Proof of the generalized Cauchy-Schwarz inequality

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In the lecture we are concerned with proving $|\mathbf{u}^T\mathbf{v}| \leq \|\mathbf{u}\|_\infty \cdot \|\mathbf{v}\|_1$ for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Here is how to do it with elementary inequalities:

$$\begin{array}{rcll} |\mathbf{u}^T\mathbf{v}| & \leq & |\sum_{i=1}^n u_i v_i| & \\ & \leq & \sum_{i=1}^n |u_i v_i| & \text{Triangle inequality} \\ & \leq & \sum_{i=1}^n |u_i| \cdot |v_i| & \text{Cauchy-Schwarz} \\ & \leq & \sum_{i=1}^n (\max_j |u_j|)|v_i| & \text{Replace by max} \\ & \leq & \underbrace{\max_i |u_i|}_{\|\mathbf{u}\|_\infty} \cdot \underbrace{\sum_{i=1}^n |v_i|}_{\|\mathbf{v}\|_1} & \text{Take out of the sum} \\ & = & \|\mathbf{u}\|_\infty \cdot \|\mathbf{v}\|_1. & \text{By definitions} \end{array} \tag{1}$$

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