2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities
Affine set

**line** through $x_1$, $x_2$: all points

\[ x = \theta x_1 + (1 - \theta)x_2 \quad (\theta \in \mathbb{R}) \]

affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{ x \mid Ax = b \}$

(conversely, every affine set can be expressed as solution set of system of linear equations)
Convex set

**line segment** between $x_1$ and $x_2$: all points

$$x = \theta x_1 + (1 - \theta)x_2$$

with $0 \leq \theta \leq 1$

**convex set**: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Rightarrow \quad \theta x_1 + (1 - \theta)x_2 \in C$$

**examples** (one convex, two nonconvex sets)
Convex combination and convex hull

**convex combination** of $x_1, \ldots, x_k$: any point $x$ of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1, \theta_i \geq 0$

**convex hull** $\text{conv } S$: set of all convex combinations of points in $S$
**Convex cone**

**conic (nonnegative) combination** of $x_1$ and $x_2$: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

**convex cone**: set that contains all conic combinations of points in the set.
Hyperplanes and halfspaces

**hyperplane:** set of the form \( \{ x \mid a^T x = b \} \) \((a \neq 0)\)

\[
a^T x = b
\]

\(x_0\)

\(x\)

**halfspace:** set of the form \( \{ x \mid a^T x \leq b \} \) \((a \neq 0)\)

\[
a^T x \geq b
\]

\(a\)

\(x_0\)

\(a^T x \leq b\)

- \(a\) is the normal vector
- hyperplanes are affine and convex; halfspaces are convex
Euclidean balls and ellipsoids

**Euclidean ball** with center $x_c$ and radius $r$:

$B(x_c, r) = \{ x \mid \|x - x_c\|_2 \leq r \} = \{ x_c + ru \mid \|u\|_2 \leq 1 \}$

**Ellipsoid**: set of the form

$\{ x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1 \}$

with $P \in S_{++}^n$ (*i.e.*, $P$ symmetric positive definite)

Other representation: $\{ x_c + Au \mid \|u\|_2 \leq 1 \}$ with $A$ square and nonsingular $A = P^{1/2}$
Norm balls and norm cones

**norm:** a function \( \| \cdot \| \) that satisfies

- \( \| x \| \geq 0; \| x \| = 0 \) if and only if \( x = 0 \)
- \( \| tx \| = |t| \| x \| \) for \( t \in \mathbb{R} \)
- \( \| x + y \| \leq \| x \| + \| y \| \)

notation: \( \| \cdot \| \) is general (unspecified) norm; \( \| \cdot \|_{\text{symb}} \) is particular norm

**norm ball** with center \( x_c \) and radius \( r \): \( \{ x \mid \| x - x_c \| \leq r \} \)

**norm cone:** \( \{ (x, t) \mid \| x \| \leq t \} \)

Euclidean norm cone is called second-order cone

norm balls and cones are convex
Polyhedra

solution set of finitely many linear inequalities and equalities

\[ Ax \preceq b, \quad Cx = d \]

\((A \in \mathbb{R}^{m \times n}, \ C \in \mathbb{R}^{p \times n}, \ \preceq \text{ is componentwise inequality})\)

polyhedron is intersection of finite number of halfspaces and hyperplanes
Positive semidefinite cone

notation:

- $S^n$ is set of symmetric $n \times n$ matrices
- $S^n_+ = \{ X \in S^n \mid X \succeq 0 \}$: positive semidefinite $n \times n$ matrices

\[ X \in S^n_+ \iff z^T X z \geq 0 \text{ for all } z \]  

(Reviewed later)

$S^n_+$ is a convex cone

- $S^{++}_n = \{ X \in S^n \mid X \succ 0 \}$: positive definite $n \times n$ matrices

example: \[
\begin{bmatrix}
x & y \\
y & z
\end{bmatrix} \in S^2_+
\]
Operations that preserve convexity

practical methods for establishing convexity of a set \( C \)

1. apply definition

\[
x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Rightarrow \quad \theta x_1 + (1 - \theta) x_2 \in C
\]

2. show that \( C \) is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . . ) by operations that preserve convexity

- intersection
- affine functions
- perspective function
- linear-fractional functions
Intersection

the intersection of (any number of) convex sets is convex

example:

(Review Example 2.7)
Affine function

suppose \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is affine \( (f(x) = Ax + b \text{ with } A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m) \)

- the image of a convex set under \( f \) is convex
  \[ S \subseteq \mathbb{R}^n \text{ convex} \quad \implies \quad f(S) = \{ f(x) \mid x \in S \} \text{ convex} \]

- the inverse image \( f^{-1}(C) \) of a convex set under \( f \) is convex
  \[ C \subseteq \mathbb{R}^m \text{ convex} \quad \implies \quad f^{-1}(C) = \{ x \in \mathbb{R}^n \mid f(x) \in C \} \text{ convex} \]

examples

- scaling, translation, projection
Perspective and linear-fractional function

**Perspective function** $P : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$:

$$P(x, t) = \frac{x}{t}, \quad \text{dom} \, P = \{(x, t) \mid t > 0\}$$

Images and inverse images of convex sets under perspective are convex.

**Linear-fractional function** $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom} \, f = \{x \mid c^T x + d > 0\}$$

Images and inverse images of convex sets under linear-fractional functions are convex.
**Example** of a linear-fractional function

\[ f(x) = \frac{1}{x_1 + x_2 + 1} \]

(Of course, C is not convex in this example)
Generalized inequalities

A convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if

- $K$ is closed (contains its boundary)
- $K$ is solid (has nonempty interior) \(^{(\text{Counterex: } x2 = 2 \times 1)}\)
- $K$ is pointed (contains no line) \(^{(\text{Counterex: } x2 = -x1)}\)

**Examples**

- Nonnegative orthant $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n\}$
- Positive semidefinite cone $K = S^n_+$
- Nonnegative polynomials on $[0, 1]$: 
  \[ K = \{x \in \mathbb{R}^n \mid x_1 + x_2t + x_3t^2 + \cdots + x_nt^{n-1} \geq 0 \ \text{for} \ t \in [0, 1]\} \]
**generalized inequality** defined by a proper cone $K$:

$$x \preceq_K y \iff y - x \in K, \quad x \prec_K y \iff y - x \in \text{int } K$$

**examples**

- componentwise inequality ($K = \mathbb{R}_+^n$)

  $$x \preceq_{\mathbb{R}_+^n} y \iff x_i \leq y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = S_+^n$)

  $$X \preceq_{S_+^n} Y \iff Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

**properties:** many properties of $\preceq_K$ are similar to $\leq$ on $\mathbb{R}$, e.g.,

$$x \preceq_K y, \quad u \preceq_K v \implies x + u \preceq_K y + v$$
Separating hyperplane theorem

if $C$ and $D$ are nonempty disjoint convex sets, there exist $a \neq 0$, $b$ s.t.

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$

the hyperplane $\{x \mid a^T x = b\}$ separates $C$ and $D$
Supporting hyperplane theorem

supporting hyperplane to set $C$ at boundary point $x_0$:

$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

supporting hyperplane theorem: if $C$ is convex, then there exists a supporting hyperplane at every boundary point of $C$
Dual cones and generalized inequalities

**Dual cone** of a cone $K$:

$$K^* = \{ y \mid y^T x \geq 0 \text{ for all } x \in K \}$$

examples

- $K = \mathbb{R}_+^n$: $K^* = \mathbb{R}_+^n$
- $K = S_+^n$: $K^* = S_+^n$
- $K = \{(x, t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_2 \leq t\}$
- $K = \{(x, t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_\infty \leq t\}$  
  (See Example 2.25)

first three examples are **self-dual** cones

dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succeq_K 0$$