Hands-On Learning Theory
Fall 2017, Homework 0
(due on Aug 24 at beginning of class - NO EXTENSION DAYS ALLOWED)

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The homework is based on a total of 10 points. Note that linear algebra was stated to be a requirement for this course. Thus, this homework will largely measure your knowledge in that area. (Some of the questions might contain mistakes or might require additional assumptions. Besides solving the homework, you should pinpoint which questions cannot be answered or require additional assumptions.)

0) [0 points] Please briefly describe the problem you will try to solve as the course project. (If you are not clear about the problem, please describe the area. If you are not clear about the area, please mention it.) Try to use up to 5 lines. Also write the name of your advisor(s).

1) [0.5 points] Fix integers $p$ and $k$, where $k < p$. How many elements are in the set $\{-1, 0, +1\}^p$. Let $1[f] = 1$ if $f$ is true, and $1[f] = 0$ if $f$ is false. How many elements are in the set $\{w \in \{-1, 0, +1\}^p : \sum_{i=1}^{p} 1[w_i \neq 0] = k\}$.

2) [0.5 points] Fix an integer $p$. Fix the set of vertices $V = \{1 \ldots p\}$. How many undirected graphs with vertex set $V$ and exactly $k$ edges there are?

3) [0.5 points] Let $f(x_1, x_2, x_3) = 5e^{x_1} + x_2e^{x_3}$. Compute $\nabla f(x) \in \mathbb{R}^3$. Show the full derivation for all the entries.

4) [0.5 points] Let $f(x_1, x_2, x_3) = x_1 \cos(x_2) + 2 \sin(x_3)$. Show the Hessian $\nabla^2 f(x)$. Additionally, you need to show the full derivation for two entries.

5) [0.5 points] Let $x, b \in \mathbb{R}^k$ and $A \in \mathbb{R}^{k \times k}$. Let $\langle \cdot, \cdot \rangle$ denote the dot product of vectors. Let $f(x) = \frac{1}{2} \langle x, Ax \rangle + \langle b, x \rangle$. Show that $\nabla f(x) = Ax + b$ by using differentiation rules.

6) [0.5 points] Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. What is $A^{-1}$, $B^{-1}$, $(AB)^{-1}$, $\det A$, $\det B$ and $\det (AB)$?

7) [1 point] Let $A \in \mathbb{R}^{k \times k}$ and $B \in (0, \infty)^{k \times k}$. Let $\|A\|_B = \max_{i=1}^{k} \max_{j=1}^{k} b_{i,j} |a_{i,j}|$. Show that $\| \cdot \|_B$ is a norm (or not a norm). Note that $B$ is a constant.
8) [0.5 points] Let $A \in \mathbb{R}^{k \times k}$. Let $\|A\|_\infty = \max_{i=1}^k \max_{j=1}^k |a_{i,j}|$. Compute the dual norm $\|\cdot\|_*$ of the norm $\|\cdot\|_\infty$. In other words, find a simple expression for:

$$\|A\|_* = \max_{B \in \mathbb{R}^{k \times k}, \|B\|_\infty \leq 1} \langle A, B \rangle$$

In the above, $\langle \cdot, \cdot \rangle$ denotes the dot product of matrices $\langle A, B \rangle = \sum_{i=1}^k \sum_{j=1}^k a_{i,j} b_{i,j}$.

9) [1 point] Let $A \in \mathbb{R}^{k \times k}$. Let $\|A\|_\infty = \max_{i=1}^k \max_{j=1}^k |a_{i,j}|$. Assume that $\|A\|_\infty \leq \varepsilon$. What can we say about the induced norm $\|A\|_{1,\infty} = \max_{i=1}^k \sum_{j=1}^k |a_{i,j}|$, the Frobenius norm and the spectral norm?

10) [0.5 points] Let $A = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix}$. Compute $A^2$ and $A^3$.

11) [0.5 points] Let $\langle \cdot, \cdot \rangle$ denote the dot product of vectors. Let $A \in \mathbb{R}^{k \times k}$ be symmetric and that $\forall x \in \mathbb{R}^k - \{0\}$ $\langle x, Ax \rangle > 0$. What can we say about the eigenvalues $\nu_1 \ldots \nu_k$ of $A$?

12) [0.5 points] Let $A \succ 0$ and $B \succeq 0$. What can we say about $A^{-1}$, $B^{-1}$, $(AB)^{-1}$, det $A$, det $B$ and det $(AB)$?

13) [1 point] Let $\sigma_1(A) \ldots \sigma_k(A)$ be the $k$ singular values of $A \in \mathbb{R}^{k \times k}$. The nuclear norm of a matrix $A$ is defined as $\|A\|_n = \sum_{i=1}^k |\sigma_i(A)|$. Show that if $A \succeq 0$, then $\|A\|_n = \text{tr}(A)$.

14) [1 point] Let $\sigma_1(A) \ldots \sigma_k(A)$ be the $k$ singular values of $A \in \mathbb{R}^{k \times k}$. Show that $\det A = \prod_{i=1}^k \sigma_i(A)$ by using the fact that $B \in \mathbb{R}^{k \times k}$ is orthonormal if and only if $B^T B = I$ and $BB^T = I$.

15) [1 point] Fix $b \in \mathbb{R}^k$ and let $g : \mathcal{X} \to \mathbb{R}^k$. Recall that for an exponential family distribution, the probability density function of $x \in \mathcal{X}$ is:

$$p(x) = e^{\langle g(x), b \rangle - f(b)}$$

where $f : \mathbb{R}^k \to \mathbb{R}$ is given by:

$$f(b) = \log \int_{x \in \mathcal{X}} e^{\langle g(x), b \rangle} dx$$

Compute $\nabla f(b)$ and the Hessian $\nabla^2 f(b)$ (these quantities should have some probabilistic interpretation). Argue that $f(b)$ is convex.

16) [0 points] Feedback: How many hours did it take you to do this assignment? (This question is only to help me calibrate and improve the assignments, and the response will not impact your grade.)