4. Convex optimization problems

- optimization problem in standard form
- convex optimization problems
- quasiconvex optimization
- linear optimization
- quadratic optimization
- geometric programming
- generalized inequality constraints
- semidefinite programming
Optimization problem in standard form

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

- \( x \in \mathbb{R}^n \) is the **optimization variable**
- \( f_0 : \mathbb{R}^n \to \mathbb{R} \) is the **objective or cost function**
- \( f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m, \) are the inequality **constraint functions**
- \( h_i : \mathbb{R}^n \to \mathbb{R} \) are the equality constraint functions

**optimal value:**

\[
p^* = \inf \{ f_0(x) \mid f_i(x) \leq 0, \ i = 1, \ldots, m, \ h_i(x) = 0, \ i = 1, \ldots, p \}
\]

- \( p^* = \infty \) if problem is infeasible (no \( x \) satisfies the constraints)
- \( p^* = -\infty \) if problem is unbounded below

Convex optimization problems
Optimal and locally optimal points

$x$ is **feasible** if $x \in \text{dom} f_0$ and it satisfies the constraints.

A feasible $x$ is **optimal** if $f_0(x) = p^*$; $X_{\text{opt}}$ is the set of optimal points.

$x$ is **locally optimal** if there is an $R > 0$ such that $x$ is optimal for

\[
\begin{align*}
\text{minimize (over } z) & \quad f_0(z) \\
\text{subject to} & \quad f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\
& \quad \|z - x\|_2 \leq R
\end{align*}
\]

**Examples** (with $n = 1$, $m = p = 0$)

- $f_0(x) = 1/x$, $\text{dom} f_0 = \mathbb{R}_{++}$: $p^* = 0$, no optimal point  \quad $f_0(x) \to 0$ as $x \to +\infty$
- $f_0(x) = -\log x$, $\text{dom} f_0 = \mathbb{R}_{++}$: $p^* = -\infty$  \quad $f_0(x) \to -\infty$ as $x \to 0^+$
- $f_0(x) = x \log x$, $\text{dom} f_0 = \mathbb{R}_{++}$: $p^* = -1/e$, $x = 1/e$ is optimal  \quad See $f_0(x)$ in $[0,2]$
- $f_0(x) = x^3 - 3x$, $p^* = -\infty$, local optimum at $x = 1$  \quad See $f_0(x)$ in $[-3, +3]$
Implicit constraints

the standard form optimization problem has an implicit constraint

\[ x \in \mathcal{D} = \bigcap_{i=0}^{m} \text{dom } f_i \cap \bigcap_{i=1}^{p} \text{dom } h_i, \]

• we call \( \mathcal{D} \) the domain of the problem
• the constraints \( f_i(x) \leq 0, h_i(x) = 0 \) are the explicit constraints
• a problem is unconstrained if it has no explicit constraints (\( m = p = 0 \))

example:

minimize \( f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x) \)

is an unconstrained problem with implicit constraints \( a_i^T x < b_i \) \iff \( b_i - a_i x > 0 \)
Feasibility problem

\[
\begin{align*}
\text{find} & \quad x \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

can be considered a special case of the general problem with \( f_0(x) = 0 \):

\[
\begin{align*}
\text{minimize} & \quad 0 \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad h_i(x) = 0, \quad i = 1, \ldots, p
\end{align*}
\]

- \( p^* = 0 \) if constraints are feasible; any feasible \( x \) is optimal
- \( p^* = \infty \) if constraints are infeasible
Convex optimization problem

standard form convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad a_i^T x = b_i, \quad i = 1, \ldots, p
\end{align*}
\]

- \( f_0, f_1, \ldots, f_m \) are convex; equality constraints are affine
- problem is quasiconvex if \( f_0 \) is quasiconvex (and \( f_1, \ldots, f_m \) convex)

often written as

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

important property: feasible set of a convex optimization problem is convex
example

minimize \[ f_0(x) = x_1^2 + x_2^2 \]
subject to \[ f_1(x) = x_1/(1 + x_2^2) \leq 0 \]
\[ h_1(x) = (x_1 + x_2)^2 = 0 \]

• \( f_0 \) is convex; feasible set \( \{(x_1, x_2) \mid x_1 = -x_2 \leq 0\} \) is convex

• not a convex problem (according to our definition): \( f_1 \) is not convex, \( h_1 \) is not affine

• equivalent to the convex problem

minimize \[ x_1^2 + x_2^2 \]
subject to \[ x_1 \leq 0 \]
\[ x_1 + x_2 = 0 \]
Local and global optima

any locally optimal point of a convex problem is (globally) optimal

proof: suppose \( x \) is locally optimal, but there exists a feasible \( y \) with \( f_0(y) < f_0(x) \) (i.e., \( x \) not globally optimal)

\( x \) locally optimal means there is an \( R > 0 \) such that

\[
\text{z feasible, } \|z - x\|_2 \leq R \implies f_0(z) \geq f_0(x)
\]

consider \( z = \theta y + (1 - \theta)x \) with \( \theta = R/(2\|y - x\|_2) \)

\[
\|y - x\|_2 > R, \text{ so } 0 < \theta < 1/2
\]

• \( z \) is a convex combination of two feasible points, hence also feasible

• \( \|z - x\|_2 = R/2 \) and

\[
f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x)
\]

which contradicts our assumption that \( x \) is locally optimal

since we found that \( f_0(z) < f_0(x) \)
Optimality criterion for differentiable $f_0$

$x$ is optimal if and only if it is feasible and

1st order condition for convexity

$$f_0(y) \geq f_0(x) + d_f(x)' (y-x)$$

I. Assume $d_f(x)' (y-x) \geq 0$
then $f_0(y) \geq f_0(x)$
then $x$ optimal

II. Assume $x$ optimal and $d_f(x)' (y-x) < 0$
Let $z(t) = t y + (1-t) x$, for $t$ in $[0,1]$
As $t \rightarrow 0$ we arrive to a contradiction
$$\frac{d}{dt} f_0(z(t)) = d_f(t y + (1-t) x)' (y-x)$$
$$\frac{d}{dt} f_0(z(t)) \text{ (at } t=0) = d_f(x)' (y-x) < 0$$

Thus, for $t \rightarrow 0$ by series expansion
$$f_0(z(t)) = f_0(x) + \frac{d}{dt} f_0(z(t)) \text{ (at } t=0) < f_0(x)$$

Thus, $x$ is not optimal

if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set $X$ at $x$
Equivalent convex problems

two problems are (informally) **equivalent** if the solution of one is readily obtained from the solution of the other, and vice-versa

some common transformations that preserve convexity:

- **eliminating equality constraints**

  \[
  \begin{align*}
  &\text{minimize } f_0(x) \\
  &\text{subject to } f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  &Ax = b
  \end{align*}
  \]

  is equivalent to

  \[
  \begin{align*}
  &\text{minimize (over } z) \quad f_0(Fz + x_0) \\
  &\text{subject to } f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m
  \end{align*}
  \]

  where \( F \) and \( x_0 \) are such that

  \[
  Ax = b \iff x = Fz + x_0 \text{ for some } z
  \]
• introducing equality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(A_0x + b_0) \\
\text{subject to} & \quad f_i(A_i x + b_i) \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } x, y_i) & \quad f_0(y_0) \\
\text{subject to} & \quad f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\
y_i = A_i x + b_i, & \quad i = 0, 1, \ldots, m
\end{align*}
\]

• introducing slack variables for linear inequalities

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to

\[
\begin{align*}
\text{minimize (over } x, s) & \quad f_0(x) \\
\text{subject to} & \quad a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\
s_i \geq 0, & \quad i = 1, \ldots m
\end{align*}
\]
\begin{itemize}
  \item **epigraph form**: standard form convex problem is equivalent to

  \[
  \begin{align*}
  \text{minimize (over } x, t) & \quad t \\
  \text{subject to} & \quad f_0(x) - t \leq 0 \\
  & \quad f_i(x) \leq 0, \quad i = 1, \ldots, m \\
  & \quad Ax = b
  \end{align*}
  \]

  \item **minimizing over some variables**

  \[
  \begin{align*}
  \text{minimize} & \quad f_0(x_1, x_2) \\
  \text{subject to} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m
  \end{align*}
  \]

  is equivalent to

  \[
  \begin{align*}
  \text{minimize} & \quad \tilde{f}_0(x_1) \\
  \text{subject to} & \quad f_i(x_1) \leq 0, \quad i = 1, \ldots, m
  \end{align*}
  \]

  where \( \tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2) \)
\end{itemize}
Linear program (LP)

minimize \( c^T x + d \)
subject to \( Gx \preceq h \)
\( Ax = b \)

• convex problem with affine objective and constraint functions
• feasible set is a polyhedron
Examples

diet problem: choose quantities $x_1, \ldots, x_n$ of $n$ foods

- one unit of food $j$ costs $c_j$, contains amount $a_{ij}$ of nutrient $i$
- healthy diet requires nutrient $i$ in quantity at least $b_i$

to find cheapest healthy diet,

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \succeq b, \quad x \succeq 0
\end{align*}$$

piecewise-linear minimization

$$\begin{align*}
\text{minimize} & \quad \max_{i=1,\ldots,m}(a_i^T x + b_i)
\end{align*}$$

equivalent to an LP

$$\begin{align*}
\text{minimize} & \quad t \\
\text{subject to} & \quad a_i^T x + b_i \leq t, \quad i = 1, \ldots, m
\end{align*}$$
Chebyshev center of a polyhedron

Chebyshev center of

\[ \mathcal{P} = \{ x \mid a_i^T x \leq b_i, \ i = 1, \ldots, m \} \]

is center of largest inscribed ball

\[ \mathcal{B} = \{ x_c + u \mid \|u\|_2 \leq r \} \]

- \( a_i^T x \leq b_i \) for all \( x \in \mathcal{B} \) if and only if

\[
\sup_u \{ a_i^T (x_c + u) \mid \|u\|_2 \leq r \} = a_i^T x_c + r \|a_i\|_2 \leq b_i
\]

since \( \sup_u (a_i u) = \|a_i\|_2 \) by norm duality

- hence, \( x_c, r \) can be determined by solving the LP

\[
\begin{align*}
\text{maximize} & \quad r \\
\text{subject to} & \quad a_i^T x_c + r \|a_i\|_2 \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]
Quadratic program (QP)

minimize \( \frac{1}{2} x^T P x + q^T x + r \)
subject to \( Gx \leq h \)
\( Ax = b \)

- \( P \in S^n_+ \), so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron
Examples

least-squares

minimize $\frac{1}{2} ||Ax - b||^2_2$

- analytical solution $x^* = A^\dagger b$ ($A^\dagger$ is pseudo-inverse)
- can add linear constraints, e.g., $l \leq x \leq u$

linear program with random cost

minimize $\bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E} c^T x + \gamma \text{var}(c^T x)$

subject to $Gx \leq h$, $Ax = b$

- $c$ is random vector with mean $\bar{c}$ and covariance $\Sigma$
- hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
- $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)
Quasiconvex optimization

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 0, \quad i = 1, \ldots, m \)
\( Ax = b \)

with \( f_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) quasiconvex, \( f_1, \ldots, f_m \) convex

can have locally optimal points that are not (globally) optimal
convex representation of sublevel sets of $f_0$

if $f_0$ is quasiconvex, there exists a family of functions $\phi_t$ such that:

- $\phi_t(x)$ is convex in $x$ for fixed $t$
- $t$-sublevel set of $f_0$ is 0-sublevel set of $\phi_t$, i.e.,

$$ f_0(x) \leq t \iff \phi_t(x) \leq 0 $$

example

$$ f_0(x) = \frac{p(x)}{q(x)} $$

with $p$ convex, $q$ concave, and $p(x) \geq 0$, $q(x) > 0$ on $\text{dom } f_0$

can take $\phi_t(x) = p(x) - tq(x)$:

- for $t \geq 0$, $\phi_t$ convex in $x$
- $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$

$$ \begin{align*}
    p(x) &\leq t q(x) \\
    p(x) - t q(x) &\leq 0
\end{align*} $$
quasiconvex optimization via convex feasibility problems

\[ \phi_t(x) \leq 0, \quad f_i(x) \leq 0, \quad i = 1, \ldots, m, \quad Ax = b \]  \hspace{1cm} (1)

- for fixed \( t \), a convex feasibility problem in \( x \)
- if feasible, we can conclude that \( t \geq p^* \); if infeasible, \( t \leq p^* \)

---

**Bisection method for quasiconvex optimization**

**given** \( l \leq p^*, \ u \geq p^* \), tolerance \( \varepsilon > 0 \).

**repeat**

1. \( t := (l + u)/2 \).
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, \( u := t \); **else** \( l := t \).

**until** \( u - l \leq \varepsilon \).

requires exactly \( \lceil \log_2((u - l)/\varepsilon) \rceil \) iterations (where \( u, l \) are initial values)
Linear-fractional program

minimize \quad f_0(x)
subject to \quad Gx \preceq h
\quad Ax = b

linear-fractional program

\[ f_0(x) = \frac{c^T x + d}{e^T x + f}, \quad \text{dom } f_0(x) = \{ x \mid e^T x + f > 0 \} \]

• a quasiconvex optimization problem; can be solved by bisection
Second-order cone programming

minimize \( f^T x \)
subject to \( \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \)
\( Fx = g \)

\((A_i \in \mathbb{R}^{n_i \times n}, \; F \in \mathbb{R}^{p \times n})\)

- inequalities are called second-order cone (SOC) constraints:

\((A_i x + b_i, c_i^T x + d_i) \in \text{second-order cone in } \mathbb{R}^{n_i+1}\)
Robust linear programming

the parameters in optimization problems are often uncertain, \textit{e.g.}, in an LP

\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & a_i^T x \leq b_i, \quad i = 1, \ldots, m,
\end{align*}

there can be uncertainty in \( c, a_i, b_i \)

two common approaches to handling uncertainty (in \( a_i \), for simplicity)

\begin{itemize}
\item deterministic model: constraints must hold for all \( a_i \in \mathcal{E}_i \)

\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & a_i^T x \leq b_i \text{ for all } a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m,
\end{align*}

\item stochastic model: \( a_i \) is random variable; constraints must hold with probability \( \eta \)

\begin{align*}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & \text{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m
\end{align*}
\end{itemize}
deterministic approach via SOCP

• choose an ellipsoid as $\mathcal{E}_i$:

\[
\mathcal{E}_i = \{ \bar{a}_i + P_i u \mid \|u\|_2 \leq 1 \} \quad (\bar{a}_i \in \mathbb{R}^n, \quad P_i \in \mathbb{R}^{n \times n})
\]

center is $\bar{a}_i$, semi-axes determined by singular values/vectors of $P_i$

• robust LP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i \quad \forall a_i \in \mathcal{E}_i, \quad i = 1, \ldots, m
\end{align*}
\]

is equivalent to the SOCP

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \bar{a}_i^T x + \|P_i^T x\|_2 \leq b_i, \quad i = 1, \ldots, m
\end{align*}
\]

(follows from $\sup_{\|u\|_2 \leq 1} (\bar{a}_i + P_i u)^T x = \bar{a}_i^T x + \|P_i^T x\|_2$)

a_i^T x is constant wrt u, only analyze $(P_i u)'x$

sup_{\{|u|_2 \leq 1\}} u' P_i' x = |P_i' x|_2 by norm duality
stochastic approach via SOCP

• assume $a_i$ is Gaussian with mean $\bar{a}_i$, covariance $\Sigma_i$ ($a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$)

• $a_i^T x$ is Gaussian r.v. with mean $\bar{a}_i^T x$, variance $x^T \Sigma_i x$; hence

$$\text{prob}(a_i^T x \leq b_i) = \Phi \left( \frac{b_i - \bar{a}_i^T x}{\|\Sigma_i^{1/2} x\|_2} \right)$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$ is CDF of $\mathcal{N}(0, 1)$

• robust LP

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \text{prob}(a_i^T x \leq b_i) \geq \eta, \quad i = 1, \ldots, m,
\end{align*}$$

with $\eta \geq 1/2$, is equivalent to the SOCP

$$\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad \bar{a}_i^T x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leq b_i, \quad i = 1, \ldots, m
\end{align*}$$
Geometric programming

monomial function

\[ f(x) = cx_1^{a_1}x_2^{a_2} \cdots x_n^{a_n}, \quad \text{dom } f = \mathbb{R}^n_{++} \]

with \( c > 0 \); exponent \( a_i \) can be any real number

posynomial function: sum of monomials

\[ f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}}x_2^{a_{2k}} \cdots x_n^{a_{nk}}, \quad \text{dom } f = \mathbb{R}^n_{++} \]

general geometric program (GP)

minimize \( f_0(x) \)
subject to \( f_i(x) \leq 1, \quad i = 1, \ldots, m \)
\( h_i(x) = 1, \quad i = 1, \ldots, p \)

with \( f_i \) posynomial, \( h_i \) monomial
Geometric program in convex form

change variables to $y_i = \log x_i$, and take logarithm of cost, constraints

- monomial $f(x) = cx_1^{a_1} \cdots x_n^{a_n}$ transforms to
  \[
  \log f(e^{y_1}, \ldots, e^{y_n}) = a^T y + b \quad (b = \log c)
  \]

- posynomial $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ transforms to
  \[
  \log f(e^{y_1}, \ldots, e^{y_n}) = \log \left( \sum_{k=1}^K e^{a_k^T y + b_k} \right) \quad (b_k = \log c_k)
  \]

- geometric program transforms to convex problem

  minimize \quad \log \left( \sum_{k=1}^K \exp (a_{0k}^T y + b_{0k}) \right)

  subject to \quad \log \left( \sum_{k=1}^K \exp (a_{ik}^T y + b_{ik}) \right) \leq 0, \quad i = 1, \ldots, m

  \quad Gy + d = 0

Convex optimization problems
Generalized inequality constraints

convex problem with generalized inequality constraints

\[
\begin{align*}
\text{minimize} & \quad f_0(x) \\
\text{subject to} & \quad f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m \\
& \quad Ax = b
\end{align*}
\]

- \( f_0 : \mathbb{R}^n \to \mathbb{R} \) convex; \( f_i : \mathbb{R}^n \to \mathbb{R}^{k_i} \) \( K_i \)-convex w.r.t. proper cone \( K_i \)
- same properties as standard convex problem (convex feasible set, local optimum is global, etc.)

conic form problem: special case with affine objective and constraints

\[
\begin{align*}
\text{minimize} & \quad c^T x \\
\text{subject to} & \quad F x + g \preceq_K 0 \\
& \quad Ax = b
\end{align*}
\]

extends linear programming (\( K = \mathbb{R}^m_+ \)) to nonpolyhedral cones
Semidefinite program (SDP)

minimize $c^T x$
subject to $x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0$
$Ax = b$

with $F_i, G \in S^k$

- inequality constraint is called linear matrix inequality (LMI)
- includes problems with multiple LMI constraints: for example,

$$x_1 \hat{F}_1 + \cdots + x_n \hat{F}_n + \hat{G} \preceq 0, \quad x_1 \tilde{F}_1 + \cdots + x_n \tilde{F}_n + \tilde{G} \preceq 0$$

is equivalent to single LMI

$$x_1 \begin{bmatrix} \hat{F}_1 & 0 \\ 0 & \tilde{F}_1 \end{bmatrix} + x_2 \begin{bmatrix} \hat{F}_2 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{F}_n & 0 \\ 0 & \tilde{F}_n \end{bmatrix} + \begin{bmatrix} \hat{G} & 0 \\ 0 & \tilde{G} \end{bmatrix} \preceq 0$$
Eigenvalue minimization

minimize $\lambda_{\text{max}}(A(x))$

where $A(x) = A_0 + x_1A_1 + \cdots + x_nA_n$ (with given $A_i \in S^k$)

equivalent SDP

minimize $t$
subject to $A(x) \preceq tI$

- variables $x \in \mathbb{R}^n$, $t \in \mathbb{R}$
- follows from

$\lambda_{\text{max}}(A) \leq t \iff A \preceq tI$
Matrix norm minimization

minimize \[ \| A(x) \|_2 = \left( \lambda_{\text{max}}(A(x)^T A(x)) \right)^{1/2} \]

where \[ A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n \] (with given \[ A_i \in \mathbb{R}^{p \times q} \])

equivalent SDP

\[ \begin{array}{c}
\text{minimize} & t \\
\text{subject to} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0
\end{array} \]

- variables \[ x \in \mathbb{R}^n, \ t \in \mathbb{R} \]
- constraint follows from

\[ \| A \|_2 \leq t \iff A^T A \preceq t^2 I, \quad t \geq 0 \]
\[ \iff \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succeq 0 \]

Let \[ X = [W \ B; \ B' \ C] \]
Schur complement:
\[ D = C - B' W^{-1} B \]

If \( W \) in \( S_{++} \) then \( X \) in \( S_+ \) if and only if \( D \) in \( S_+ \)

Assume this is \( X \) and you will see Convex optimization problems 4–39