Approximating Cumulative Pebbling Cost is Unique Games Hard

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(Parallel) Graph Pebbling and Cumulative Pebbling Cost cc(G)

Overview

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(Parallel) Graph Pebbling.

- Pebbling example
- Cumulative Pebbling Cost of G

Problem Statement.

- Given a DAG G find the (approx.) minimum cost pebbling

Significance of cc(G)**.**

- Analysis of data-independent memory-hard functions
- Amortization / Parallelism

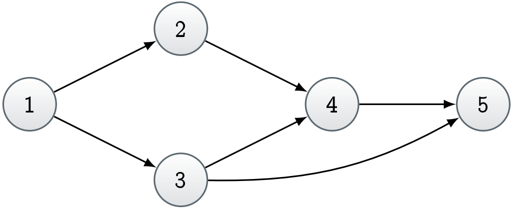
Results.

- Unique Games Hard to approximate cc(G) for any constant factor

Technical Ingredients.

- Indegree reduction using γ-extreme depth robust graphs
- Superconcentrator overlay

(Parallel) Pebbling Example.



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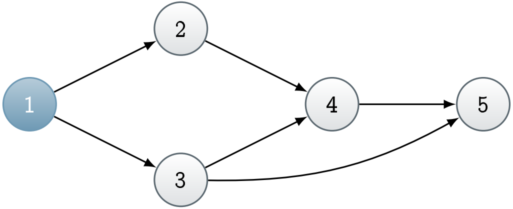

$$P_1 = \{1\}$$

(Parallel) Graph Pebbling and Cumulative Pebbling Cost $\mathrm{cc}(G)$

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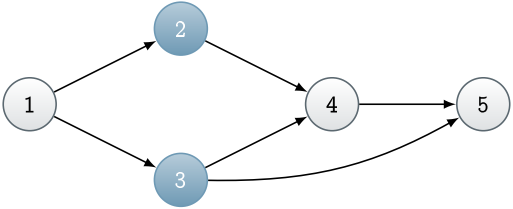

$$P_1 = \{1\}, P_2 = \{2, 3\}$$

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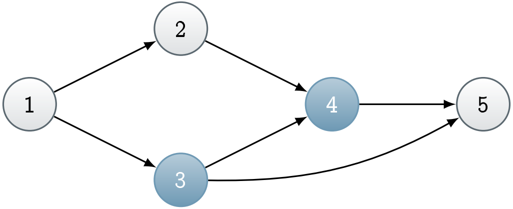

$$P_1 = \{1\}, P_2 = \{2,3\}, P_3 = \{3,4\}$$

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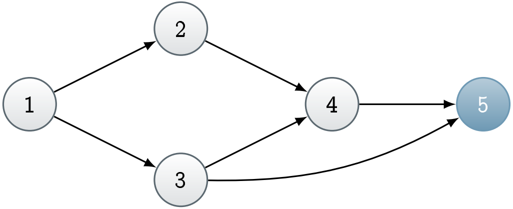

$$P_1 = \{1\}, P_2 = \{2,3\}, P_3 = \{3,4\}, P_4 = \{5\}$$

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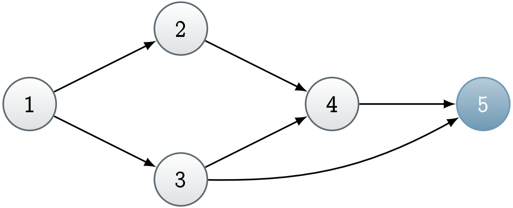

$$P_1 = \{1\}, P_2 = \{2,3\}, P_3 = \{3,4\}, P_4 = \{5\}$$

$$\mathrm{cc}(G) := \min_P \{|P_1| + \cdots + |P_t|\}$$

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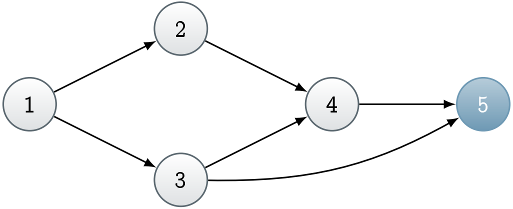

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$$\text{cc}(G) := \min_P \{|P_1| + \cdots + |P_t|\}$$

$$\therefore \text{cc}(G) \leq \sum_{i=1}^{t} |P_i| = 1$$

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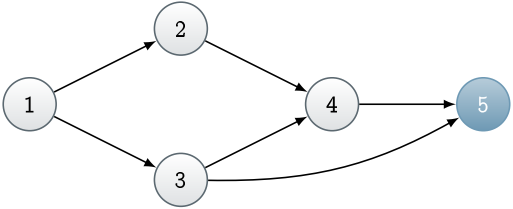

$$P_1 = \{1\}, P_2 = \{2, 3\}, P_3 = \{3, 4\}, P_4 = \{5\}$$

$$\text{cc}(G) := \min_P \{|P_1| + \cdots + |P_t|\}$$

$$\therefore \text{cc}(G) \leq \sum_{i=1}^{t} |P_i| = 1 + 2$$

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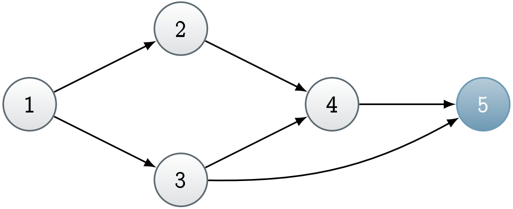

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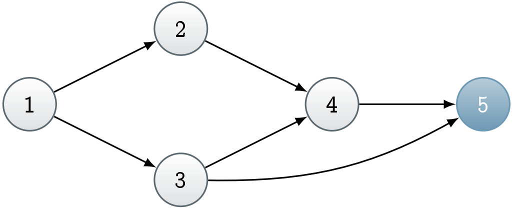

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$$\text{cc}(G) := \min_P \{|P_1| + \cdots + |P_t|\}$$

$$\therefore \text{cc}(G) \le \sum_{i=1}^t |P_i| = 1 + 2 + 2 + 1 = 6.$$

Significance of cc(G) and a Challenging Problem

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Challenging Problem.

- Given a DAG G, find the (approximately) minimum cost pebbling

Why We Care About cc(G)**?**

- Analysis of data-independent Memory-Hard Functions (iMHFs)
- [AS15] For a secure iMHF, it suffices to find a DAG G with *constant indegree* and *maximum* cc(G)
- Amortization / Parallelism ($\mathrm{cc}(G^{\times n}) = n \times \mathrm{cc}(G)$)

Challenges.

- We don't know how to compute cc(G) exactly for any given G
- Large gaps between upper/lower bounds for known constructions

Example

DRSample: one practical instantiation of an iMHF

$$\frac{10^{-6} \cdot N^2}{\log N} \leq \mathrm{cc}(\mathrm{DRSample}) \leq \frac{1 \cdot N^2}{\log N}.$$

Our Main Result: Hardness of Approximating cc(G)

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Our Result.

- [BZ18] proved that computing cc(G) is NP-Hard
- This did not rule out the existence of a constant-factor approximation algorithm for cc(G)
- Our result is the hardness of any constant factor approximation to the cost of graph pebbling even for DAGs with constant indegree.

Theorem

Given a DAG G with constant indegree, it is Unique Games hard to approximate cc(G) *within any constant factor.*

Our Main Result: Hardness of Approximating cc(G)

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Theorem

Given a DAG G with constant indegree, it is Unique Games hard to approximate cc(G) *within any constant factor.*

Implication.

- Cryptanalysis of iMHFs is Hard!

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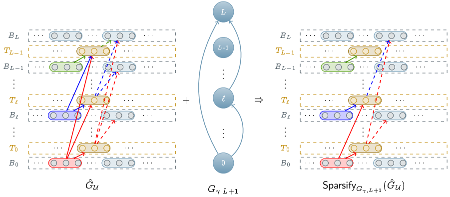
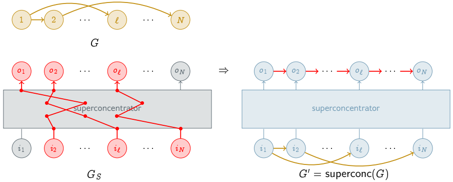
Svensson's Result [Sve12].

- cc(G) is related to the combinatorial property called Depth-Robustness
- Unique Games Hard to approximately test DAGs for Depth-Robustness
  - Challenge 1: Svensson's reduction dœsn't work for constant indegree graphs
  - Challenge 2: Connection between Depth-Robustness and cc(G) is not tight

Indegree Reduction Procedure using γ-Extreme DR Graph $G_{\gamma,L+1}$.



Superconcentrator Overlay.



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