CS 580-Spring 2019
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I collaborated with (...). I affirm that I wrote the solutions in my own words and that I understand the solutions I am submitting.
Name:

Homework 5

Due Date: March 28, 2019 at 11:59 PM on Gradescope.

Question 1

Consider the following two decision problems:

- OR(G,k): Either G contains no odd length cycle OR G has a vertex cover $S \subseteq V(G)$ of size $|S| \leq k$.
- AND(G,k): G contains no odd length cycle AND G has a vertex cover $S \subseteq V(G)$ of size $|S| \leq k$.

One of the problems is NP-Complete and the other one is in P. Your mission – you better choose to accept it – is to figure out which of the above problems is NP-Complete. You should prove that your answer is correct by providing a polynomial time algorithm to solve one problem and by proving that the other problem is NP-Complete.

Question 2

A sorted sequence of real numbers is a sequence of the form $a_1 \leq a_2 \leq \ldots \leq a_n$. A sequence of real numbers $b_1, b_2, \ldots b_n$ is said to be ε-close to being sorted if there is a sorted sequence $a_1 \leq a_2 \leq \ldots \leq a_n$ such that $\sum |a_i - b_i| \leq \varepsilon$.

Of course, you know what a sorted sequence of numbers looks like. Unfortunately, your manager at a software development company does not. He has a sequence of real numbers

$$\boldsymbol{Y} = y_1, y_2, \ldots y_n$$

which is not quite sorted which will be fed as an input to `TheGreatExperiment`. There is some hope that this will not lead to any disaster as long as the sequence $\boldsymbol{Y}$ is (pretty darn) ε-close for some tiny $\varepsilon > 0$. Use linear programming to determine whether the sequence $\boldsymbol{Y}$ is ε-close to being sorted.

Question 3

Recall that a 2-SAT Instance Φ consists of n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$ where each clause $C_i = \ell_{i_1} \vee \ell_{i_2}$ is the disjunction of two *distinct* literals. In this problem you will develop a linear time algorithm to solve the problem by reducing the 2-SAT instance to a directed graph G_Φ defined as follows:

Nodes: For each literal ℓ we add a node v_ℓ ($2n$ nodes total)
Notation: Given a node v_ℓ we use $v_{\overline{\ell}}$ to denote the node corresponding to $\overline{\ell}$. For example

if $\ell = \overline{x}$ then $v_{\overline{\ell}} = v_x$.
Edges: For each clause $C = \ell \vee \ell'$ we add two directed edges $(v_{\overline{\ell}}, v_{\ell'})$ and $(v_{\overline{\ell'}}, v_\ell)$.

Part 0 Suppose that we have a directed path $P = v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_k}$ in G_Φ. Explain why $\overline{P} = v_{\overline{\ell}_k}, \ldots, , v_{\overline{\ell}_2}, v_{\overline{\ell}_1}$ is a valid directed path in G_Φ.

Part 1: Suppose that for some literal ℓ there is directed path from node v_ℓ to $v_{\overline{\ell}}$ in G_Φ *and* that there is also a directed path from $v_{\overline{\ell}}$ to v_ℓ in G_Φ. Show that the 2-SAT instance Φ is not satisfiable.

Part 2: Suppose that for every literal ℓ the nodes v_ℓ and $v_{\overline{\ell}}$ are in separate strongly connected components in G_Φ. Show that the 2-SAT instance Φ is satisfiable.

Part 3: Describe an O(n+m) time algorithm which solves the 2-SAT problem. If the 2-SAT formula Φ is not satisfiable then the algorithm should output no-solution otherwise the algorithm should output a satisfying assignment. (Hint: You may use Tarjan's Algorithm as a blackbox.).

Question 4

So, we are back to city Z which is frequented by villanous monsters. The city has started a Hero Association to deal with these troublemakers. There is a set V of n villains and a set H of m heroes. Each hero $h \in H$ is a resource that can be allocated to fight off exactly one villain. Each villain $v \in V$ has a subset $H_v \subseteq H$ of heroes that must be assigned to capture v. The Hero Association needs you to solve the decision problem $ZHERO(n, m, k)$ which asks whether they can allocate the heroes to villains such that at least k villains get captured.

Let us try to get a feel for how this problem behaves through the following subparts.

1. Assume $k = O(1)$ and m and n are large. Help the Hero Association come up with a polynomial time algorithm to solve $ZHERO(n, m, k)$ in this setting.

2. Assume all of n, m, k are all very large. Answer the following questions.
   (a) Show that $ZHERO(n, m, k)$ is in NP.
   (b) Show that $ZHERO(n, m, k)$ is NP-Hard by giving a reduction from the matching problem in 3-uniform hypergraphs. The matching problem in 3-uniform hypergraphs is as follows: You are given a triple of vertex sets A, B, C where $|A| = |B| = |C| = m$. You are told that a subset $E \subseteq A \times B \times C$ form hyperedges each of which span 3 vertices – one each from A, B and C. We want to decide whether there exists a subset $M \subseteq E$ with $|M| \geq k$ where no two hyperedges in M share any vertex.
   (c) Assume you have an oracle which solves the decision problem $ZHERO(n, m, k)$. Give an algorithm that calls this oracle a polynomial number of times and finds the biggest subset $S \subseteq V$ of villains that can be safely captured.