CS 580: Algorithm Design and Analysis

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Announcement: Homework 2 due on Tuesday, February 5th at 11:59PM (Gradescope)

Recap: Minimum Weight Spanning Trees

Cut Property: Minimum weight edge crossing a cut must be in the MST (assume edge weights are distinct)

Cycle Property: Maximum weight edge in a cycle must not be in the MST (assuming edge weights are distinct)

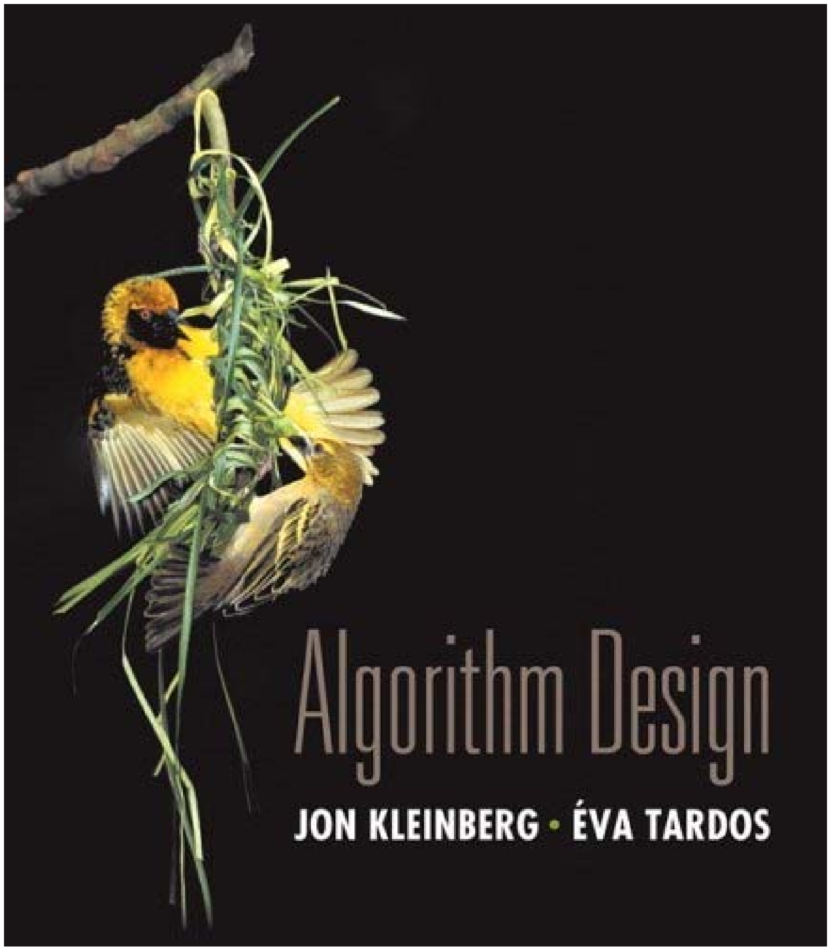
Prim's Algorithm

- **Repeatedly applies cut property to expand tree**
- **O(m log n) time with Binary Heap**
- **O(m+n log n) time with Fibonacci Heap**

Kruskal's Algorithm

- Consider edges in increasing order of weight
- O(m log n) running time.

Union-Find Data Structure



Divide and Conquer

Divide-and-Conquer

Divide-and-conquer.

- Break up problem into several parts.
- Solve each part recursively.
- Combine solutions to sub-problems into overall solution.

Most common usage.

- Break up problem of size n into two equal parts of size $\frac{1}{2}n$.
- Solve two parts recursively.
- Combine two solutions into overall solution in linear time.

Consequence.

- Brute force: n^2.
- Divide-and-conquer: $n \log n$.

> Divide et impera.
> Veni, vidi, vici.
> - Julius Caesar

5.1 Mergesort

Sorting

Sorting. Given n elements, rearrange in ascending order.

Applications.

- Sort a list of names.
- Organize an MP3 library.
- Display Google PageRank results.
- List RSS news items in reverse chronological order.

obvious applications

- Find the median.
- Find the closest pair.
- Binary search in a database.
- Identify statistical outliers.
- Find duplicates in a mailing list.

problems become easy once items are in sorted order

- Data compression.
- Computer graphics.
- Computational biology.
- Supply chain management.
- Book recommendations on Amazon.
- Load balancing on a parallel computer.

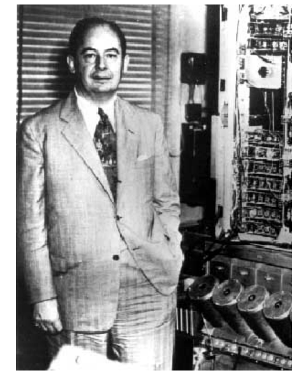
. . .

non-obvious applications

Mergesort

Mergesort.

- Divide array into two halves.
- Recursively sort each half.
- Merge two halves to make sorted whole.



Jon von Neumann (1945)

A	L	G	O	R	I	T	H	M	S

A	L	G	O	R	I	T	H	M	S	divide	O(1)
A	G	L	O	R	H	I	M	S	T	sort	2T(n/2)

A	G	H	I	L	M	O	R	S	T	merge	O(n)

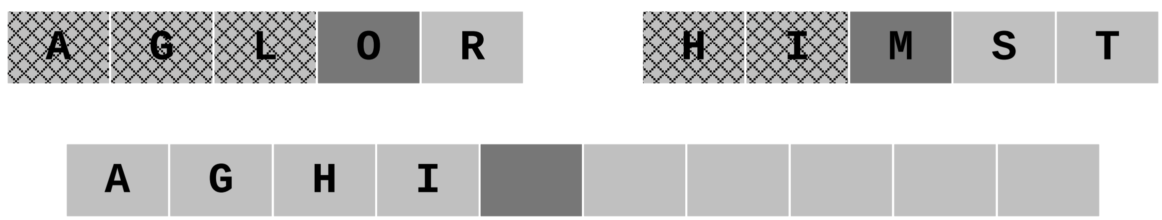
$$T(n) \leq 2\,T\left(\frac{n}{2}\right) + O(n)$$

Merging

Merging. Combine two pre-sorted lists into a sorted whole.

How to merge efficiently?
- Linear number of comparisons.
- Use temporary array.

05demo-merge.ppt



Challenge for the bored. In-place merge. [Kronrud, 1969]

using only a constant amount of extra storage

A Useful Recurrence Relation

Def. T(n) = number of comparisons to mergesort an input of size n.

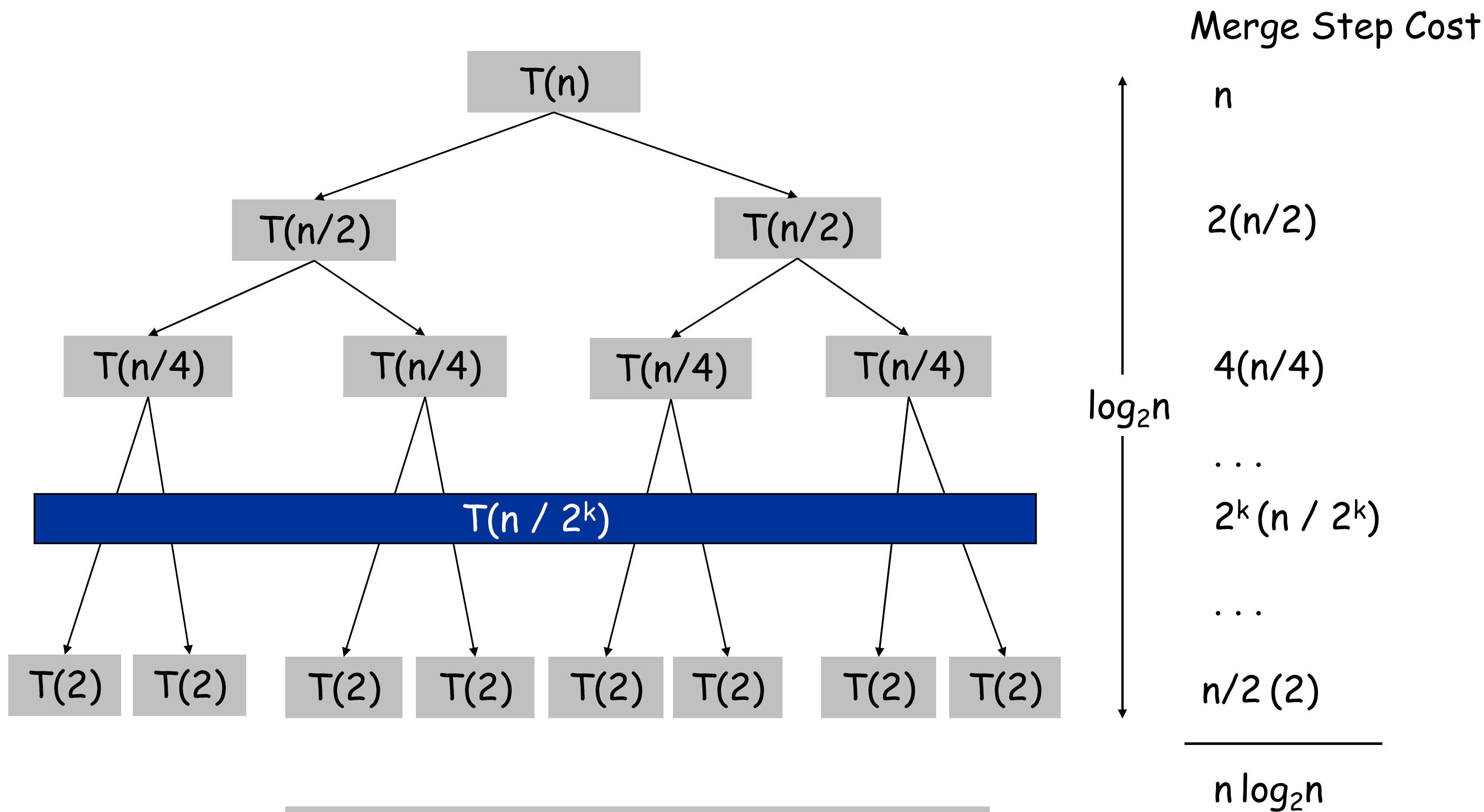
Mergesort recurrence.

$$\mathrm{T}(n) \;\le\; \begin{cases} 0 & \text{if } n=1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Solution. $T(n) = O(n \log_2 n)$.

Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace $\le$ with $=$.

Proof by Recursion Tree



$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Proof by Telescoping

Claim. If T(n) satisfies this recurrence, then $T(n) = n \log_2 n$.

assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. For n > 1:

$$\begin{aligned} \frac{T(n)}{n} &= \frac{2T(n/2)}{n} + 1 \\ &= \frac{T(n/2)}{n/2} + 1 \\ &= \frac{T(n/4)}{n/4} + 1 + 1 \\ &\cdots \\ &= \frac{T(n/n)}{n/n} + \underbrace{1 + \cdots + 1}_{\log_2 n} \\ &= \log_2 n \end{aligned}$$

Proof by Induction

Claim. If T(n) satisfies this recurrence, then $T(n) = n \log_2 n$.

assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

Pf. (by induction on n)

- Base case: n = 1.
- Inductive hypothesis: $T(n) = n \log_2 n$.
- Goal: show that $T(2n) = 2n \log_2 (2n)$.

$$\begin{aligned} T(2n) &= 2T(n) + 2n \\ &= 2n\log_2 n + 2n \\ &= 2n\big(\log_2(2n) - 1\big) + 2n \\ &= 2n\log_2(2n) \end{aligned}$$

Analysis of Mergesort Recurrence

Claim. If T(n) satisfies the following recurrence, then $T(n) \le n \lceil \lg n \rceil$.

$\uparrow$ $\log_2 n$

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

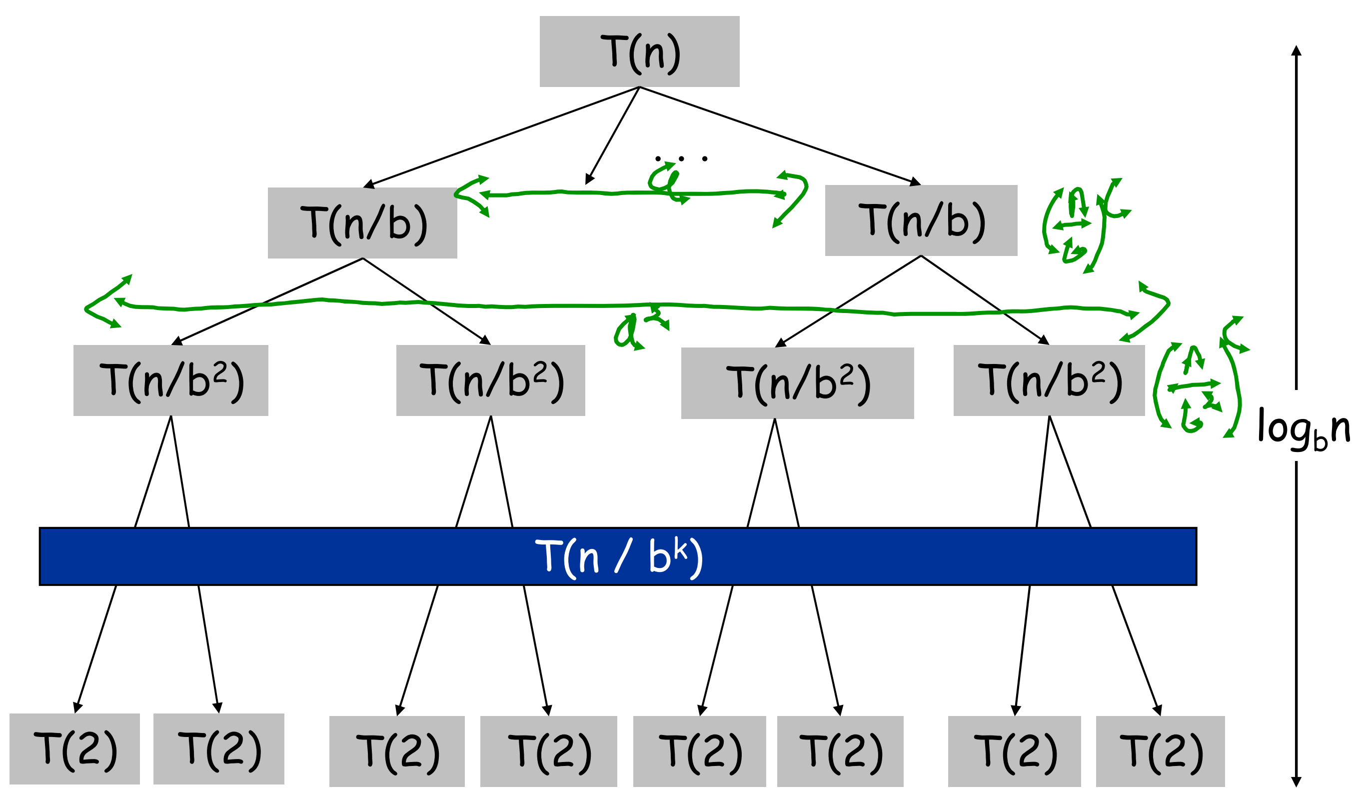
Pf. (by induction on n)

- Base case: n = 1.
- Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$.
- Induction step: assume true for 1, 2, ... , n–1.

$$\begin{aligned} T(n) &\le T(n_1) + T(n_2) + n \\ &\le n_1 \lceil \lg n_1 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &\le n_1 \lceil \lg n_2 \rceil + n_2 \lceil \lg n_2 \rceil + n \\ &= n \lceil \lg n_2 \rceil + n \\ &\le n (\lceil \lg n \rceil - 1) + n \\ &= n \lceil \lg n \rceil \end{aligned}$$

$$\begin{aligned} n_2 &= \lceil n/2 \rceil \\ &\le \left\lceil 2^{\lceil \lg n \rceil} / 2 \right\rceil \\ &= 2^{\lceil \lg n \rceil} / 2 \\ \Rightarrow \lg n_2 &\le \lceil \lg n \rceil - 1 \end{aligned}$$

More General Analysis



Merge Step Cost

n^c

$a(n/b)^c = n^c(a/b^c)$

$a^2(n/b^2)^c = \mathbf{n^c(a/b^c)^2}$

. . .

$n^c(a/b^c)^k$

. . .

$n^c(a/b^c)^{\log_b n}$

$$T(n) \le n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i$$

$$T(n) \le \begin{cases} 1 & if\ n = 1 \\ a \times T\left(\frac{n}{b}\right) + n^c & otherwise \end{cases}$$

A Helpful Identity

Fact: If $\gamma \neq 1$ then

$$1 + \gamma^1 + \gamma^2 \ldots + \gamma^k = \frac{1 - \gamma^{k+1}}{1 - \gamma}$$

Proof

$$1 + \gamma^1 + \gamma^2 \ldots + \gamma^k = \frac{1 - \gamma}{1 - \gamma}\left(1 + \gamma^1 + \gamma^2 \ldots + \gamma^k\right)$$

$$= \frac{\left(1 + \gamma^1 + \gamma^2 \ldots + \gamma^k\right)}{1 - \gamma} - \frac{\gamma\left(1 + \gamma^1 + \gamma^2 \ldots + \gamma^k\right)}{1 - \gamma}$$

$$= \frac{1 + \gamma^1 + \gamma^2 \ldots + \boldsymbol{\gamma^k}}{1 - \gamma} + \frac{-\gamma^1 - \gamma^2 \ldots - \boldsymbol{\gamma^k} - \gamma^{k+1}}{1 - \gamma}$$

$$= \frac{1 - \gamma^{k+1}}{1 - \gamma}$$

A Helpful Identity

Fact: If $\gamma \neq 1$ then

$$1 + \gamma^1 + \gamma^2 \ldots + \gamma^k = \frac{1 - \gamma^{k+1}}{1 - \gamma}$$

Observation 1: If $\gamma = 1$ then $1 + \gamma^1 + \gamma^2 \ldots + \gamma^k = k + 1 \in \Theta(k)$

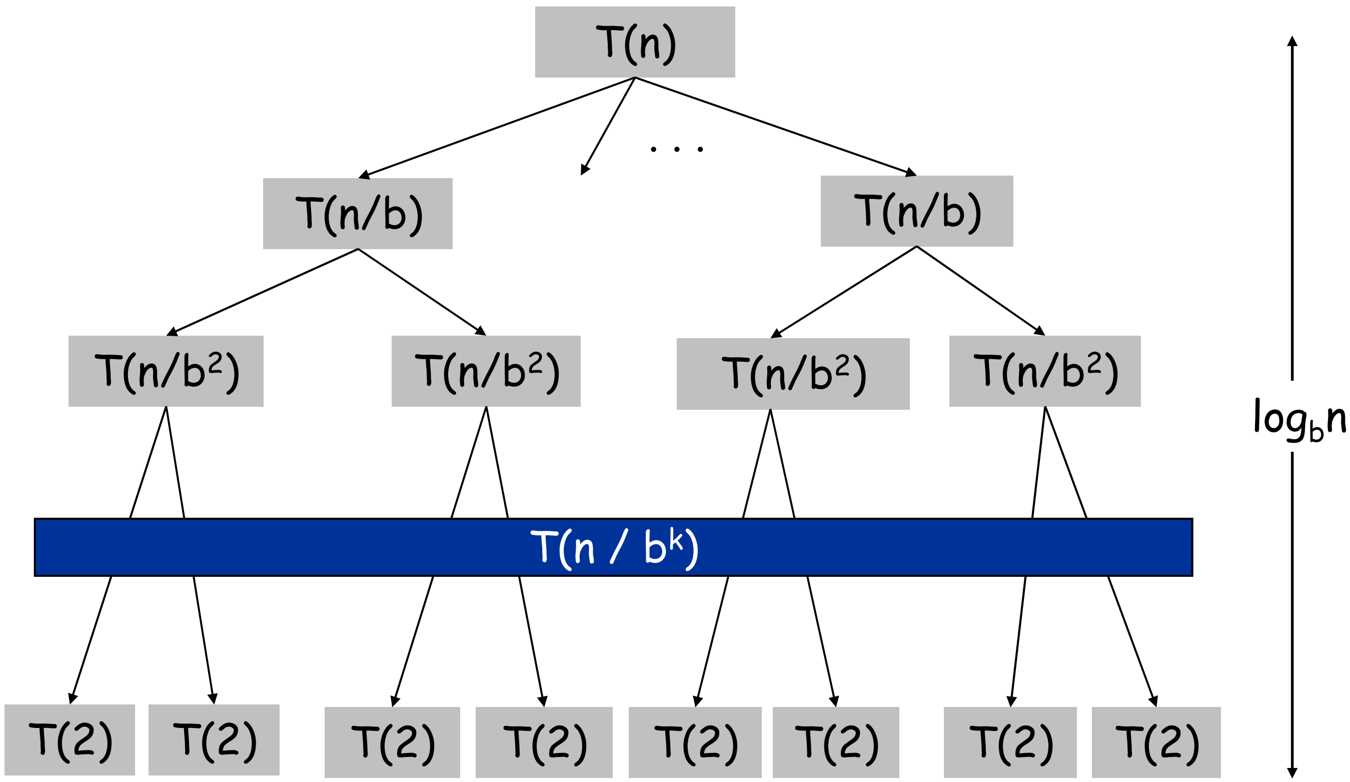
Observation 2: If $0 < \gamma < 1$ then $\mathbf{1} + \gamma^1 + \gamma^2 \ldots + \gamma^k \approx \frac{1}{1-\gamma} \in \Theta(1)$

Observation 3: If $1 < \gamma$ then $1 + \gamma^1 + \gamma^2 \ldots + \boldsymbol{\gamma^k} \approx \frac{\gamma^{k+1}}{\gamma-1} \in \Theta(\gamma^k)$

Observation 4: In our case $k = \log_b n$ and $\gamma = \left(\frac{a}{b^c}\right)$

$$T(n) \leq n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i = n^c \left(\frac{1 - \gamma^{k+1}}{1 - \gamma}\right)$$

More General Analysis



Merge Step Cost

n^c

$a(n/b)^c = n^c(a/b^c)$

$n^c(a/b^c)^2$

. . .

$n^c(a/b^c)^k$

. . .

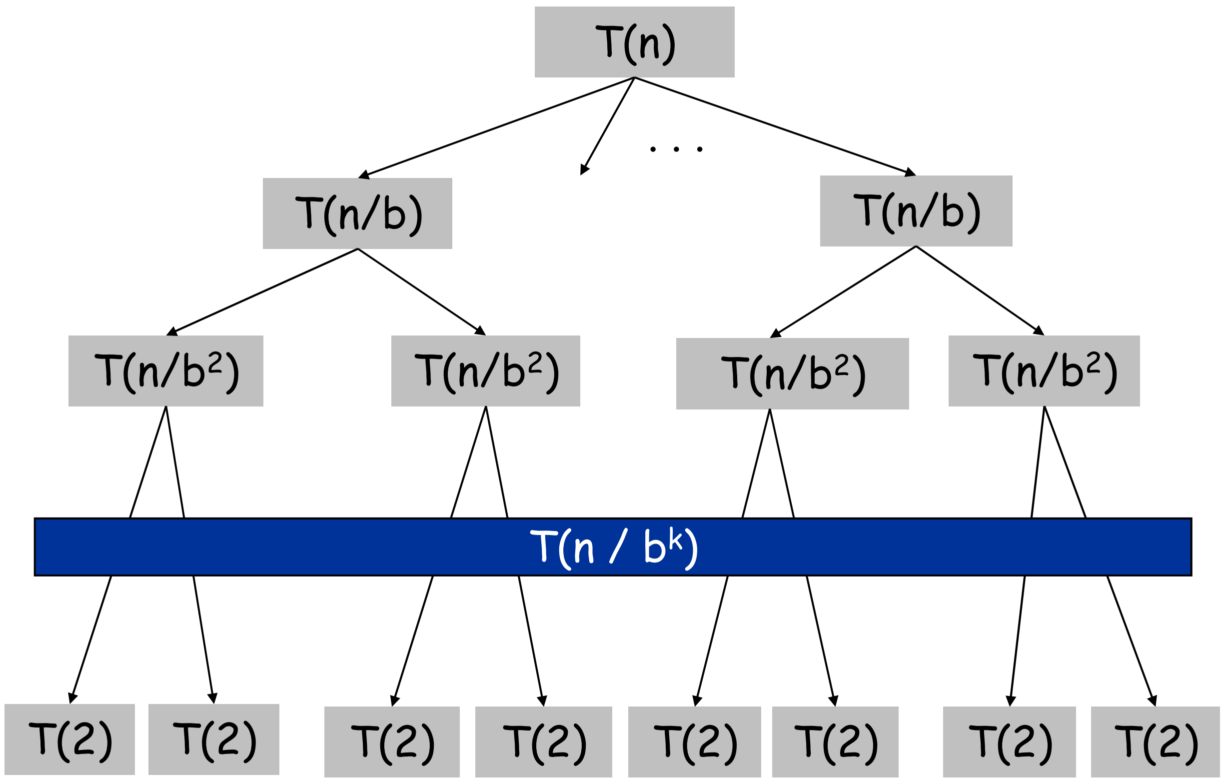
$n^c(a/b^c)^{\log_b n}$

$$T(n) \le \begin{cases} 1 & if\ n = 1 \\ a \times T\left(\frac{n}{b}\right) + n^c & otherwise \end{cases}$$

Case 1: $\gamma = \left(\frac{a}{b^c}\right) = 1$

$$T(n) \le n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i$$
$$= n^c \log_b n$$

More General Analysis



$\log_b n$

Merge Step Cost

n^c

$a(n/b)^c = n^c(a/b^c)$

$n^c(a/b^c)^2$

. . .

$n^c(a/b^c)^k$

. . .

$n^c(a/b^c)^{\log_b n}$

$$T(n) \le \begin{cases} 1 & \text{if } n = 1 \\ a \times T\left(\frac{n}{b}\right) + n^c & \text{otherwise} \end{cases}$$

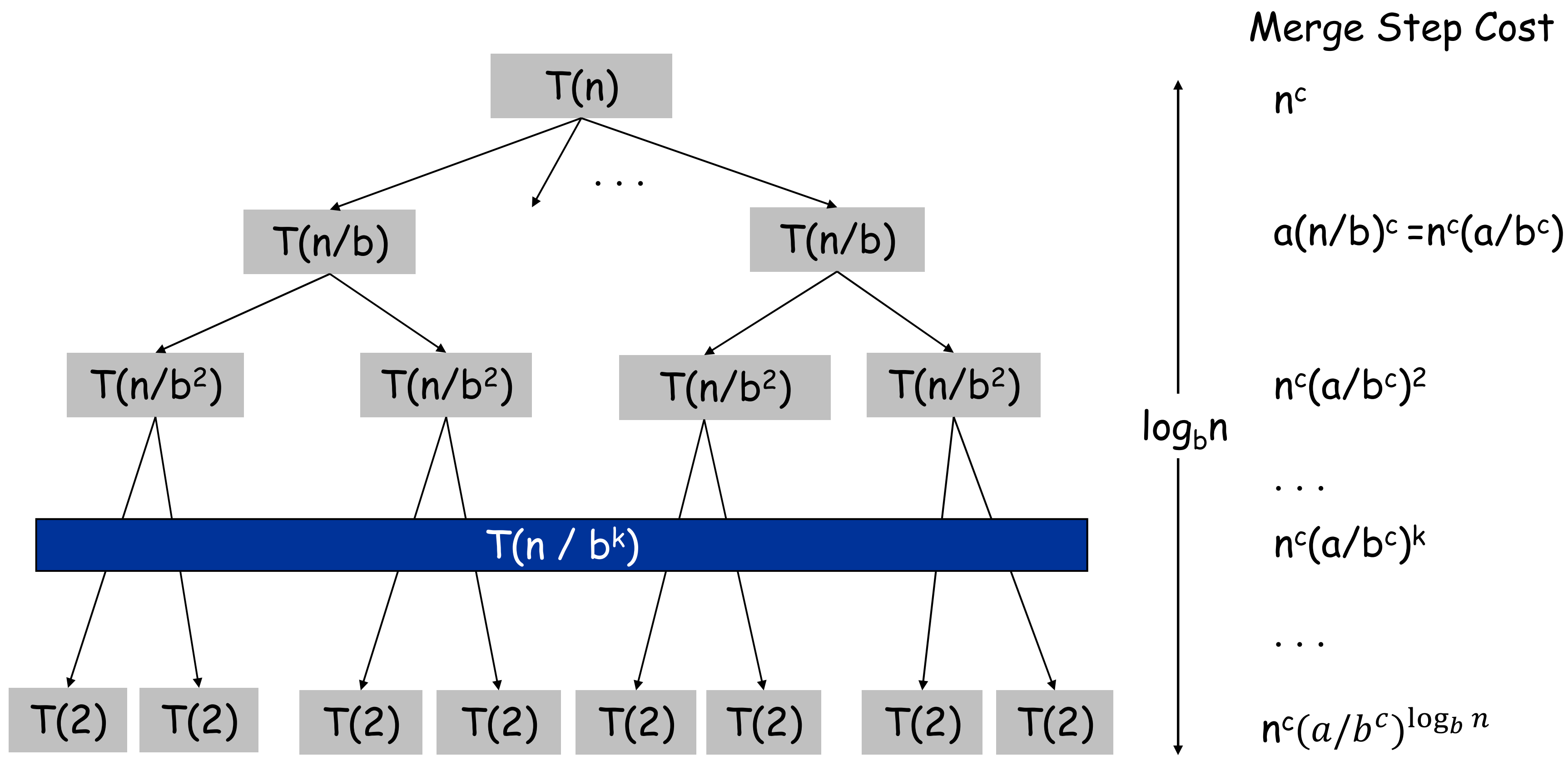
Case 2: $\gamma = \left(\frac{a}{b^c}\right) < 1$

$$T(n) \le n^c \sum_{i=0}^{\log_b n} \left(\frac{a}{b^c}\right)^i$$

$$= \Theta(n^c)$$

$$n^c \frac{(1-\gamma)}{(1-\gamma)} \sum_{i=0}^{\infty} \gamma^i = \frac{n^c}{(1-\gamma)}$$

More General Analysis



$$T(n) \leq \begin{cases} 1 & if\ n = 1 \\ a \times T\left(\frac{n}{b}\right) + n^c & otherwise \end{cases}$$

Case 3: $\gamma = \left(\frac{a}{b^c}\right) > 1$

$$T(n) \leq \sum_{i=0}^{\log_b n} n^c \left(\frac{a}{b^c}\right)^i = \Theta\left(n^{\log_b a}\right)$$

$$n^c \left(\frac{a}{b^c}\right)^{\log_b n} = \left(\frac{n^c}{b^{c \log_b n}}\right) \times a^{\log_b n} = n^{\log_b a}$$

Implications for Divide and Conquer Analysis

- Merge Cost: $O(n^c)$ (want c to be small)
- Branching Factor: a (smaller branching factor → faster)
- Reduction in Input Size: b (bigger is better)
  - Key Ratio: a/b^c

$$T(n) \le \begin{cases} 1 & if\ n = 1 \\ a \times T\left(\frac{n}{b}\right) + n^c & otherwise \end{cases}$$

Case 1: $\left(\frac{a}{b^c}\right) < 1$ $\quad T(n) = \Theta(n^c)$

Case 2: $\gamma = \left(\frac{a}{b^c}\right) = 1$ $\quad T(n) = \Theta(n^c \log n)$

Case 3: $\left(\frac{a}{b^c}\right) > 1$ $\quad T(n) = \Theta\left(n^{\log_b a}\right)$

Implications for Divide and Conquer Analysis

- Merge Cost: $O(n^c)$ (want c to be small)
- Branching Factor: a (smaller branching factor → faster)
- Reduction in Input Size: b (bigger is better)
  - Key Ratio: a/b^c

$$T(n) \le \begin{cases} \mathbf{1000000} & \boldsymbol{if\ n \le 100} \\ a \times T\left(\frac{n}{b} + \mathbf{50}\right) + n^c & otherwise \end{cases}$$

Case 1: $\left(\frac{a}{b^c}\right) < 1 \qquad T(n) = \Theta(n^c)$

Case 2: $\gamma = \left(\frac{a}{b^c}\right) = 1 \quad T(n) = \Theta(n^c \log n)$

Case 3: $\left(\frac{a}{b^c}\right) > 1 \quad T(n) = \Theta\left(n^{\log_b a}\right)$

Other Recurrences

- $\mathrm{T(n) = T(n-1) + 1}$ (Unroll: $\mathrm{T(n) = n}$)

- $\mathrm{T(n) = 2 \times T(n-10)}$ (Exponential)

Two branches   Only constant reduction in input size

- $\mathrm{T(n) = T}\left(\frac{\mathrm{n}}{4}\right) + T\left(\frac{2n}{3}\right) + n$

  (Solution: $\mathrm{T(n)} \in \Theta(n)$)

- $\mathrm{T(n) = T}\left(\frac{\mathrm{n}}{4}\right) + T\left(\frac{3n}{4}\right) + n$

  (Solution: $\mathrm{T(n)} \in \Theta(n \log n)$)

Other Recurrences

- $\mathrm{T(n) = 2 \times T(n-10)}$ **(Exponential)**

Two branches

Only constant reduction in input size

$$\mathrm{T(n) = \Theta(c^n)}$$

- How to find c? [Trick]

$$2 = \frac{\mathrm{T(n)}}{\mathrm{T(n-10)}} = \frac{\mathrm{c^n}}{c^{n-10}}$$

$$\rightarrow \mathrm{c^{10} = 2}$$

$$\rightarrow \mathrm{c} = \sqrt[10]{2} \approx 1.07177$$

- ***Must verify solution by induction***

5.3 Counting Inversions

Counting Inversions

Music site tries to match your song preferences with others.

- You rank n songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: 1, 2, …, n.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j inverted if i < j, but $a_i > a_j$.

Songs

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5

Inversions

3-2, 4-2

Brute force: check all $\Theta(n^2)$ pairs i and j.

Applications

Applications.

- Voting theory.
- Collaborative filtering.
- Measuring the "sortedness" of an array.
- Sensitivity analysis of Google's ranking function.
- Rank aggregation for meta-searching on the Web.
- Nonparametric statistics (e.g., Kendall's Tau distance).

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

1	5	4	8	10	2	6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.

1	5	4	8	10	2	6	9	12	11	3	7

Divide: $O(1)$.

1	5	4	8	10	2

6	9	12	11	3	7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.

1	5	4	8	10	2	6	9	12	11	3	7

Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

Conquer: 2T(n / 2)

5 blue-blue inversions

5-4, 5-2, 4-2, 8-2, 10-2

8 green-green inversions

6-3, 9-3, 9-7, 12-3, 12-7, 12-11, 11-3, 11-7

Counting Inversions: Divide-and-Conquer

Divide-and-conquer.

- Divide: separate list into two pieces.
- Conquer: recursively count inversions in each half.
- Combine: count inversions where a_i and a_j are in different halves, and return sum of three quantities.

1	5	4	8	10	2	6	9	12	11	3	7

Divide: O(1).

1	5	4	8	10	2

6	9	12	11	3	7

Conquer: 2T(n / 2)

5 blue-blue inversions

8 green-green inversions

9 blue-green inversions
5-3, 4-3, 8-6, 8-3, 8-7, 10-6, 10-9, 10-3, 10-7

Combine: ???

Total = 5 + 8 + 9 = 22.

Counting Inversions: Combine

Combine: count blue-green inversions

- Assume each half is sorted.
- Count inversions where a_i and a_j are in different halves.
- Merge two sorted halves into sorted whole.

to maintain sorted invariant

play

3	7	10	14	18	19

2	11	16	17	23	25
6	3	2	2	0	0

13 blue-green inversions: 6 + 3 + 2 + 2 + 0 + 0

Count: O(n)

2	3	7	10	11	14	16	17	18	19	23	25

Merge: O(n)

$$T(n) \le T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) \Rightarrow \mathrm{T}(n) = O(n \log n)$$

Counting Inversions: Implementation

Pre-condition. [Merge-and-Count] A and B are sorted.
Post-condition. [Sort-and-Count] L is sorted.

```
Sort-and-Count(L) {
   if list L has one element
      return 0 and the list L

   Divide the list into two halves A and B
   (rA, A) ← Sort-and-Count(A)
   (rB, B) ← Sort-and-Count(B)
   (r , L) ← Merge-and-Count(A, B)

   return r = rA + rB + r and the sorted list L
}
```

5.4 Closest Pair of Points

Closest Pair of Points

Closest pair. Given n points in the plane, find a pair with smallest Euclidean distance between them.

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of nearest neighbor, Euclidean MST, Voronoi.

fast closest pair inspired fast algorithms for these problems

Brute force. Check all pairs of points p and q with $\Theta(n^2)$ comparisons.

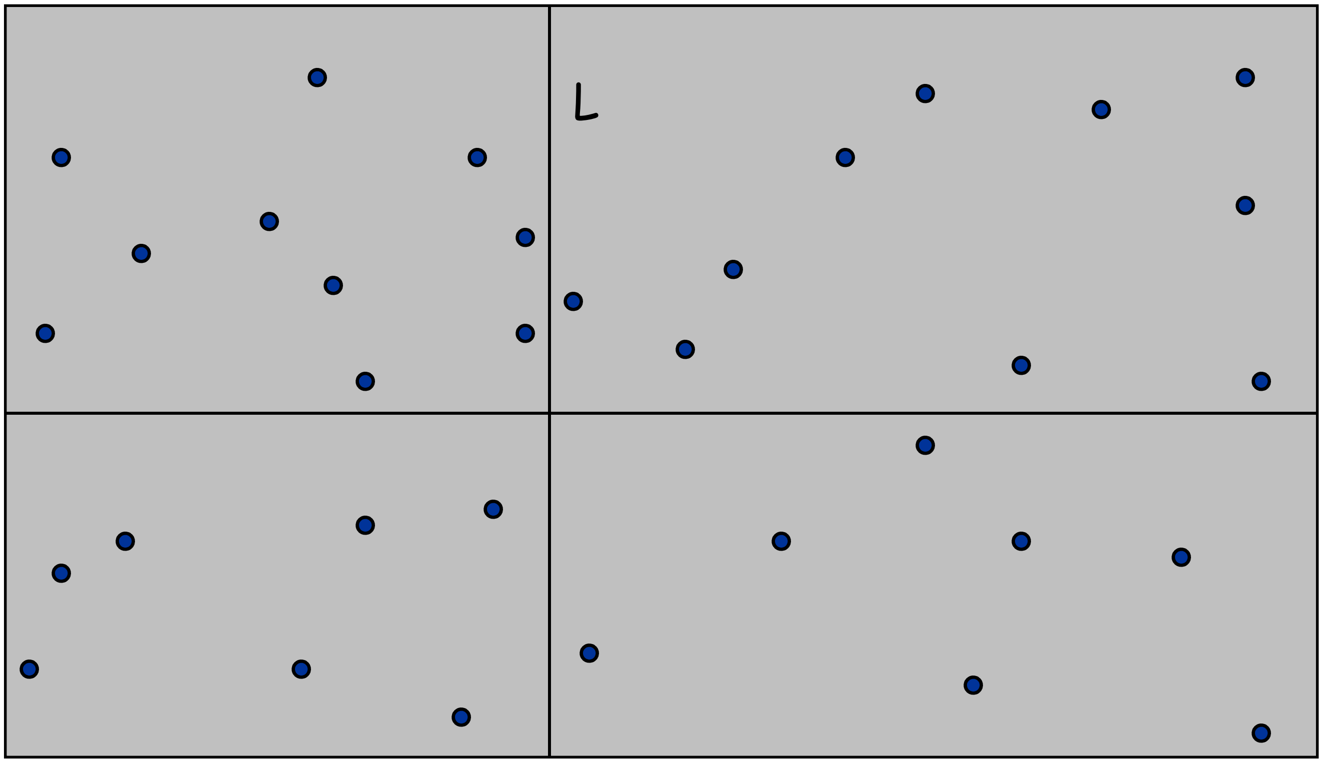
1-D version. O(n log n) easy if points are on a line.

Assumption. No two points have same x coordinate.

to make presentation cleaner

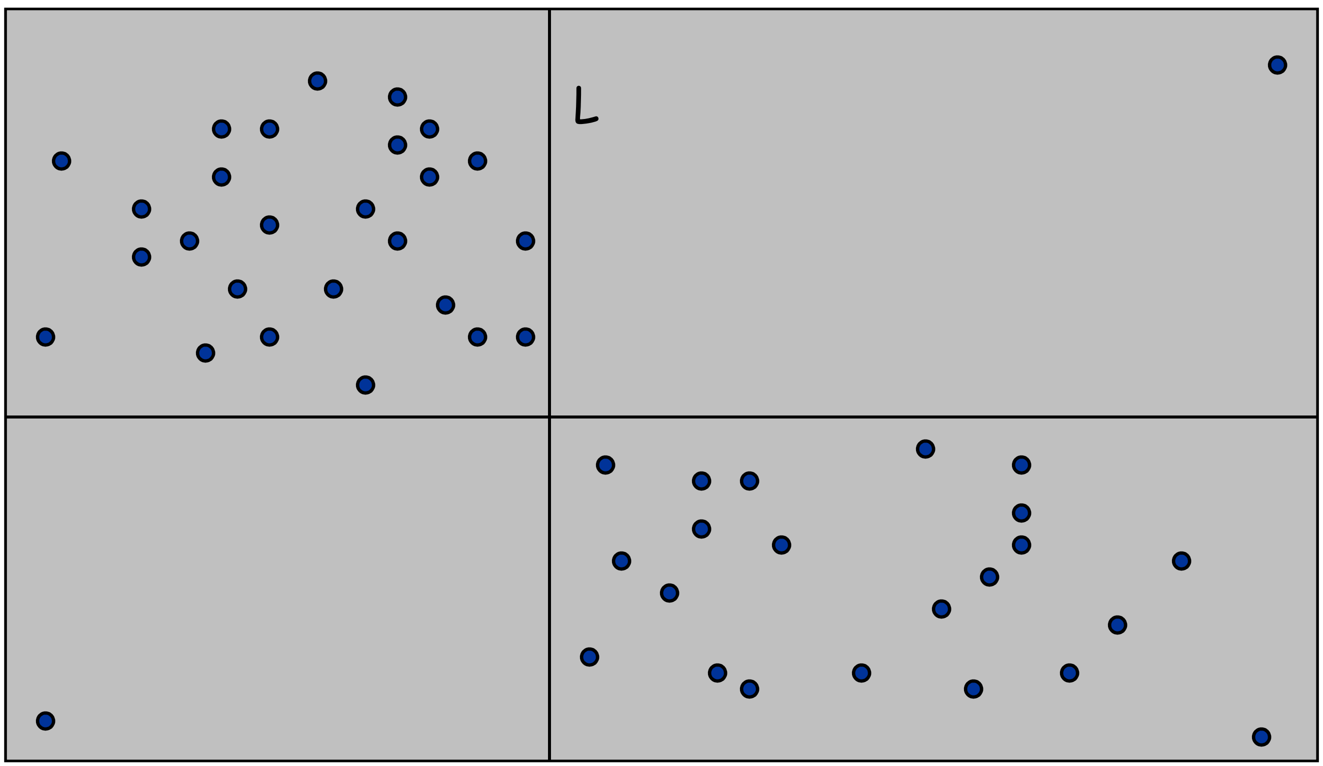
Closest Pair of Points: First Attempt

Divide. Sub-divide region into 4 quadrants.



Closest Pair of Points: First Attempt

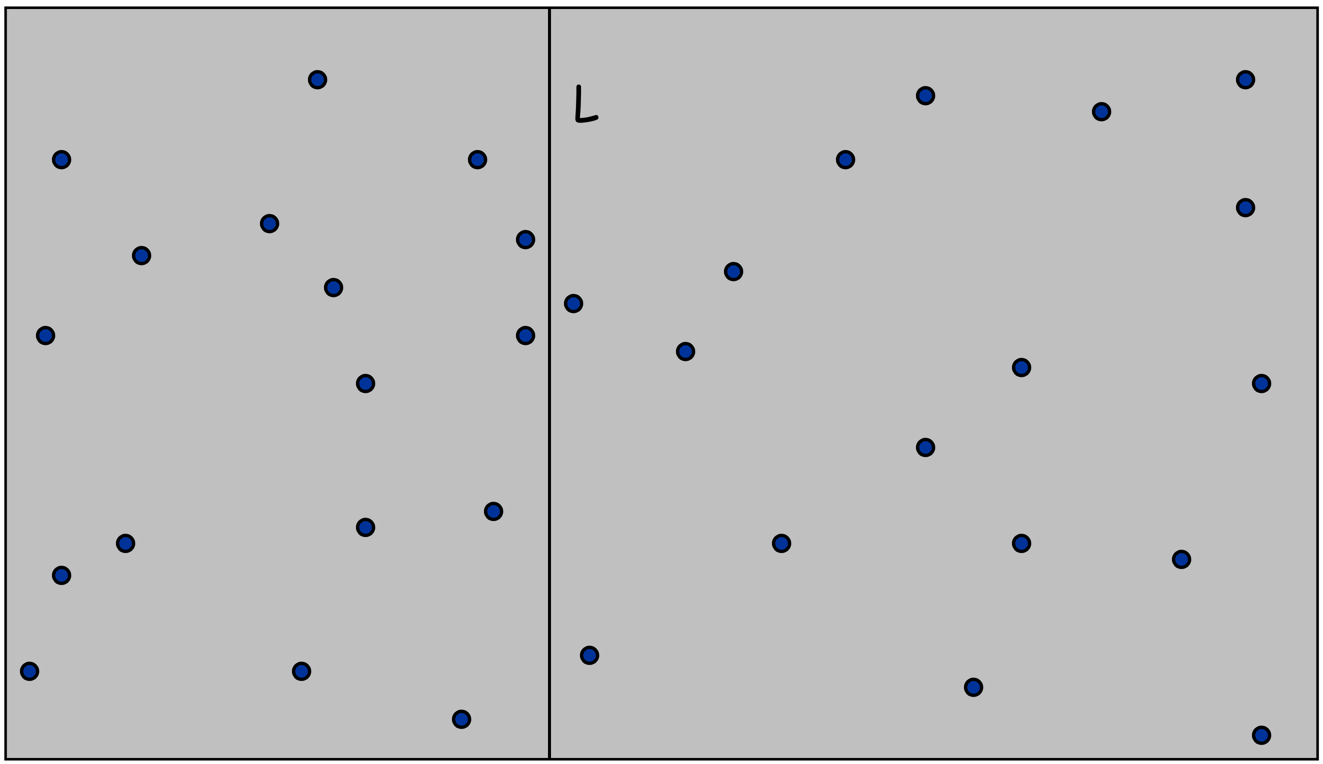
Divide. Sub-divide region into 4 quadrants.
Obstacle. Impossible to ensure n/4 points in each piece.



Closest Pair of Points

Algorithm.

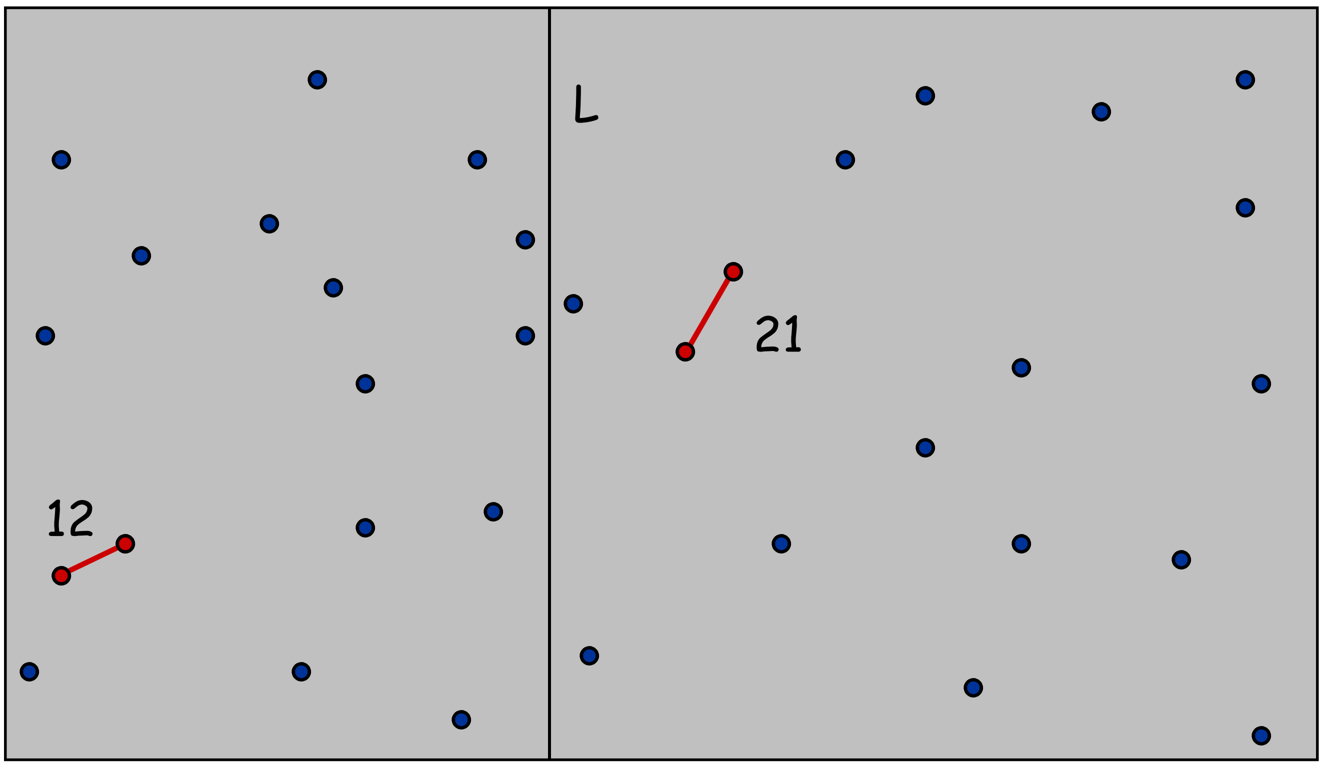
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.



Closest Pair of Points

Algorithm.

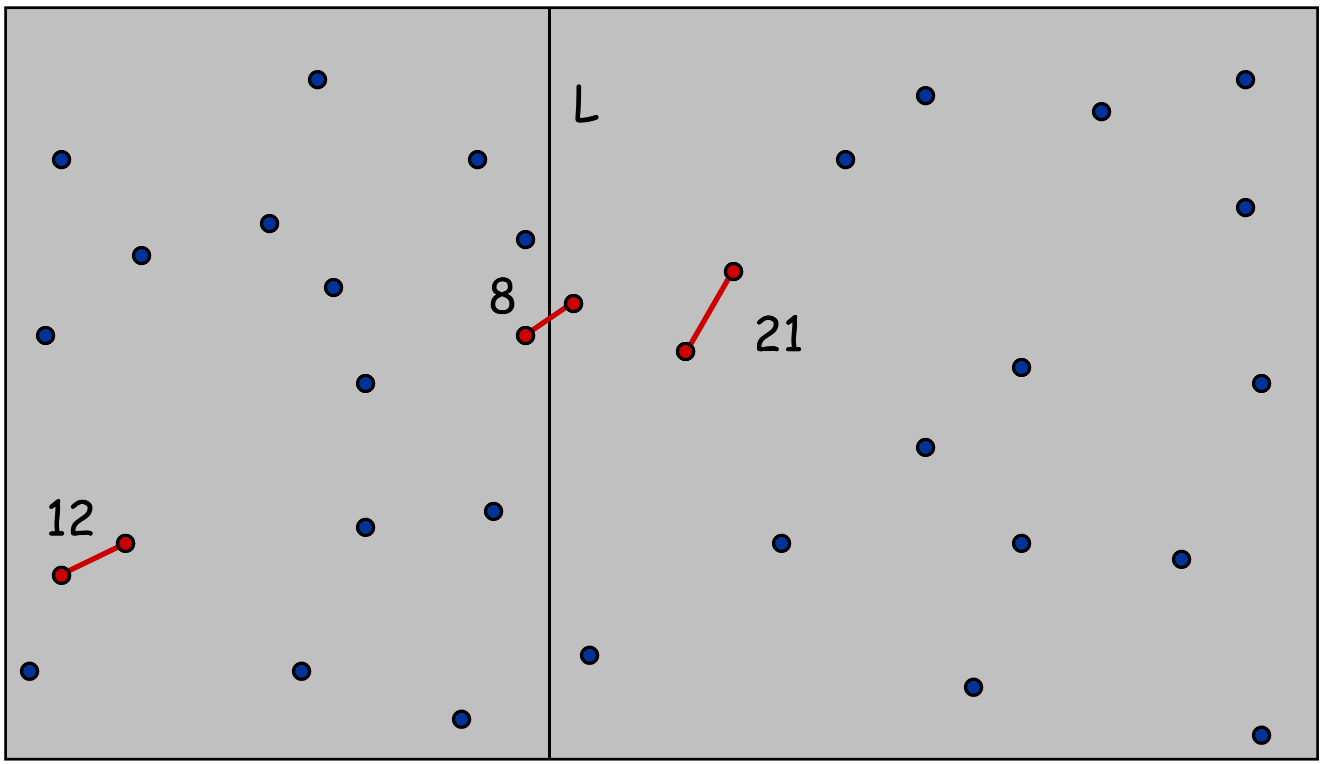
- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.



Closest Pair of Points

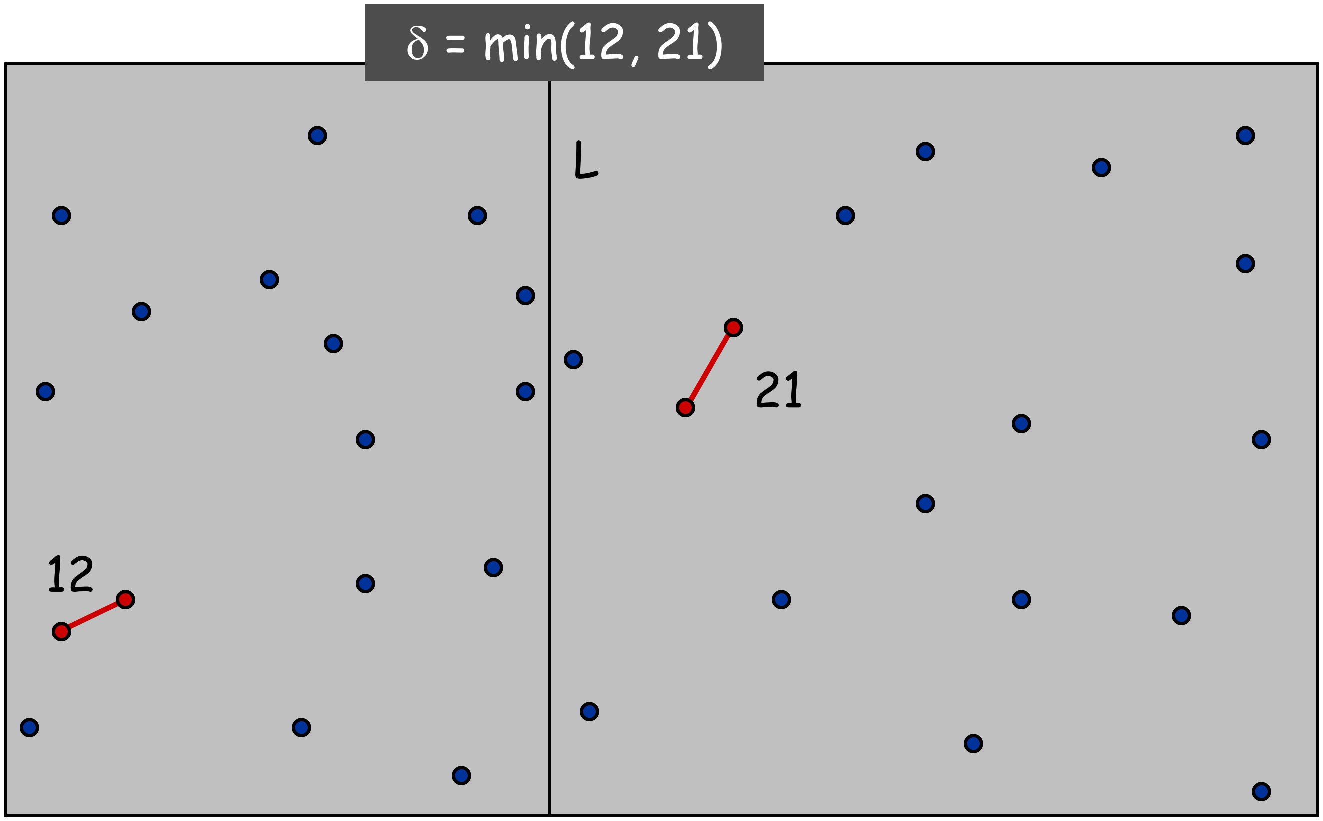
Algorithm.

- Divide: draw vertical line L so that roughly $\frac{1}{2}n$ points on each side.
- Conquer: find closest pair in each side recursively.
- Combine: find closest pair with one point in each side. ← seems like $\Theta(n^2)$
- Return best of 3 solutions.



Closest Pair of Points

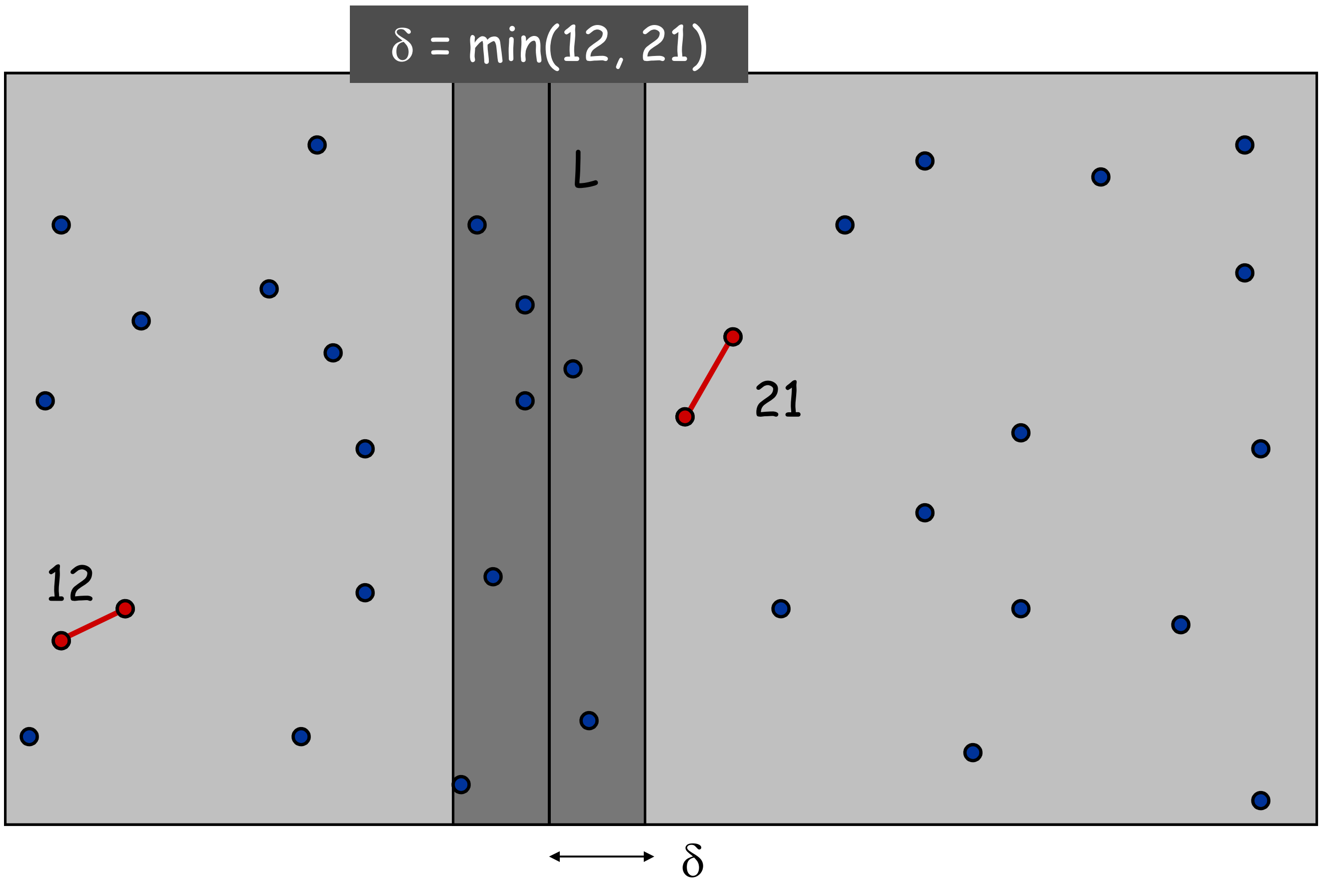
Find closest pair with one point in each side, assuming that distance < δ.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

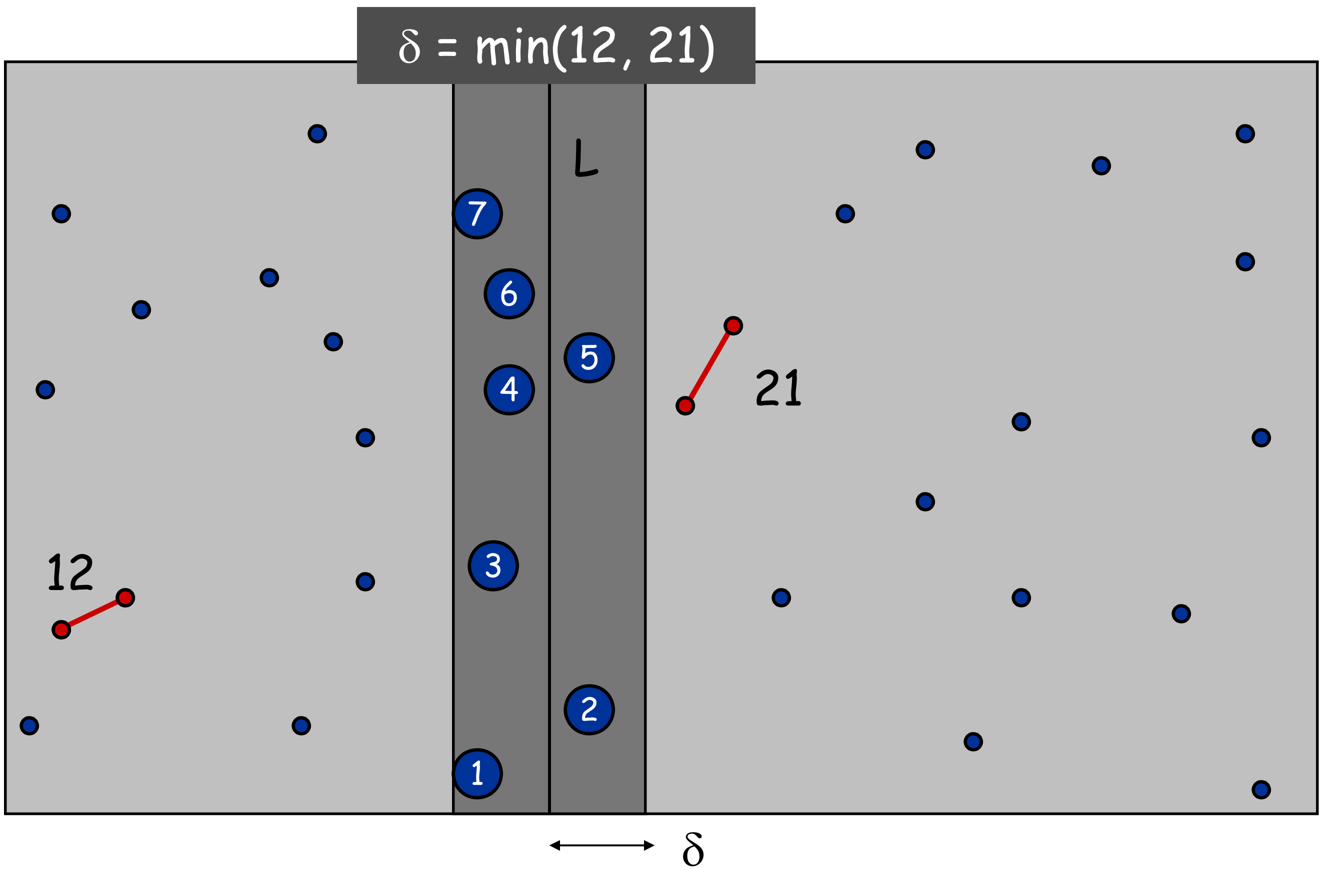
- Observation: only need to consider points within δ of line L.



Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

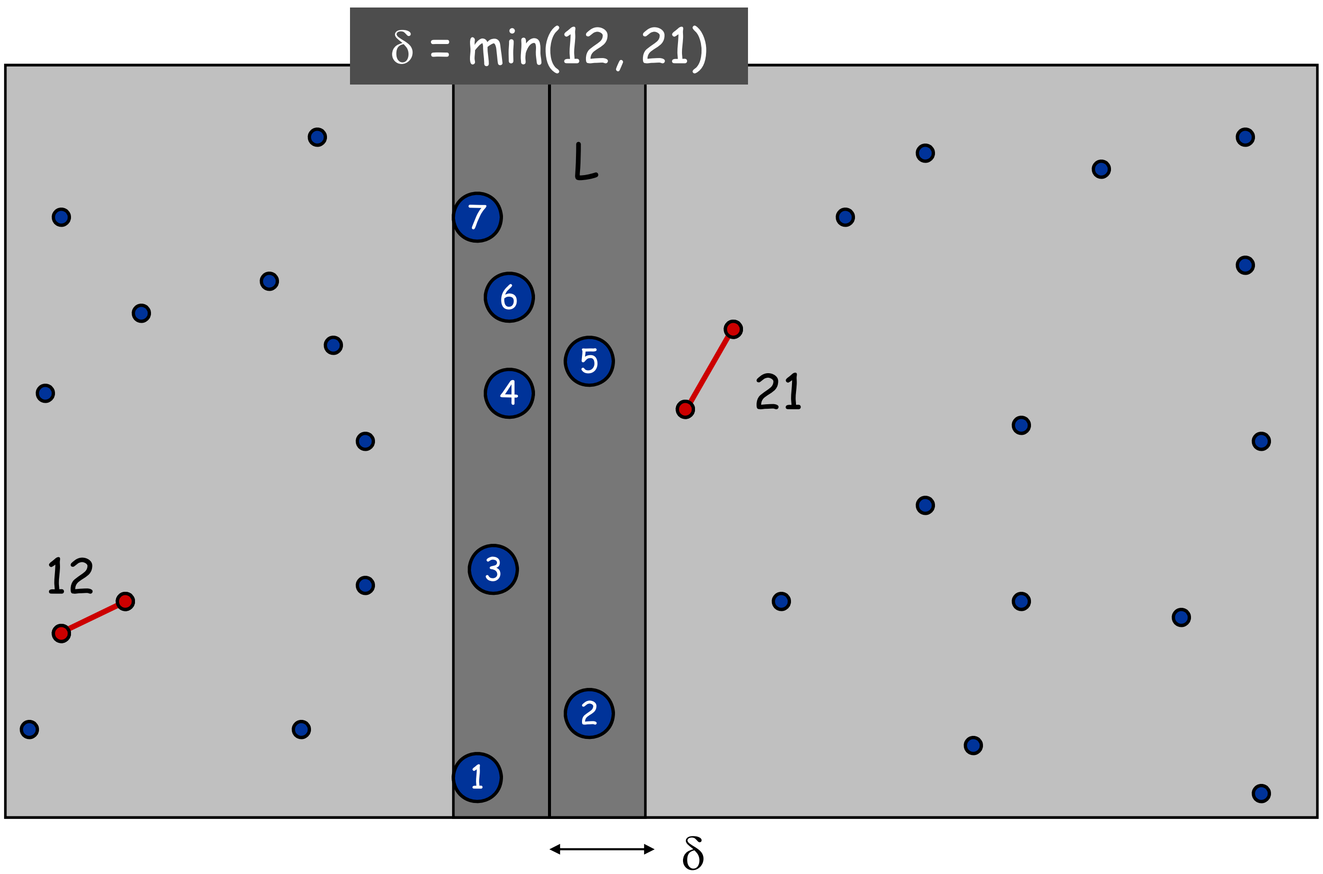
- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.



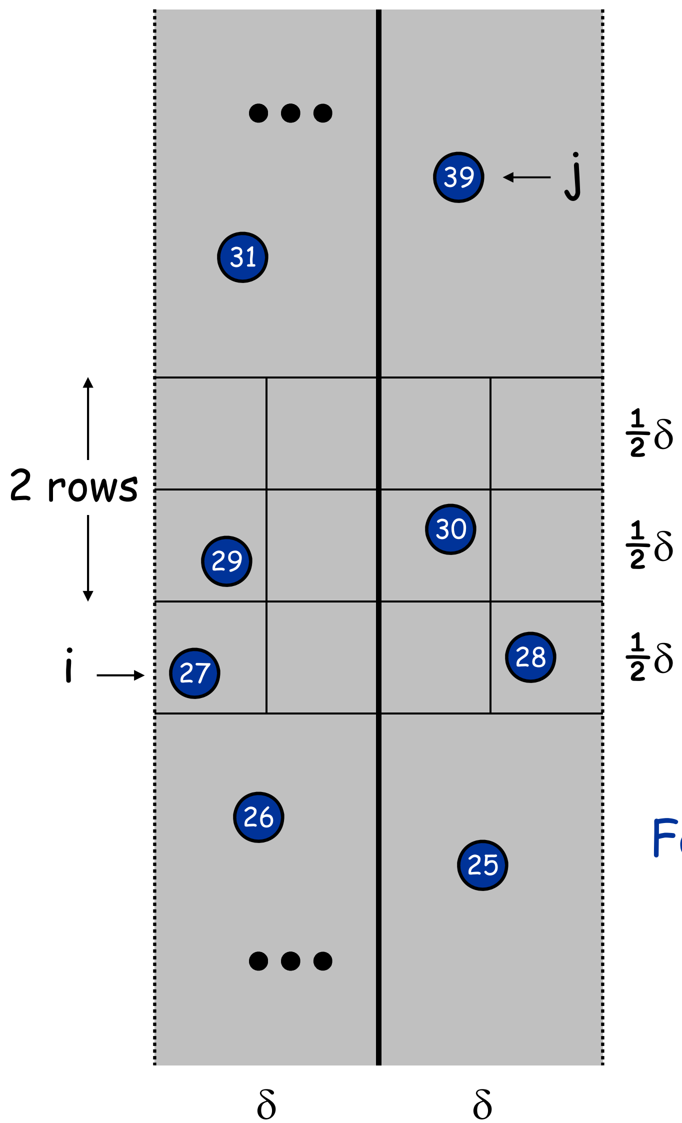
Closest Pair of Points

Find closest pair with one point in each side, assuming that distance < δ.

- Observation: only need to consider points within δ of line L.
- Sort points in 2δ-strip by their y coordinate.
- Only check distances of those within 11 positions in sorted list!



Closest Pair of Points



Def. Let s_i be the point in the 2δ-strip, with the i^{th} smallest y-coordinate.

Claim. If $|i - j| \geq 12$, then the distance between s_i and s_j is at least δ.

Pf.

- No two points lie in same $\frac{1}{2}\delta$-by-$\frac{1}{2}\delta$ box.
- Two points at least 2 rows apart have distance $\geq 2(\frac{1}{2}\delta)$. ▪

Fact. Still true if we replace 12 with 7.

Closest Pair Algorithm

```
Closest-Pair(p1, …, pn) {
   Compute separation line L such that half the points
   are on one side and half on the other side.

   δ1 = Closest-Pair(left half)
   δ2 = Closest-Pair(right half)
   δ  = min(δ1, δ2)

   Delete all points further than δ from separation line L

   Sort remaining points by y-coordinate.

   Scan points in y-order and compare distance between
   each point and next 11 neighbors. If any of these
   distances is less than δ, update δ.

   return δ.
}
```

$O(n \log n)$

$2T(n / 2)$

$O(n)$

$O(n \log n)$

$O(n)$

Closest Pair of Points: Analysis

Running time.

$$T(n) \le 2T(n/2) + O(n \log n) \Rightarrow T(n) = O(n \log^2 n)$$

Q. Can we achieve O(n log n)?

A. Yes. Don't sort points in strip from scratch each time.

- Each recursive returns two lists: all points sorted by y coordinate, and all points sorted by x coordinate.
- Sort by merging two pre-sorted lists.

$$T(n) \le 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$