Fair Division of Goods, Bads, and Mixed

Colloquium@Purdue

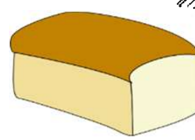
Nov 17, 2021

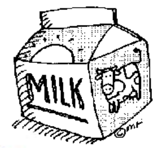
Ruta Mehta

ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

(Based on joint work with S. Boodaghians, B. R. Chaudhary, J. Garg, and P. McGlaughlin)

Divisible items



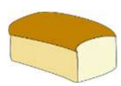


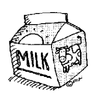
Goal: Find *fair* and *efficient* allocation

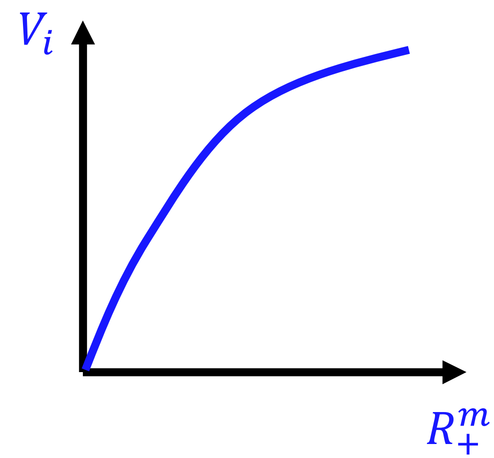

ASSETS
LIABILITIES
U.S.ARMY
U.S.AIR FORCE

Model

- A: set of n agents
- M: set of m divisible items (manna)
  - Supply of every item is one





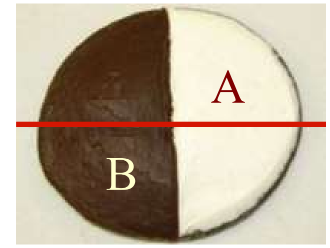


- Each agent i has
  - Concave valuation function $V_i: R_+^m \to R$ over bundles of items

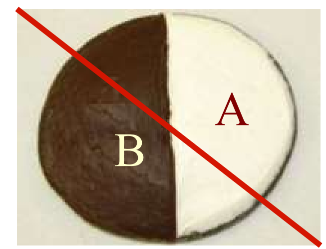
Goal: Find *fair* and *efficient* allocation

Example: Half moon cookie

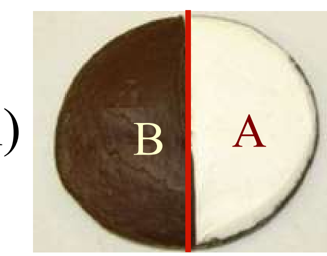
(i)



(ii)



(iii)



Agreeable (Fair)

Envy-free: No agent *envies* other's allocation over her own.

Proportional: Each agent i gets value at least $\frac{v_i(M)}{n}$

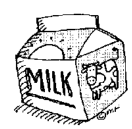
Non-wasteful (Efficient)

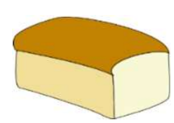
Pareto-optimal: No other allocation is better for all.

(Nash) Welfare Maximizing

[3, 2, 2] 

[20, 20, 30] 





Agreeable (Fair)	Non-wasteful (Efficient)
Envy-free	Pareto-optimal
Proportional	(Nash) Welfare Maximizing

Competitive Equilibrium
(with equal income)

Competitive Equilibrium (CE)

traditional setting…

Goods

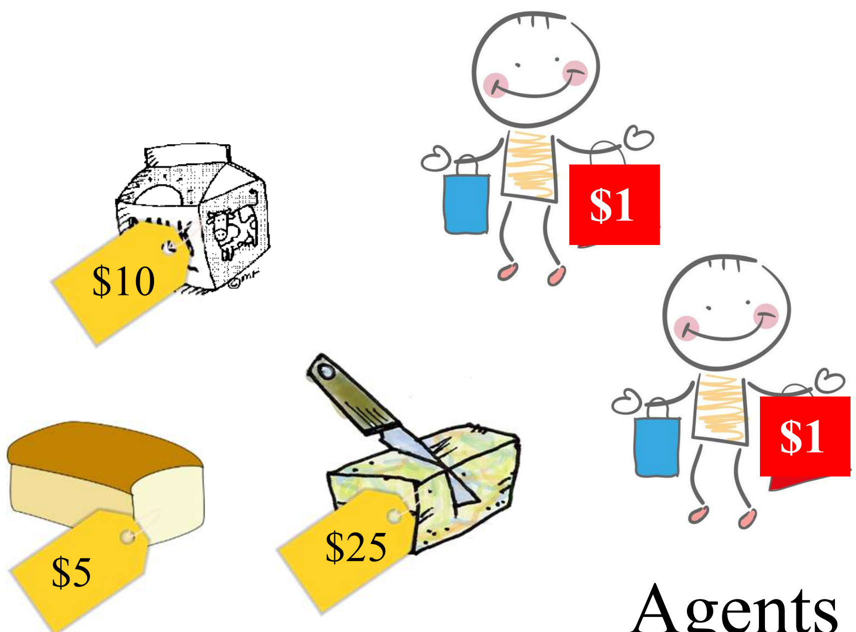
Agents
(buyers)

Buy optimal bundle ⟶ Demand

Competitive Equilibrium:
Demand = Supply

w/ equal income (CEEI):
Each agent has one dollar to spend

CEEI: Properties



Goods

Agents
(buyers)

Demand optimal bundle

Competitive Equilibrium:
Demand = Supply

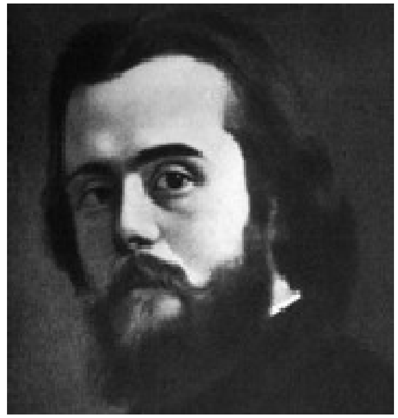
An agent can afford anyone's bundle, but demands hers
⇒ Envy-free

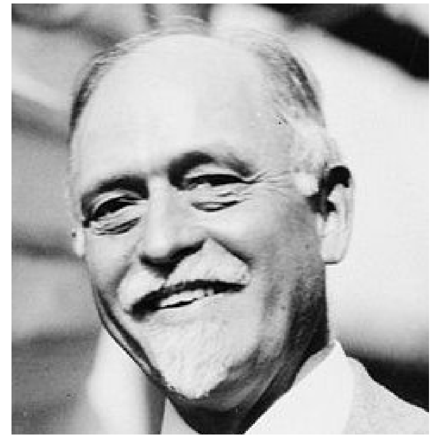
Envy-free, everything allocated
⇒ Proportional

1^{st} welfare theorem
⇒ Pareto-optimal

CE History


Adam Smith (1776)


Leon Walras (1880s)


Irving Fisher (1891)


Arrow-Debreu (1954)
(Nobel prize)

(Existence of CE in the exchange model w/ firms)

…

Computation of CE (w/ goods)

Algorithms

- Convex programming formulations
  - Eisenberg-Gale (1959): CEEI w/ 1-homogeneous valuations
  - Shmyrev (2009), DGV (2013), CDGJMVY (2017) …
- (Strongly) Poly-time algorithms (linear valuations)
  - DPSV (2002), Orlin (2010), DM (2015), GV (2019) …
- Simplex-like algorithms: Eaves (1976), GM.SV (2011), GM.V (2014), …

Complexity

- PPAD: Papadimitrou'92, CDDT'09, VY'11, CPY'17, Rubinstein'18, …
- FIXP: EY'09, GM.VY'17, F-RHHH'21 …

*Chen, Cole, Deng, Devanur, Duan, Dai, Etessami, Filos-Ratsikas, Garg, Gkatzelis, Hansen, Hogh, Hollender, Jain, Mai, Mehlhorn, Papadimitriou, Paparas, Saberi, Sohoni, Vazirani, Vegh, Yazdanbod, Yannakakis, …

CE with Bads/Chores

Demand optimal bundle

Competitive Equilibrium:
Demand = Supply

Agents

w/ equal income (CEEI):
Each agent *has to earn* one dollar

CE with Mixed Items

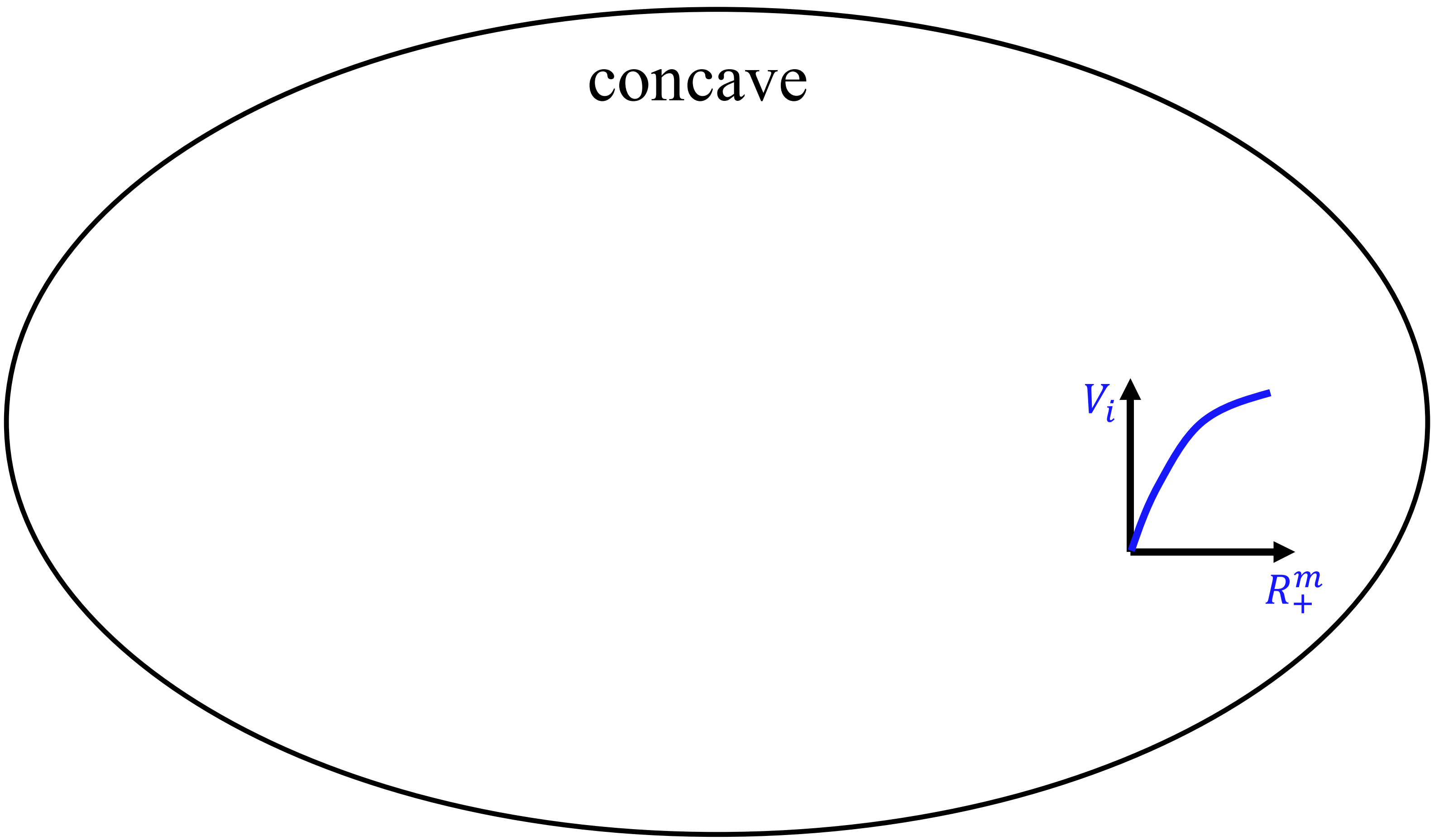
Demand optimal bundle

Agents

Competitive Equilibrium:
Demand = Supply

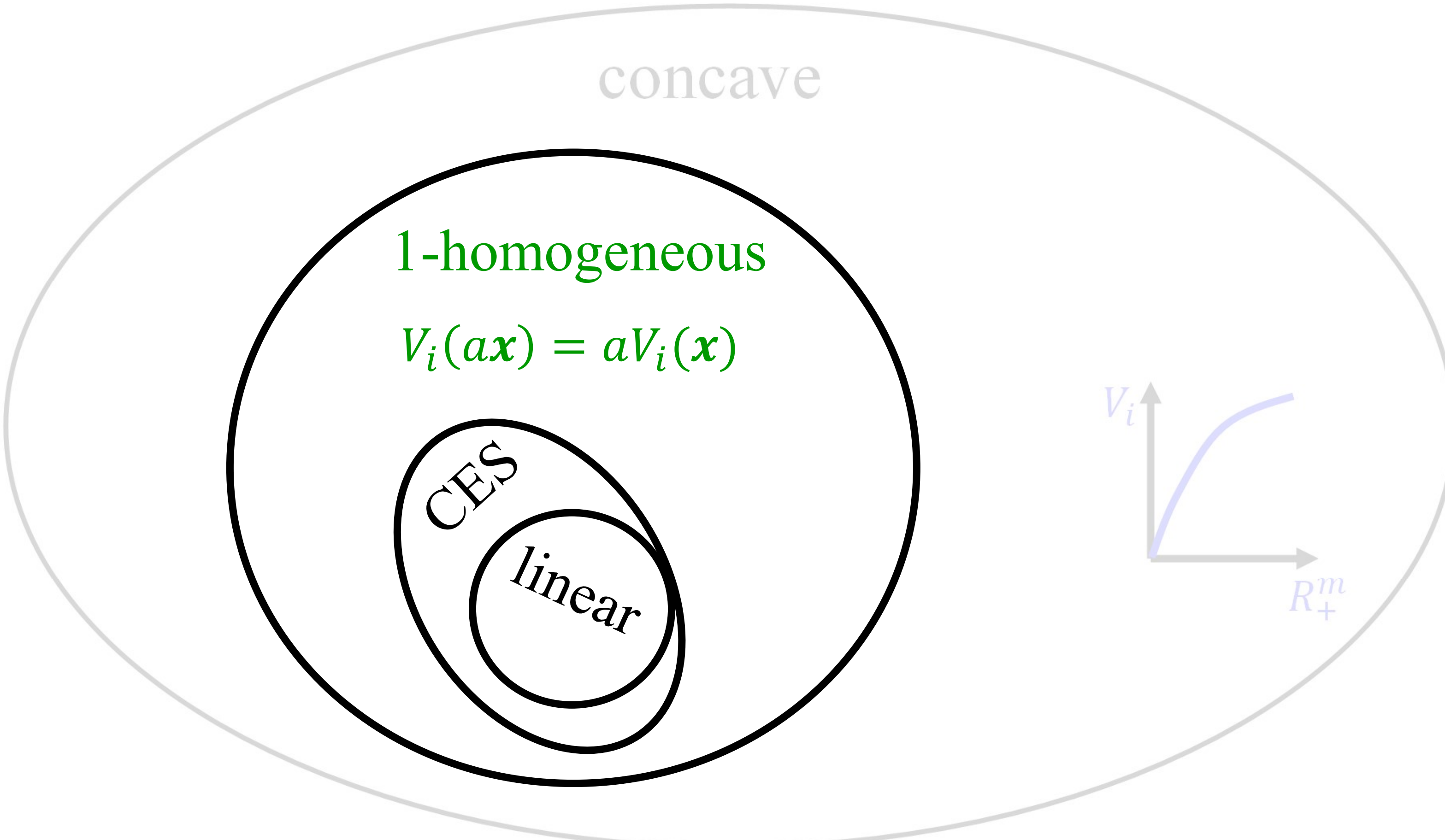
Bad/Mixed Manna: Known Results

- Bogomolnaia-Moulin-Sandomirskiy-Yanovskaia (2017)
  - $V_i s$ are 1-homogeneous



Bad/Mixed Manna: Known Results

- Bogomolnaia-Moulin-Sandomirskiy-Yanovskaia (2017)
  - $V_i s$ are 1-homogeneous



Bad/Mixed Manna: Known Results

- Bogomolnaia-Moulin-Sandomirskiy-Yanovskaia (Econometrica'17)
  - $V_i s$ are 1-homogeneous: CEEI characterization
  - CEEI set is non-convex even with linear vals, bads only.
    - In contrast, for goods: convex CE set, (strongly) poly-time algos.
  - **"expect computational difficulties"**

- Linear V_is, #agents or #items is a constant:
  - Branzei-Sandomiskiy'19 (to appear in OR): poly-time for bad manna
  - Garg-McGlaughlin (AAMAS'20): poly-time for mixed manna

Questions: Complexity w/ linear valuations?

Is even efficient approximation possible?

Our Results

Bads manna, linear valuations

CEEI

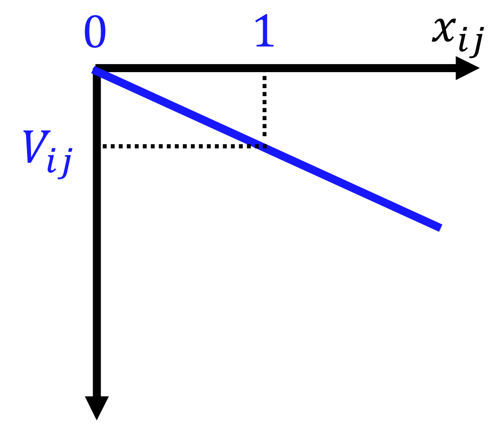
- Boodaghians-Chaudhury-M. (SODA 2022):
  - An *exterior-point method* to find an approximate CEEI (FPTAS)
  - Extends to 1-homogeneous valuations.
  - Extends to mixed manna and to CE.

General method: coordinate-wise monotone functions.

Linear Bads: Exterior-point Method

Bads w/ Linear Valuations

- A: set of n agents
  - Each agent needs to earn \$1
- M: set of m divisible bads/chores
  - Supply of every chore is one



- Each agent i has
  - Valuations: $V_i(x_i) = \sum_j V_{ij} x_{ij}$
  - Disutility: $V_{ij} \leq 0$. $D_{ij} = |V_{ij}|, D_i(x_i) = \sum_j D_{ij} x_{ij}$

CEEI: KKT Points of

x feasible $\equiv \sum_{i\in A} x_{ij} = 1,\ \forall j \in M;\quad \boldsymbol{x} \geq \boldsymbol{0}$

Goods ($V_{ij} \geq 0$) [EG'59]

$$\max_{x \text{ feasible}} \Pi_{i\in N} V_i(x_i)$$

Bads [BMSY'17]

$$\min_{x \text{ feasible}} \Pi_{i\in N} D_i(x_i)$$

$$\text{s.t. } D_i(x_i) > 0,\ \forall i$$

CEEI: KKT Points of

Goods [EG’59]

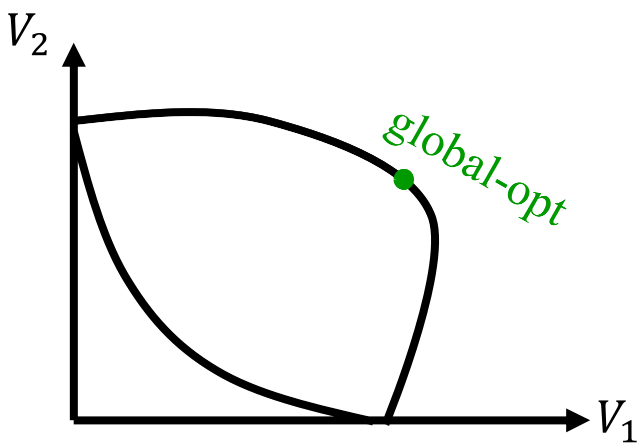
$$\max_{x \text{ feasible}} \sum_{i\in N} \log V_i(x_i)$$

Bads [BMSY’17]

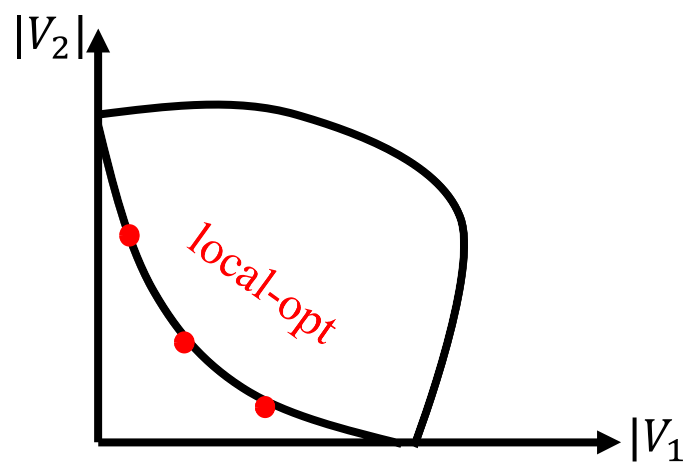
$$\min_{x \text{ feasible}} \sum_{i\in N} \log D_i(x_i)$$
$$\text{s.t. } \boxed{D_i(x_i) > 0, \ \forall i}$$

Open constraints

Convex program

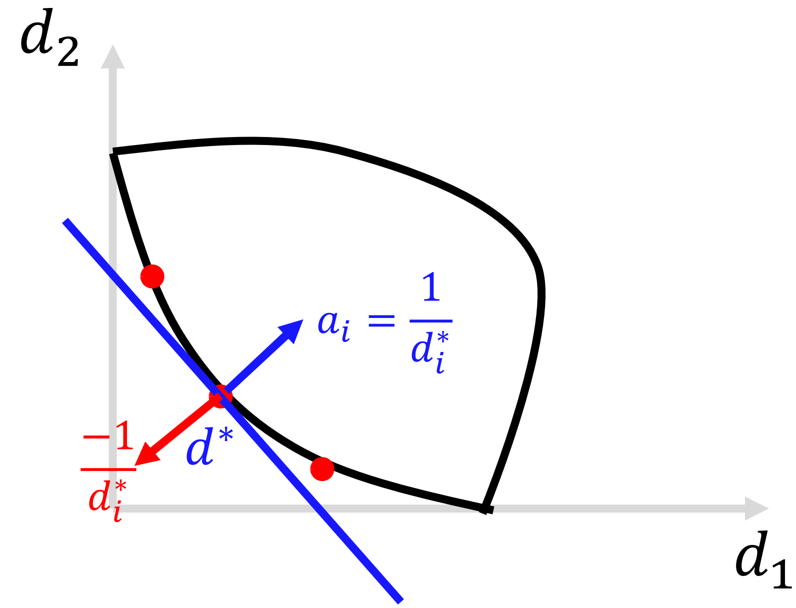


Non-convex program



KKT Points: Geometric view

$$\underset{\substack{d \text{ feasible} \\ d>0}}{\text{local min}} \sum_i \log(d_i) \qquad \text{Gradient} \propto \frac{1}{d_\mathrm{i}}$$



Why gives CEEI?

Bads CEEI

Given prices $p = (p_1, \dots, p_m)$ of items

- ***Optimal bundle:*** agent i demands $x'_i \in \underset{x:\ p \cdot x \geq 1}{argmin} \ \sum_j D_{ij} x_j$
  - Pain-per-buck-earned: $\frac{D_{ij}}{p_j}$ from chore j
  - Demand j only if it gives minimum pain-per-buck (MPB_i)= $\min_k \frac{D_{ik}}{p_k}$

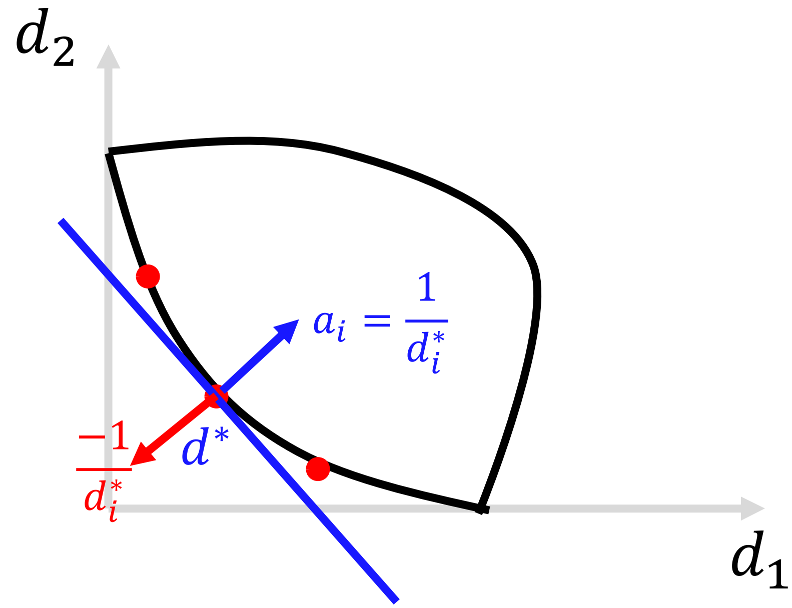
$$\forall j \in M,\ x'_{ij} > 0 \Rightarrow \frac{D_{ij}}{p_j} = MPB_i \ \Rightarrow \ D_i(x'_i) = MPB_i * 1 (=\text{earning})$$

- **Demand = Supply**

$$\forall j, \quad \sum_i x'_{ij} = 1$$

KKT Points → CEEI [BMSY'17]

$$\underset{\substack{d \text{ feasible} \\ d>0}}{\text{local min}} \sum_i \log(d_i) \qquad \text{Gradient} \propto \frac{1}{d_i}$$



Proof idea: let $p_j = \min_i a_i D_{ij}$

$\Rightarrow p_j \le a_i D_{ij}, \ \forall (i,j)$

$\Rightarrow \frac{1}{a_i} \le \frac{D_{ij}}{p_j}, \ \forall (i,j)$

$\Rightarrow$ for each agent i, $\frac{1}{a_i} \le \min_j \frac{D_{ij}}{p_j} = \mathrm{MPB_i}$

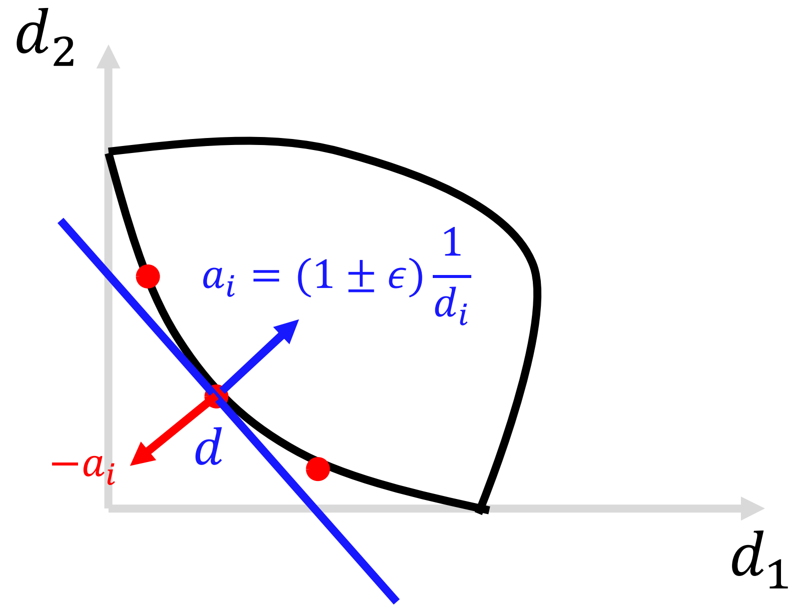
$\Rightarrow$ using supp. hyp. property, show that $\frac{1}{a_i} = \min_j \frac{D_{ij}}{p_j} = \mathrm{MPB_i}$

$\Rightarrow (\because KKT)\ d_i^* = \frac{1}{a_i} = MPB_i \Rightarrow$ CEEI!

KKT Points → CEEI

$$\underset{\substack{d \text{ feasible} \\ d>0}}{\text{local min}} \sum_i \log(d_i)$$

Gradient $\propto \frac{1}{d_i}$



Extension [BCM.'21]: ϵ-KKT gives ϵ-CEEI

(where all agents earns $(1 \pm \epsilon)$)

Easy, apply gradient-decent!

Pitfalls of Local Search (GD)

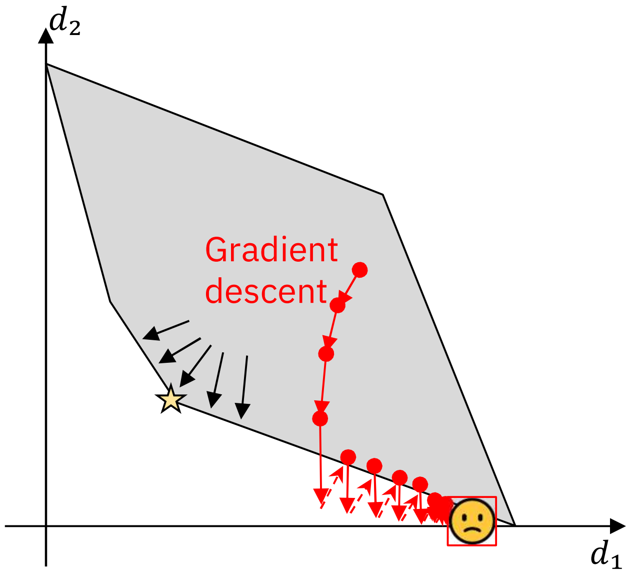
Tricky to ensure, open constraint $\boldsymbol{d_i > 0}$

- to $\boldsymbol{min} \sum_i \boldsymbol{log}(\boldsymbol{d_i})$ move in $(-1/d_i)$ direction.
  - Smaller d_is reduce fast.

- Experimentally, even with log-barriers

$$\min_x \sum_i \log(D_i(x_i)) + 0.01 \sum_i \log(\sum_j x_{ij})$$

- Intuitively: *unstable* local min



An exterior approach

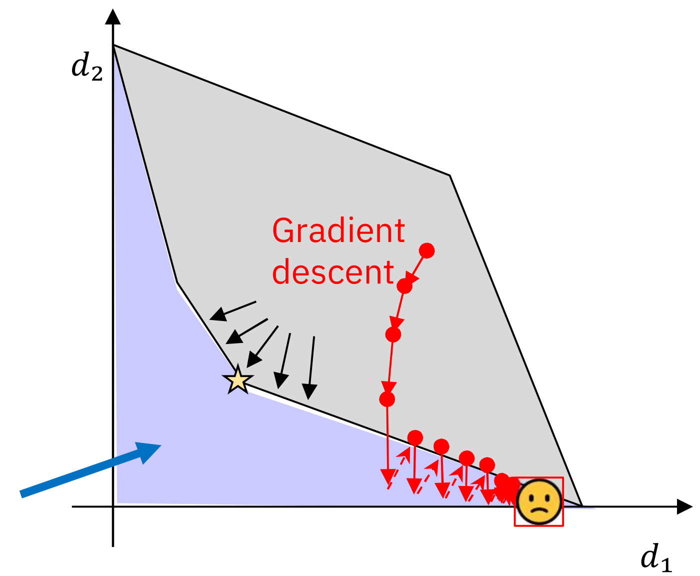
Observation: Local max from the exterior

- Move in $\frac{1}{d_i}$ direction increases smaller $d_i s$!

Idea: Approach from exterior.

- **How? Nonconvex region!**
- Any potential function?
- How to show fast convergence?

The exterior



Approach from the exterior

How? Nonconvex region! Any potential function?

Claim 1. d be the nearest feasible point to e, then $d \geq e$.

⇩

$$\sum_i \log(e_i) \leq \sum_i \log(d_i)$$

e': maximize sum-log on H

⇩

$$\sum_i \log(d_i) < \sum_i \log(e'_i)$$

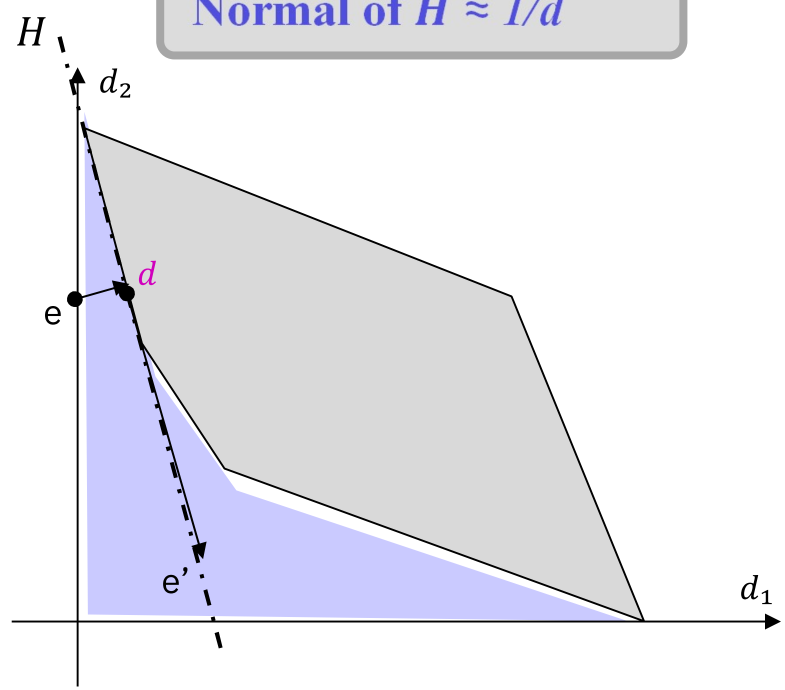
Potential function!

Goal

$$\underset{d \text{ in exterior}}{\text{local max}} \sum_i \log(d_i)$$

$\equiv$

Normal of $H \approx 1/d$

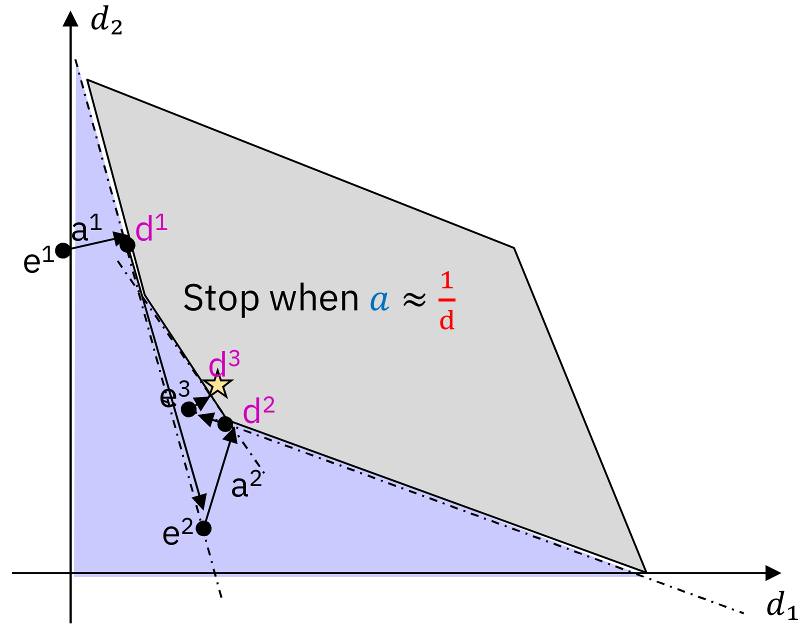


Exterior Points Method

Goal: Hyperplane normal ≈ Gradient

- e^t is t^{th} exterior infeasible point.
- d^t is t^{th} point on boundary
- a^t is t^{th} supp. hyperplane normal

1. Set $d^t \leftarrow$ *nearest feasible point* to e^t
2. $a^t \leftarrow (e^t - d^t)$ supp. hyperplane normal
3. Stop if $a^t \approx \left({}^1\!/_{d_1^t}, \ldots, {}^1\!/_{d_n^t}\right)$
4. Set $e^{t+1} \leftarrow \operatorname{argmax} \sum_i \log(d_i)$ along the supp. hyperplane



Iterative ⇒ Approximate KKT. **Sufficient.**

Correctness & Convergence

Lemma: Approximate KKT gives approximate CEEI

Lemma: Objective $\sum_i \log(d_i(x_i))$ always increasing

$\Rightarrow$ *Potential Function*

Lemma: Either

1. Increase potential by $\geq \Omega\left(\varepsilon^2/_{n^2}\right)$

   – OR –

2. Terminate with $(1+\varepsilon)$-approximate KKT

Convergence Rate

Lemma: Either

1. Increase potential by $\geq \Omega\left(\varepsilon^2/_{n^2}\right)$

   – OR –

2. Terminate with $(1+\varepsilon)$-approximate KKT

Poly-time: FPTAS

Lemma: Approximate KKT gives approximate CEEI

Lemma: Objective $\sum_i \log(d_i(x_i))$ always increasing

$\Rightarrow$ *Potential Function*

Lemma: Either

1. Increase potential by $\geq \Omega\left(\varepsilon^2/_{n^2}\right)$

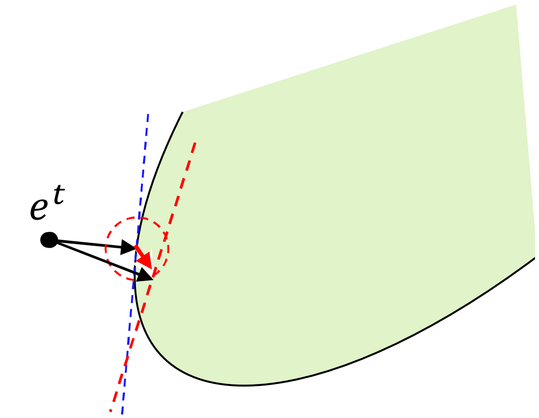
– OR –

2. Terminate with $(1+\varepsilon)$-approximate KKT

Theorem: If $\log(d_i) \leq L \;\; \forall i$, search terminates in $O\left(n^3 L/_{\varepsilon^2}\right)$ steps.

Extensions

1-Homogeneous (Obstacles)



- Function access: value oracle
- D space need not be convex. $D + R_+^n$
- *"Set d^t ← nearest feasible point to e^t"*
  - If exact, will get exactly supp. hyp.
  - If approximate, (GD), error gives δ-approx. supp. hyp.

- Algorithm needs pre-images and feasibility $d_i \mapsto x_i$
  - If *linear*, everything is LP → Fully explicit, efficient, algo in linear setting
  - In general w/ only oracle access, introduces errors.

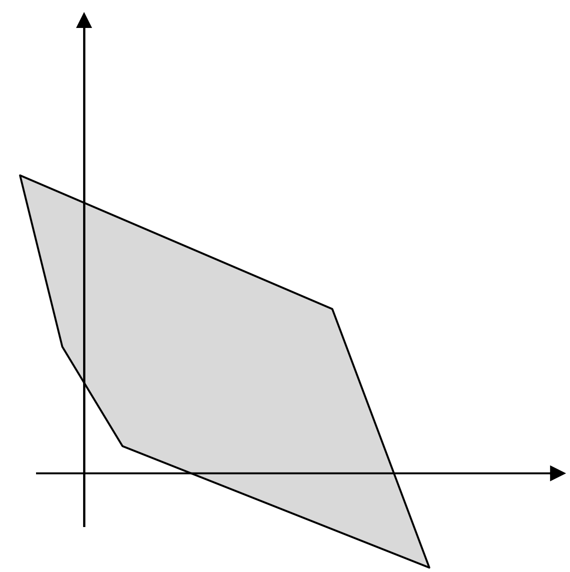
Extensions

CE (unequal weights): w_i for agent i

$$L(d) = \sum_i w_i \log d_i$$

Mixed manna: [BMSY'17]

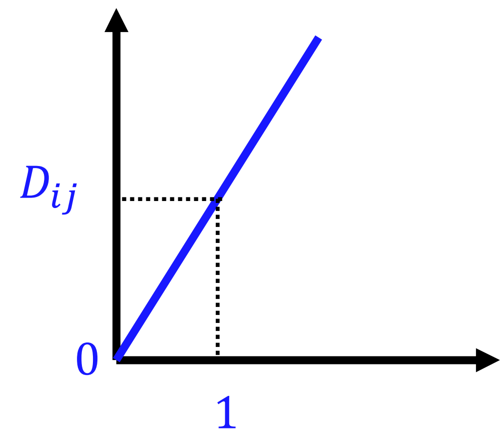
1. +ve instance ~ goods manna
2. Null instance ~ feasibility problem
3. -ve instance ~ bads manna
   - Infeasible starting point is tricky.



Exchange Model
(Barter system, re-allocation)

Exchange Model

- A: set of n agents
- M: set of m divisible chores
  - Supply of every item is one



- Each agent i has
  - Linear disutility $D_i: R_+^m \rightarrow R.\ D_i(x_i) = \sum_j D_{ij} x_{ij}$
  - **Exchange: W_{ij} units of chore j**
    - Need to earn enough to pay for her chores.

Motivation/Examples

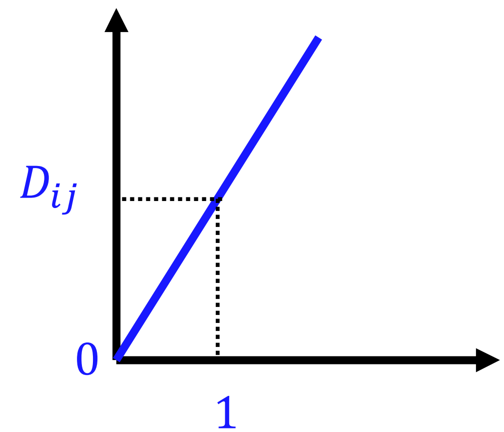
Living for help

Timebank

Students teaching each other

Exchange Model (w/ ∞)

- A: set of n agents
- M: set of m divisible bads
  - Supply of every item is one

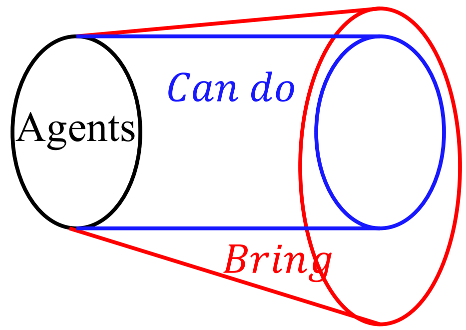


- Each agent i has
  - Linear disutility $D_i: R_+^m \rightarrow R.\ D_i(x_i) = \sum_j D_{ij} x_{ij}$
  - Exchange: W_{ij} units of item j

What if agent i does not have skills to do chore j? **Set D_{ij} to ∞**

Existence of CE? [BGMM. ITCS'22]

- Need not exist



- Assumption 1: Strongly connected economy graph
  - Suffices for goods manna, but not for bads manna
- Still checking existence is strongly NP-hard.
  - Even for CEEI. And $\frac{11+\epsilon}{12}$-approximation.
- Assumption 2: A pair of agents can either do the same set of chores or a disjoint sets of chores.
  - Unavoidable: If the sets differ by one chore, CE may not exist.

Existence of CE? [BGMM. ITCS'22]

- Assumption 1: Strongly connected economy graph (standard)
- Assumption 2: A pair of agents can either do the same set of chores or a disjoint sets of chores. (Unavoidable)

Theorem: CE always exists under the two assumptions.

Difficulty: Undefined optimal bundles at zero prices.

Solution: Modify price domain s.t. o.b. are always defined.

Proof uses both Kakutani and Brouwer fixed-point theorems. The latter nested inside the former.

Computation in Exchange?

Difficulty: Not easy! With linear valuations, no hardness results known.

Combinatorial Algorithms? (faster, flow-based, intuitive price-update dynamics, …)

Difficulty: The *surplus-decrease approach,* central for the goods case, fails.

Recent Work [BGMM.'21]

Exchange: PPAD-hard, even $\frac{1}{poly(n)}$ approximation.

CEEI (CE): Combinatorial algorithm

- FPTAS (faster).
- Exact CEEI if $D_{ij} = (1+\alpha)^{k_{ij}}$ (quadratic time)
- Puts the problem in PLS ⇒ in CLS [BGMM.'21, FGHS'21]

Under linear valuations
First separation: CEEI vs Exchange

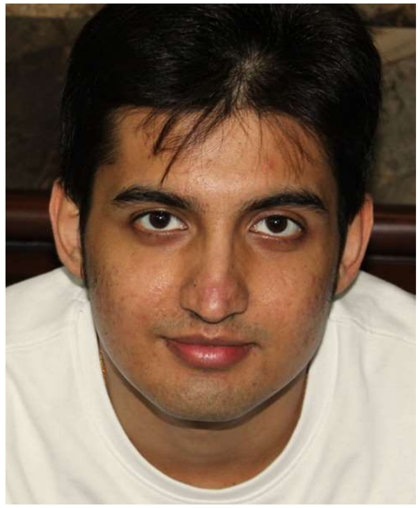
(solution set is non-convex in both)

Open Questions

Exact CEEI: Hard or easy?

Strategic analysis: Price-of-Anarchy

Dynamics: Proportional response?

THANK YOU