11. Equality constrained minimization

- equality constrained minimization
- eliminating equality constraints
- Newton's method with equality constraints
- infeasible start Newton method
- implementation

Equality constrained minimization

minimize
$$f(x)$$
 subject to $Ax = b$

- f convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\operatorname{\mathbf{rank}} A = p < \mathbf{n}$ (fewer constraints than unknowns)
- ullet we assume p^{\star} is finite and attained

optimality conditions: x^* is optimal iff there exists a ν^* such that

$$abla f(x^\star) + A^T \nu^\star = 0, \qquad Ax^\star = b$$
(stationarity) (primal feasibility)

equality constrained quadratic minimization (with $P \in S_+^n$)

minimize
$$(1/2)x^TPx + q^Tx + r$$
 subject to
$$Ax = b$$

$$\lim_{0 = dL/dx = P} \frac{L(x,v) = \frac{1}{2}x^*Px + q^*x + r + v^*(Ax - b)}{0 = dL/dx = Px + q + A^*v}$$
 optimality condition:
$$\left[\begin{array}{cc} P & A^T \\ A & 0 \end{array} \right] \left[\begin{array}{c} x^* \\ \nu^* \end{array} \right] = \left[\begin{array}{c} -q \\ b \end{array} \right]$$
 equivalent to:
$$Px^* + A^*v^* + q = 0$$

$$Ax^* = b$$

- coefficient matrix is called KKT matrix, if non-singular => unique primal-dual pair (x*,v*) Recall that a matrix Q is nonsingular iff y = 0 is the only solution of Qy = 0
- KKT matrix is nonsingular if and only if

$$Ax = 0, \quad x \neq 0 \qquad \Longrightarrow \qquad x^T P x > 0$$

Assume Ax=0, $x \neq 0$, Px=0, then [P A'] [x] = [0] and thus, the KKT matrix is singular [A 0] [0] [0]

Assume KKT is singular, there exists x in Rⁿ, z in R^p such that [P A'][x] = [0]

[A 0] [z] [0]

thus, Ax=0 and Px+A'z=0 => 0 = x'(Px+A'z) = x'Px + (Ax)'z = x'Px => Px = 0 (which contradicts P pos.semidef. unless x=0) $Then we must have <math>z \neq 0$, but then 0 = Px+A'z = A'z (which contradicts rank A = p)

Newton step

Newton step $\Delta x_{\rm nt}$ of f at feasible x is given by solution v of

$$\left[\begin{array}{cc} \nabla^2 f(x) & A^T \\ A & 0 \end{array} \right] \left[\begin{array}{c} v \\ w \end{array} \right] = \left[\begin{array}{cc} -\nabla f(x) \\ 0 \end{array} \right] \quad \begin{array}{c} \text{equivalent to:} \\ \text{d^2f(x) v + A'w + df(x) = 0} \\ \text{A v = 0} \end{array} \right]$$

interpretations

• $\Delta x_{\rm nt}$ solves second order approximation (with variable v) assume x is feasible: Ax=b we want Av=0

minimize
$$\widehat{f}(x+v) = f(x) + \nabla f(x)^T v + (1/2) v^T \nabla^2 f(x) v$$
 subject to
$$A(x+v) = b$$

$$\text{L(v,w)} = \text{df(x)'v} + \frac{1}{2} \text{ v' d^2f(x) v + w (Av)}$$

$$\text{0} = \text{dL/dv} = \text{df(x)} + \text{d^2f(x) v + A'w}$$

ullet $\Delta x_{
m nt}$ equations follow from linearizing optimality conditions

$$\nabla f(x+v) + A^T w \approx \nabla f(x) + \nabla^2 f(x)v + A^T w = 0, \qquad A(x+v) = b$$

Newton decrement

$$\lambda(x) = \left(\Delta x_{\rm nt}^T \nabla^2 f(x) \Delta x_{\rm nt}\right)^{1/2} = \left(-\nabla f(x)^T \Delta x_{\rm nt}\right)^{1/2}$$

properties

 $p^* = \inf_{Ay=b} f(y)$

ullet gives an estimate of $f(x)-p^\star$ using quadratic approximation \widehat{f} :

$$f(x) - \inf_{Ay=b} \widehat{f}(y) = \frac{1}{2}\lambda(x)^2$$
 Let $H = d^2 f(x)$
$$d = df(x)$$

$$\lambda = \lambda (x)$$

$$\Delta x = \Delta x_n t = v \text{ in previous slide}$$

$$= f(x) + d'\Delta x + \frac{1}{2}\Delta x' H \Delta x \dots \text{ since } d = -H \Delta x - A'w$$

$$= f(x) - \Delta x' H \Delta x - w'A \Delta x + \frac{1}{2}\Delta x' H \Delta x \dots \text{ since } A \Delta x = 0$$

$$[H A'] [\Delta x] = [-d]$$

$$[A 0] [w] [0]$$

$$f(x) - \inf_{Ay=b} f^{(y)} = f^{(x)} + d'\Delta x + \frac{1}{2}\Delta x' H \Delta x \dots \text{ since } A \Delta x = 0$$

$$f(x) - \inf_{Ay=b} f^{(y)} = \frac{1}{2}\Delta x' H \Delta x + \frac{1}{2}\Delta x' H \Delta x + \frac{1}{2}\Delta x' H \Delta x$$

$$f(x) - \inf_{Ay=b} f^{(y)} = f^{(x)} + d'\Delta x + \frac{1}{2}\Delta x' H \Delta x$$

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$$f(x) - \int_{Ay=b} f^{(y)} = f^{(y)} + \int_{Ay=b} f$$

 $f(x) - \inf_y f^*(y) = -\frac{1}{2} d' \Delta x = \frac{1}{2} \lambda^2$

Thus $\lambda = \operatorname{sqrt}(-d' \Delta x)$

 $= f(x) + \frac{1}{2} d' \Delta x$

Then $d = -H \Delta x - A'w$

 $H \Delta x = -d - A'w$

... since A $\Delta x = 0$

Newton's method with equality constraints

given starting point $x \in \operatorname{dom} f$ with Ax = b, tolerance $\epsilon > 0$. repeat

- 1. Compute the Newton step and decrement Δx_{nt} , $\lambda(x)$.
- 2. Stopping criterion. quit if $\lambda^2/2 \leq \epsilon$.
- 3. *Line search.* Choose step size t by backtracking line search.
- 4. Update. $x := x + t\Delta x_{\rm nt}$.

```
if df(x) \neq 0
A \Delta x = 0, then A t \Delta x = 0 for any t>0
                                                                         df(x)' \Delta x = -\lambda(x)^2 < 0 \text{ (see slide 10-5)}
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- a feasible descent method: $x^{(k)}$ feasible and $f(x^{(k+1)}) < f(x^{(k)})$
- $\min f^{\sim}(y) = f(T y)$ then $\Delta y = T^{-1} \Delta x$, $y^{(k)} = y + \Delta y = T^{-1} (x + \Delta x) = T^{-1} x^{(k)}$ affine invariant s.t. ATy = b

Thus $\Delta x = T \Delta y$

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Newton step at infeasible points

2nd interpretation of page 11–6 extends to infeasible x (i.e., $Ax \neq b$)

linearizing optimality conditions at infeasible x (with $x \in \operatorname{dom} f$) gives

$$\begin{bmatrix} \nabla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{\rm nt} \\ w \end{bmatrix} = - \begin{bmatrix} \nabla f(x) \\ Ax - b \end{bmatrix}$$
 (1)

Although $Ax \neq b$, we want $A(x+\Delta x) = b$, thus $A \Delta x = -(Ax-b)$

primal-dual interpretation

• write optimality condition as r(y) = 0, where s.t. Ax=b

L(x,v) = f(x) + v(Ax-b)dL/dx = df(x) + A'v = 0

$$y = (x, \nu),$$
 $r(y) = (\nabla f(x) + A^T \nu, Ax - b)$

• linearizing r(y)=0 gives $r(y+\Delta y)\approx r(y)+Dr(y)\Delta y=0$: (1st order Taylor) Since Dr(y) Δy = -r(y) we have:

$$\begin{bmatrix}
abla^2 f(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{
m nt} \\ \Delta
u_{
m nt} \end{bmatrix} = - \begin{bmatrix}
abla f(x) + A^T
u \\ Ax - b \end{bmatrix}$$

same as (1) with $w=
u+\Delta
u_{
m nt}$

 $r(y)_1 = df(x)+A'v$ $r(y)_2 = Ax-b$ $Dr(y)_{11} = d(r(y)_1)/dx = d(df(x)+A'v)/dx = d^2f(x)$ $Dr(y)_{12} = d(r(y)_1)/dv = d(df(x)+A'v)/dv = A'$ $Dr(y)_{21} = d(r(y)_2)/dx = d(Ax-b)/dx = A$

Dr(y) {22} = d(r(y) 2)/dv = d(Ax-b)/dv = 0

 $y=(x,v) \Rightarrow \Delta y=(\Delta x, \Delta v)$

Infeasible start Newton method

Since we want r(y) = 0, it is natural to try to decrease the norm of r(y)

given starting point $x \in \operatorname{dom} f$, ν , tolerance $\epsilon > 0$, $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$. repeat

- 1. Compute primal and dual Newton steps $\Delta x_{
 m nt}$, $\Delta
 u_{
 m nt}$.
- 2. Backtracking line search on $||r||_2$.

$$t := 1$$
.

while
$$||r(x + t\Delta x_{\rm nt}, \nu + t\Delta \nu_{\rm nt})||_2 > (1 - \alpha t)||r(x, \nu)||_2$$
, $t := \beta t$.

3. Update. $x:=x+t\Delta x_{\rm nt},\ \nu:=\nu+t\Delta \nu_{\rm nt}.$

until
$$Ax = b$$
 and $||r(x, \nu)||_2 \le \epsilon$.

- not a descent method: $f(x^{(k+1)}) > f(x^{(k)})$ is possible
- directional derivative of $||r(y)||_2$ in direction $\Delta y = (\Delta x_{\rm nt}, \Delta \nu_{\rm nt})$ is

$$\left.\frac{d}{dt}\left\|r(y+t\Delta y)\right\|_2\right|_{t=0}=-\|r(y)\|_2\qquad \text{Thus, the norm of } r \text{ decreases in the Newton direction}$$

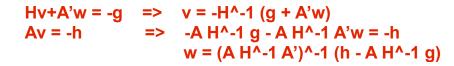
Solving KKT systems

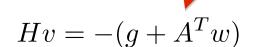
$$\left[\begin{array}{cc} H & A^T \\ A & 0 \end{array}\right] \left[\begin{array}{c} v \\ w \end{array}\right] = - \left[\begin{array}{c} g \\ h \end{array}\right]$$

solution methods

- LDL^T factorization
- elimination (if *H* nonsingular)

$$AH^{-1}A^Tw = h - AH^{-1}q, \qquad Hv = -(q + A^Tw)$$





• elimination with singular H: write as

Originally: Hv+A'w = -g, Av = -hNow: (H+A'QA)v + A'w = -g - A'Qh, Av = -hEquivalent if: A'QAv = -A'Qh ... true since Av = -h

$$\begin{bmatrix} H + A^T Q A & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = - \begin{bmatrix} g + A^T Q h \\ h \end{bmatrix}$$

with $Q \succeq 0$ for which $H + A^TQA \succ 0$, and apply elimination

Recall: Ax=0, $x \neq 0 \Rightarrow xPx>0$ Therefore $x(P+A'QA)x = xPx + |Q^{1}/2|Ax| 2^{2} > 0$