf{PT}{=}m \mid \mathbf{CT}{=}c]$

This is also equivalent to: When $\Pr[\mathbf{PT} = m] \neq 0$, we have

$\Pr[\mathbf{CT} = c] = \Pr[\mathbf{PT}{=}m \wedge \mathbf{CT}{=}c] / \Pr[\mathbf{PT} = m] = \Pr[\mathbf{CT}{=}c \mid \mathbf{PT}{=}m]$

Example for Information Theoretical Security

- Consider an example of encrypting the result of a 6-side dice (1 to 6).
  - Method 1: randomly generate K=[0..5], ciphertext is result + K.
    - What is plaintext distribution? After seeing that the ciphertext is 6, what could be the plaintext. After seeing that the ciphertext is 11, what could be the plaintext?
  - Method 2: randomly generate K=[0..5], ciphertext is (result + K) mod 6.
    - Same questions.
    - Can one do a brute-force attack?

Perfect Secrecy

- Fact: When keys are uniformly chosen in a cipher, the cipher has perfect secrecy iff. the number of keys encrypting M to C is the same for any (M,C)
  - This implies that
    $\forall c \forall m_1 \forall m_2 \; \Pr[\mathbf{CT}=c \mid \mathbf{PT}=m_1] = \Pr[\mathbf{CT}=c \mid \mathbf{PT}=m_2]$

- One-time pad has perfect secrecy when limited to messages over the same length (Proof?)

Key Randomness in One-Time Pad

- One-Time Pad uses a very long key, what if the key is not chosen randomly, instead, texts from, e.g., a book are used as keys.
  - this is not One-Time Pad anymore
  - this does not have perfect secrecy
  - this can be broken
  - How?
- The key in One-Time Pad should never be reused.
  - If it is reused, it is Two-Time Pad, and is insecure!
  - Why?

Usage of One-Time Pad

- To use one-time pad, one must have keys as long as the messages.
- To send messages totaling certain size, sender and receiver must agree on a shared secret key of that size.
  - typically by sending the key over a secure channel
- This is difficult to do in practice.
- Can't one use the channel for send the key to send the messages instead?
- Why is OTP still useful, even though difficult to use?

Usage of One-Time Pad

- The channel for distributing keys may exist at a different time from when one has messages to send.

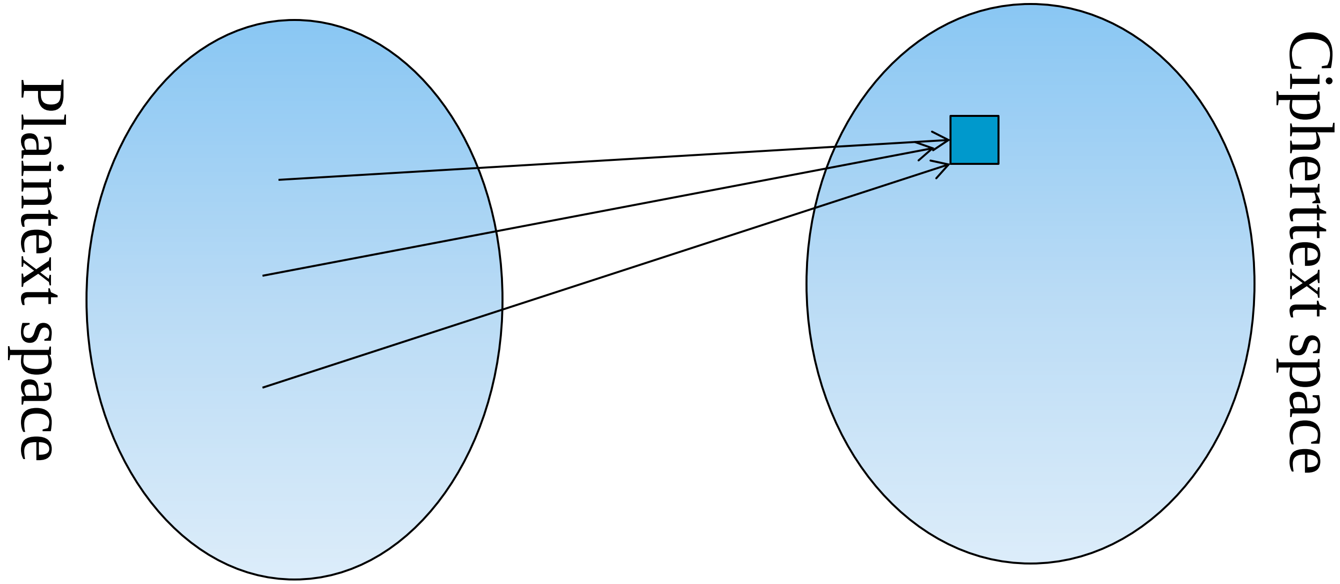
- The channel for distributing keys may have the property that keys can be leaked, but such leakage will be detected
  - Such as in Quantum cryptography

The "Bad News" Theorem for Perfect Secrecy

- Question: OTP requires key as long as messages, is this an inherent requirement for achieving perfect secrecy?
- Answer. Yes. Perfect secrecy implies that

  key-length ≥ msg-length

Proof:



- Implication: Perfect secrecy difficult to achieve in practice

Stream Ciphers

- In One-Time Pad, a key is a random string of length at least the same as the message
- Stream ciphers:
  - Idea: replace “rand” by “pseudo rand”
  - Use Pseudo Random Number Generator
  - PRNG: $\{0,1\}^s \rightarrow \{0,1\}^n$
    - expand a short (e.g., 128-bit) random seed into a long (e.g., 10^6 bit) string that “looks random”
  - Secret key is the seed
  - $E_{key}[M] = M \oplus PRNG(key)$

The RC4 Stream Cipher

- A proprietary cipher owned by RSA, designed by Ron Rivest in 1987.
- Became public in 1994.
- Simple and effective design.
- Variable key size (typical 40 to 256 bits),
- Output unbounded number of bytes.
- Widely used (web SSL/TLS, wireless WEP).
- Extensively studied, not a completely secure PRNG, first part of output biased, when used as stream cipher, should use RC4-Drop[n]
  - Which drops first n bytes before using the output
  - Conservatively, set n=3072

Pseudo Random Number Generator

- Useful for cryptography, simulation, randomized algorithm, etc.
  - Stream ciphers, generating session keys
- The same seed always gives the same output stream
  - Why is this necessary for stream ciphers?
- Simulation requires uniform distributed sequences
  - E.g., having a number of statistical properties
- **Cryptographically secure pseudo-random number generator** requires unpredictable sequences
  - satisfies the "next-bit test“: given consecutive sequence of bits output (but not seed), next bit must be hard to predict
- Some PRNG’s are weak: knowing output sequence of sufficient length, can recover key.
  - Do not use these for cryptographic purposes

Properties of Stream Ciphers

- Typical stream ciphers are very fast

- Widely used, often incorrectly
  - Content Scrambling System (uses Linear Feedback Shift Registers incorrectly),
  - Wired Equivalent Privacy (uses RC4 incorrectly)
  - SSL (uses RC4, SSLv3 has no known major flaw)

Security Properties of Stream Ciphers

- Under known plaintext, chosen plaintext, or chosen ciphertext, the adversary knows the key stream (i.e., PRNG(key))
  - Security depends on PRNG
  - PRNG must be “unpredictable”
- Do stream ciphers have perfect secrecy?
- How to break a stream cipher in a brute-force way?
- If the same key stream is used twice, then easy to break.
  - This is a fundamental weakness of stream ciphers; it exists even if the PRNG used in the ciphers is strong

Using Stream Ciphers in Practice

- If the same key stream is used twice, then easy to break.
  - This is a fundamental weakness of stream ciphers; it exists even if the PRNG used in the ciphers is strong
- In practice, one key is used to encrypt many messages
  - Example: Wireless communication
  - Solution: Use Initial vectors (IV).
  - $E_{key}[M] = [IV, M \oplus PRNG(key \,||\, IV)]$
    - IV is sent in clear to receiver;
    - IV needs integrity protection, but not confidentiality protection
    - IV ensures that key streams do not repeat, but does not increase cost of brute-force attacks
    - Without key, knowing IV still cannot decrypt
  - Need to ensure that IV never repeats! How?

Coming Attractions …

- Cryptography: Semantic Security, Block ciphers, encryption modes, cryptographic functions