2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities
Affine set

line through $x_1$, $x_2$: all points

$$x = \theta x_1 + (1 - \theta) x_2 \quad (\theta \in \mathbb{R})$$

affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{ x \mid Ax = b \}$

(conversely, every affine set can be expressed as solution set of system of linear equations)
Convex set

**line segment** between $x_1$ and $x_2$: all points

$$x = \theta x_1 + (1 - \theta)x_2$$

with $0 \leq \theta \leq 1$

**convex set**: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Rightarrow \quad \theta x_1 + (1 - \theta)x_2 \in C$$

**examples** (one convex, two nonconvex sets)
Convex combination and convex hull

convex combination of $x_1, \ldots, x_k$: any point $x$ of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\text{conv } S$: set of all convex combinations of points in $S$
Convex cone

**conic (nonnegative) combination** of $x_1$ and $x_2$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0, \theta_2 \geq 0$

**convex cone**: set that contains all conic combinations of points in the set
Hyperplanes and halfspaces

**hyperplane**: set of the form \( \{ x \mid a^T x = b \} \) \((a \neq 0)\)

\[ a^T x = b \]

\[ x_0 \]

\[ x \]

- \( a \) is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

**halfspace**: set of the form \( \{ x \mid a^T x \leq b \} \) \((a \neq 0)\)

\[ a^T x \geq b \]

\[ a^T x \leq b \]
Euclidean balls and ellipsoids

(Euclidean) ball with center $x_c$ and radius $r$:

$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$

ellipsoid: set of the form

$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$

with $P \in \mathbb{S}^n_{++}$ (i.e., $P$ symmetric positive definite)

other representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$ with $A$ square and nonsingular

$A = P^{\frac{1}{2}}$
**Norm balls and norm cones**

**norm:** a function $\| \cdot \|$ that satisfies

- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \|x\|$ for $t \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|$

notation: $\| \cdot \|$ is general (unspecified) norm; $\| \cdot \|_{\text{symb}}$ is particular norm

**norm ball** with center $x_c$ and radius $r$: $\{ x \mid \|x - x_c\| \leq r \}$

**norm cone:** $\{(x, t) \mid \|x\| \leq t\}$

Euclidean norm cone is called second-order cone

norm balls and cones are convex
Polyhedra

solution set of finitely many linear inequalities and equalities

\[ Ax \preceq b, \quad Cx = d \]

\((A \in \mathbb{R}^{m \times n}, \ C \in \mathbb{R}^{p \times n}, \ \preceq \text{ is componentwise inequality})\)

polyhedron is intersection of finite number of halfspaces and hyperplanes

Convex sets
Positive semidefinite cone

notation:
• \( S^n \) is set of symmetric \( n \times n \) matrices
• \( S_+^n = \{ X \in S^n \mid X \succeq 0 \} \): positive semidefinite \( n \times n \) matrices

\[ X \in S_+^n \iff z^T X z \geq 0 \text{ for all } z \]  

\( S_+^n \) is a convex cone
• \( S_{++}^n = \{ X \in S^n \mid X \succ 0 \} \): positive definite \( n \times n \) matrices

example: \( \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_+^2 \)
Operations that preserve convexity

practical methods for establishing convexity of a set $C$

1. apply definition

\[ x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \implies \theta x_1 + (1 - \theta)x_2 \in C \]

2. show that $C$ is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . . ) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions
Intersection

the intersection of (any number of) convex sets is convex

example:

(Review Example 2.7)
Affine function

suppose \( f : \mathbb{R}^n \to \mathbb{R}^m \) is affine (\( f(x) = Ax + b \) with \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m \))

• the image of a convex set under \( f \) is convex

\[ S \subseteq \mathbb{R}^n \text{ convex} \implies f(S) = \{ f(x) \mid x \in S \} \text{ convex} \]

• the inverse image \( f^{-1}(C) \) of a convex set under \( f \) is convex

\[ C \subseteq \mathbb{R}^m \text{ convex} \implies f^{-1}(C) = \{ x \in \mathbb{R}^n \mid f(x) \in C \} \text{ convex} \]

examples

• scaling, translation, projection
Perspective and linear-fractional function

**perspective function** \( P : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n : \)

\[
P(x, t) = \frac{x}{t}, \quad \text{dom} \ P = \{(x, t) \mid t > 0\}
\]

images and inverse images of convex sets under perspective are convex

**linear-fractional function** \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m : \)

\[
f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom} \ f = \{x \mid c^T x + d > 0\}
\]

images and inverse images of convex sets under linear-fractional functions are convex
example of a linear-fractional function

\[
f(x) = \frac{1}{x_1 + x_2 + 1} x
\]

(Of course, C is not convex in this example)
Generalized inequalities

A convex cone \( K \subseteq \mathbb{R}^n \) is a **proper cone** if

- \( K \) is closed (contains its boundary)
- \( K \) is solid (has nonempty interior) \hspace{1cm} (Counterex: \( x_2 = 2 \times 1 \))
- \( K \) is pointed (contains no line) \hspace{1cm} (Counterex: \( x_2 = -x_1 \))

**Examples**

- Nonnegative orthant \( K = \mathbb{R}^n_+ = \{ x \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n \} \)
- Positive semidefinite cone \( K = \mathbb{S}^n_+ \)
- Nonnegative polynomials on \([0, 1]\):

\[
K = \{ x \in \mathbb{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1] \}
\]
**generalized inequality** defined by a proper cone $K$:

$$x \lesssim_K y \iff y - x \in K,$$

$$x \prec_K y \iff y - x \in \text{int } K$$

**examples**

- componentwise inequality ($K = \mathbb{R}^n_+$)

  $$x \leq_{\mathbb{R}^n_+} y \iff x_i \leq y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbb{S}^n_+$)

  $$X \preceq_{\mathbb{S}^n_+} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\lesssim_K$

**properties:** many properties of $\lesssim_K$ are similar to $\leq$ on $\mathbb{R}$, e.g.,

$$x \lesssim_K y, \quad u \lesssim_K v \implies x + u \lesssim_K y + v$$
Separating hyperplane theorem

if $C$ and $D$ are nonempty disjoint convex sets, there exist $a \neq 0$, $b$ s.t.

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$

the hyperplane $\{x \mid a^T x = b\}$ separates $C$ and $D$
Supporting hyperplane theorem

**supporting hyperplane** to set $C$ at boundary point $x_0$:

$$
\{x \mid a^T x = a^T x_0\}
$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

**supporting hyperplane theorem**: if $C$ is convex, then there exists a supporting hyperplane at every boundary point of $C$
Dual cones and generalized inequalities

dual cone of a cone $K$:

$$K^* = \{ y \mid y^T x \geq 0 \text{ for all } x \in K \}$$

eamples

- $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
- $K = S^n_+$: $K^* = S^n_+$
- $K = \{(x, t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_2 \leq t\}$
- $K = \{(x, t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_\infty \leq t\}$  \hspace{1cm} (See Example 2.25)

first three examples are self-dual cones
dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succeq_K 0$$