CS 580: Algorithm Design and Analysis

Jeremiah Blocki
Purdue University
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Recap: Stable Matching Problem

- Definition of Stable Matching Problem
- Gale-Shapley Algorithm
  - Unengaged men propose to the top remaining woman w on their preference list
  - The proposal is (temporarily) accepted if the woman w is currently unengaged or if the proposer m is preferred to current fiancé m'
- Analysis of Gale-Shapley Algorithm
  - Proof of correctness
    - Everyone is matched when algorithm terminates
    - Gale-Shapley Matching is stable
  - Implementation + Running Time Analysis:
    - Runs in at most $O(n^2)$ steps
- Implies Stable Matching Always exists
  - (Contrast with Stable-Roommate Problem)
- Gale-Shapley Matching is Optimal for Men

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a valid partner of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield man-optimal assignment, which is a stable matching!
- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S* is man-optimal.
Pf. (by contradiction)
- Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference ⇒ some man is rejected by valid partner.
- Let Y be first such man, and let A be first valid woman that rejects him.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms) engagement with a man, say Z, whom she prefers to Y.
- Let B be Z's partner in S.
- Z not rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.
  since this is first rejection by a valid partner
- But A prefers Z to Y.
- Thus A-Z is unstable in S. ▪

S
Amy-Yancey
Bertha-Zeus
. . .

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.
no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.
w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching S*.

Pf.
- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a man, say Y, whom she likes less than Z.
- Let B be Z's partner in S.
- Z prefers A to B. ← man-optimality
- Thus, A-Z is an unstable in S. ▪

S
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. . .

Extensions: Matching Residents to Hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

hospital X wants to hire 3 residents

Gale-Shapley Algorithm Still Works. Minor modifications to code to handle variations!

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Def. Matching S unstable if there is a hospital h and resident r such that:

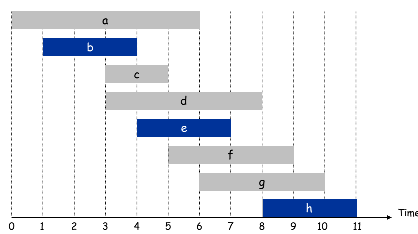
- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to *at least one* of its assigned residents.

1.2 Five Representative Problems

Interval Scheduling

Input. Set of jobs with start times and finish times.
Goal. Find maximum cardinality subset of mutually compatible jobs.

jobs don't overlap



Interval Scheduling

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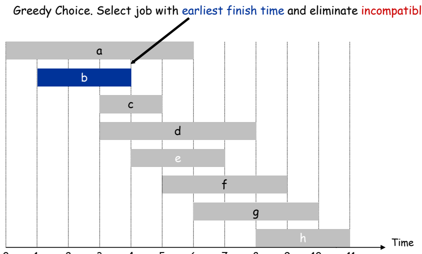
Greedy Choice. Select job with earliest finish time and eliminate incompatible jobs.



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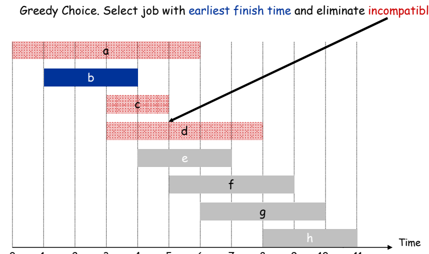

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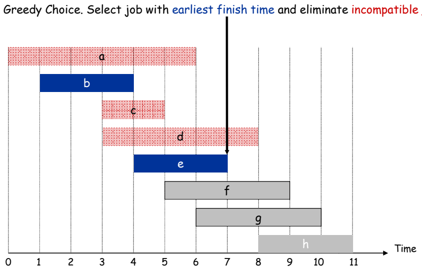

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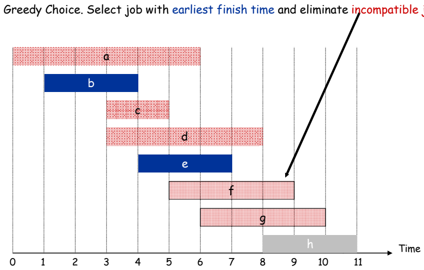

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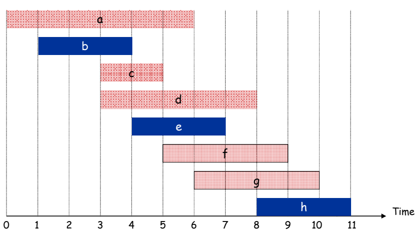
Interval Scheduling

Input. Set of jobs with start times and finish times.
Goal. Find maximum cardinality subset of mutually compatible jobs.

↑ jobs don't overlap

Chapter 4: We will prove that this greedy algorithm always finds the optimal solution!



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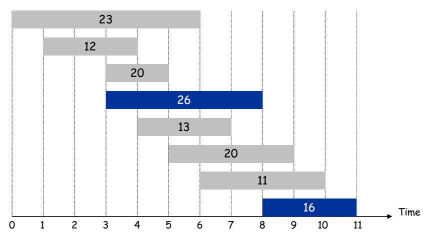
Weighted Interval Scheduling

Input. Set of jobs with start times, finish times, and weights.
Goal. Find maximum weight subset of mutually compatible jobs.

Greedy Algorithm No Longer Works!



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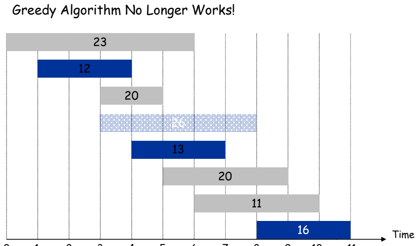

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Weighted Interval Scheduling

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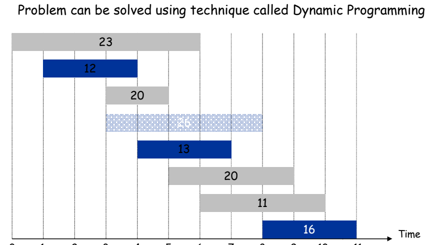
Problem can be solved using technique called Dynamic Programming



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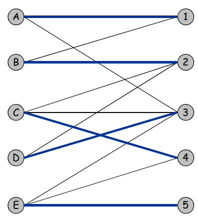
Bipartite Matching

Input. Bipartite graph.
Goal. Find maximum cardinality matching.



Different from Stable Matching Problem! How?

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Bipartite Matching

Input. Bipartite graph.
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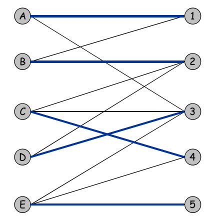

Problem can be solved using Network Flow Algorithms

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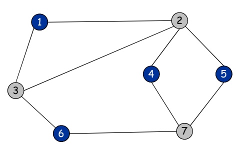
Independent Set

Input. Graph.
Goal. Find maximum cardinality independent set.

subset of nodes such that no two joined by an edge



Brute-Force Algorithm: Check every possible subset.
Running Time: $\geq 2^n$ steps

NP-Complete: Unlikely that efficient algorithm exists!

Positive: Can easily check that there is an independent set of size k

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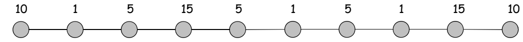
Competitive Facility Location

Input. Graph with weight on each node.
Game. Two competing players alternate in selecting nodes.
Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.



Second player can guarantee 20, but not 25.

22

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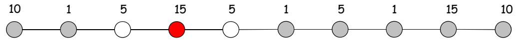

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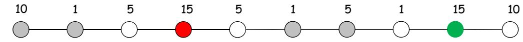

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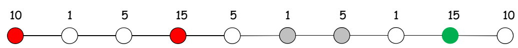

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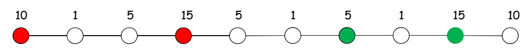

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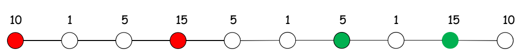
Competitive Facility Location

Input. Graph with weight on each node.
Game. Two competing players alternate in selecting nodes.
Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.

PSPACE-Complete: Even harder than NP-Complete!

No short proof that player can guarantee value B. (Unlike previous problem)



Second player can guarantee 20, but not 25.

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Five Representative Problems

Variations on a theme: independent set.

Interval scheduling: n log n greedy algorithm.
Weighted interval scheduling: n log n dynamic programming algorithm.
Bipartite matching: n^k max-flow based algorithm.
Independent set: NP-complete.
Competitive facility location: PSPACE-complete.

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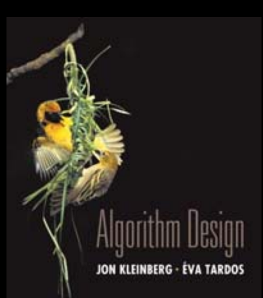


Chapter 2

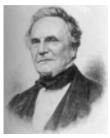
Basics of Algorithm Analysis

2.1 Computational Tractability

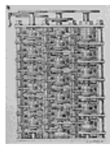
"For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing." - *Francis Sullivan*

Computational Tractability

> As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - *Charles Babbage*



Charles Babbage (1864)



Analytic Engine (schematic)

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Polynomial-Time

Brute force. For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.
- Typically takes 2^N time or worse for inputs of size N.
- Unacceptable in practice.

n! for stable matching with n men and n women

Desirable scaling property. When the input size doubles, the algorithm should only slow down by some constant factor C.

> There exists constants c > 0 and d > 0 such that on every input of size N, its running time is bounded by $c N^d$ steps.

Def. An algorithm is poly-time if the above scaling property holds.

choose $C = 2^d$

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Worst-Case Analysis

Worst case running time. Obtain bound on largest possible running time of algorithm on input of a given size N.
- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

Average case running time. Obtain bound on running time of algorithm on random input as a function of input size N.
- Hard (or impossible) to accurately model real instances by random distributions.
- Algorithm tuned for a certain distribution may perform poorly on other inputs.

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Worst-Case Polynomial-Time

Def. An algorithm is efficient if its running time is polynomial.

Justification: It really works in practice!
- Although $6.02 \times 10^{23} \times N^{20}$ is technically poly-time, it would be useless in practice.
- In practice, the poly-time algorithms that people develop almost always have low constants and low exponents.
- Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

Exceptions.
- Some poly-time algorithms do have high constants and/or exponents, and are useless in practice.
- Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare.

simplex method
Unix grep

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Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1,000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10,000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100,000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1,000,000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

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2.2 Asymptotic Order of Growth

Asymptotic Order of Growth

Upper bounds. T(n) is O(f(n)) if there exist constants $c > 0$ and $n_0 \ge 0$ such that for all $n \ge n_0$ we have $T(n) \le c \cdot f(n)$.

Lower bounds. T(n) is Ω(f(n)) if there exist constants $c > 0$ and $n_0 \ge 0$ such that for all $n \ge n_0$ we have $T(n) \ge c \cdot f(n)$.

Tight bounds. T(n) is Θ(f(n)) if T(n) is both O(f(n)) and Ω(f(n)).

Ex: $T(n) = 32n^2 + 17n + 32$.
- T(n) is $O(n^2)$, $O(n^3)$, $\Omega(n^2)$, $\Omega(n)$, and $\Theta(n^2)$.
- T(n) is not $O(n)$, $\Omega(n^3)$, $\Theta(n)$, or $\Theta(n^3)$.

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Notation

Slight abuse of notation. T(n) = O(f(n)).
- Not transitive:
  - $f(n) = 5n^3$; $g(n) = 3n^2$
  - $f(n) = O(n^3) = g(n)$
  - but $f(n) \ne g(n)$.
- Better notation: $T(n) \in O(f(n))$.

Meaningless statement. Any comparison-based sorting algorithm requires at least O(n log n) comparisons.
- Statement doesn't "type-check."
- Use Ω for lower bounds.

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Properties

Transitivity.
- If $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$.
- If $f \in \Omega(g)$ and $g \in \Omega(h)$ then $f \in \Omega(h)$.
- If $f \in \Theta(g)$ and $g \in \Theta(h)$ then $f \in \Theta(h)$.

Additivity.
- If $f \in O(h)$ and $g \in O(h)$ then $f + g \in O(h)$
- If $f \in \Omega(h)$ and $g \in \Omega(h)$ then $f + g \in \Omega(h)$.
- If $f \in \Theta(h)$ and $g \in \Theta(h)$ then $f + g \in \Theta(h)$.

Proof of A1 (If $f \in O(h)$ and $g \in O(h)$ then $f + g \in O(h)$)
- $f \in O(h)$ means that for some constants c_1, N_1 we have $f(n) \le c_1 \times h(n)$ for all $n \ge N_1$
- $g \in O(h)$ means that for some constants c_2, N_2 we have $g(n) \le c_2 \times h(n)$ for all $n \ge N_2$
- Set $c = c_1 + c_2$ and $N = \max\{N_1, N_2\}$ for all $n \ge N = \max(N_1, N_2)$

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Asymptotic Bounds for Some Common Functions

Polynomials. $a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$.

Polynomial time. Running time is $O(n^d)$ for some constant d independent of the input size n.

Logarithms. $O(\log_a n) = O(\log_b n)$ for any constants $a, b > 0$.

can avoid specifying the base

Logarithms. For every $x > 0$, $\log n = O(n^x)$.

log grows slower than every polynomial
(even if x=0.000000001)

Exponentials. For every $r > 1$ and every $d > 0$, $n^d = O(r^n)$.

every exponential grows faster than every polynomial

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2.4 A Survey of Common Running Times

Linear Time: O(n)

Linear time. Running time is proportional to input size.

Computing the maximum. Compute maximum of n numbers $a_1, \ldots, a_n$.

```
max ← a1
for i = 2 to n {
   if (ai > max)
      max ← ai
}
```

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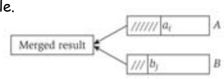
Linear Time: O(n)

Merge. Combine two sorted lists `A = a1,a2,…,an` with `B = b1,b2,…,bn` into sorted whole.




```
i = 1, j = 1
while (both lists are nonempty) {
   if (ai ≤ bj) append ai to output list and increment i
   else         append bj to output list and increment j
}
append remainder of nonempty list to output list
```

Claim. Merging two lists of size n takes O(n) time.
Pf. After each comparison, the length of output list increases by 1.

O(n log n) Time

O(n log n) time. Arises in divide-and-conquer algorithms.

also referred to as linearithmic time

Sorting. Mergesort and heapsort are sorting algorithms that perform O(n log n) comparisons.

Largest empty interval. Given n time-stamps $x_1, \ldots, x_n$ on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

O(n log n) solution. Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.

Quadratic Time: $O(n^2)$

Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

```
min ← (x1 - x2)² + (y1 - y2)²
for i = 1 to n {
   for j = i+1 to n {
      d ← (xi - xj)² + (yi - yj)²
      if (d < min)
         min ← d
   }
}
```

don't need to take square roots

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion.

see chapter 5

Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Given n sets $S_1, \ldots, S_n$ each of which is a subset of 1, 2, …, n, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

```
foreach set Si {
   foreach other set Sj {
      foreach element p of Si {
         determine whether p also belongs to Sj
      }
      if (no element of Si belongs to Sj)
         report that Si and Sj are disjoint
   }
}
```


Polynomial Time: $O(n^k)$ Time

Independent set of size k. Given a graph, are there k nodes such that no two are joined by an edge?

k is a constant

$O(n^k)$ solution. Enumerate all subsets of k nodes.

```
foreach subset S of k nodes {
   check whether S in an independent set
   if (S is an independent set)
      report S is an independent set
   }
}
```

- Check whether S is an independent set = $O(k^2)$.
- Number of k element subsets = $\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k\,(k-1)\,(k-2)\cdots(2)\,(1)} \le \frac{n^k}{k!}$
- $O(k^2 n^k / k!) = O(n^k)$.

poly-time for k=17, but not practical

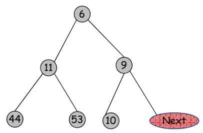
Exponential Time

Independent set. Given a graph, what is maximum size of an independent set?

$O(n^2 2^n)$ solution. Enumerate all subsets.

```
S* ← φ
foreach subset S of nodes {
   check whether S in an independent set
   if (S is largest independent set seen so far)
      update S* ← S
   }
}
```

Review: Heap Data Structure



Min Heap Order: For each node v in the tree
Parent(v).Value ≤ v.Value

Max Heap Order: For each node v in the tree
Parent(v).Value ≥ v.Value

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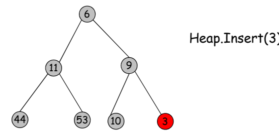
Heap Insertion



Min Heap Order: For each node v in the tree
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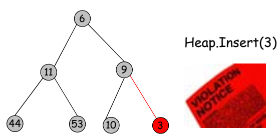
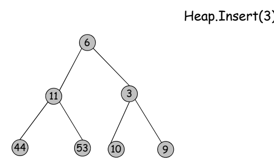
Heap Insertion



Min Heap Order: For each node v in the tree
Parent(v).Value ≤ v.Value

51

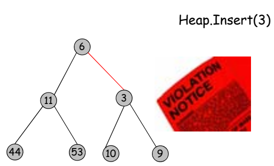
Heap Insertion



Min Heap Order: For each node v in the tree
Parent(v).Value ≤ v.Value

52

Heap Insertion



Min Heap Order: For each node v in the tree
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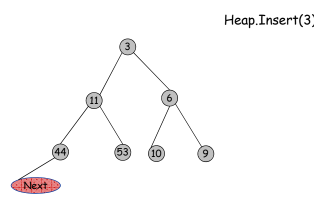
53

Heap Insertion



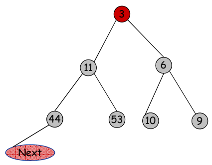
Min Heap Order: For each node v in the tree
Parent(v).Value ≤ v.Value

Theorem 2.12 [KT]: The procedure Heapify-up fixes the heap property and allows us to insert a new element into a heap of n elements in O(log n) time.

54

Heap Extract Minimum

Heap.ExtractMin()



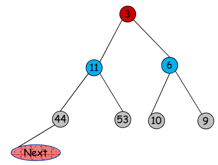
Min Heap Order: For each node v in the tree
Parent(v).Value ≤ v.Value

Theorem 2.13 [KT]: The procedure Heapify-down fixes the heap property and allows us to delete an element in a heap of n elements in O(log n) time.

55

Heap Extract Minimum

Heap.ExtractMin()



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56

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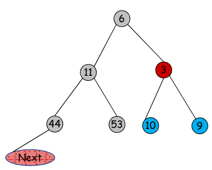

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57

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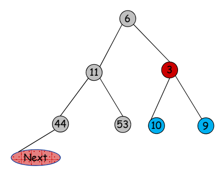

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58

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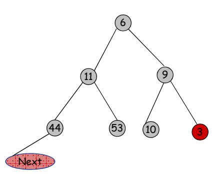
Heap.ExtractMin()



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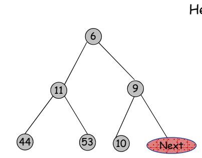
59

Heap Extract Minimum

Heap.ExtractMin()



Min Heap Order: For each node v in the tree
Parent(v).Value ≤ v.Value

Theorem 2.13 [KT]: The procedure Heapify-down fixes the heap property and allows us to delete an element in a heap of n elements in O(log n) time.

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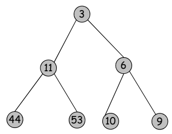
Heap Summary



Insert: O(log n)
FindMin: O(1)
Delete: O(log n) time
ExtractMin: O(log n) time

Thought Question: O(n log n) time sorting algorithm using heaps?

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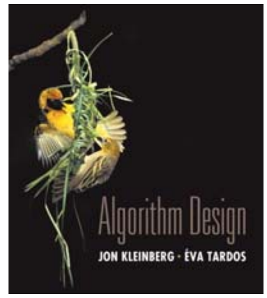

Graphs

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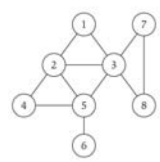
3.1 Basic Definitions and Applications

Undirected Graphs

Undirected graph. G = (V, E)
- V = nodes.
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: n = |V|, m = |E|.



V = { 1, 2, 3, 4, 5, 6, 7, 8 }
E = { 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6 }
n = 8
m = 11

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Some Graph Applications

Graph	Nodes	Edges
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

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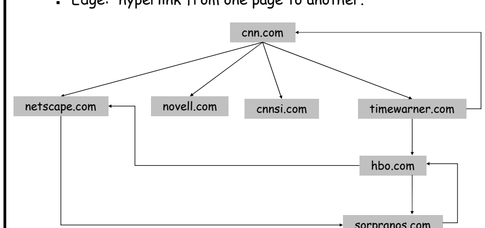
World Wide Web

Web graph.
- Node: web page.
- Edge: hyperlink from one page to another.



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9-11 Terrorist Network

Social network graph.
- Node: people.
- Edge: relationship between two people.

Reference: Valdis Krebs, http://www.firstmonday.org/issues/issue7_4/krebs

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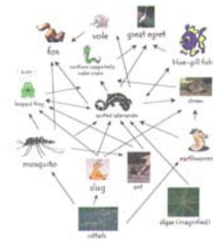
Ecological Food Web

Food web graph.
- Node = species.
- Edge = from prey to predator.



Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff

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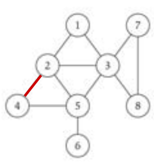
Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.
- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	1	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

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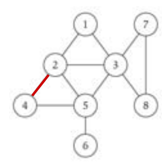
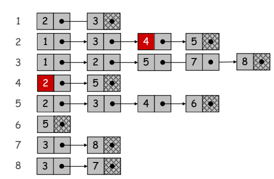
Graph Representation: Adjacency List

Adjacency list. Node indexed array of lists.
- Two representations of each edge.
- Space proportional to m + n.
- Checking if (u, v) is an edge takes O(deg(u)) time.
- Identifying all edges takes $\Theta(m + n)$ time.

degree = number of neighbors of u





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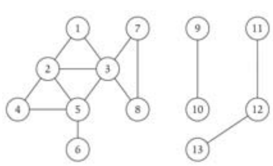
Paths and Connectivity

Def. A path in an undirected graph G = (V, E) is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.

Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



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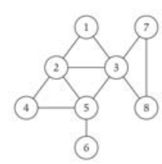
Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first k-1 nodes are all distinct.



cycle C = 1-2-4-5-3-1

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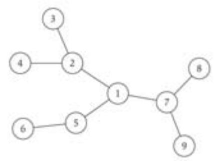
Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.
- G is connected.
- G does not contain a cycle.
- G has n-1 edges.



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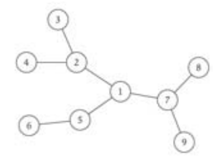
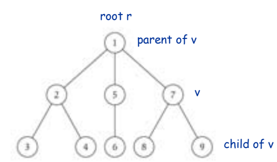
Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.

Importance. Models hierarchical structure.



a tree



the same tree, rooted at 1

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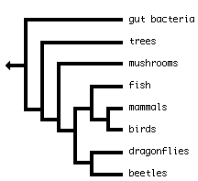
Phylogeny Trees

Phylogeny trees. Describe evolutionary history of species.



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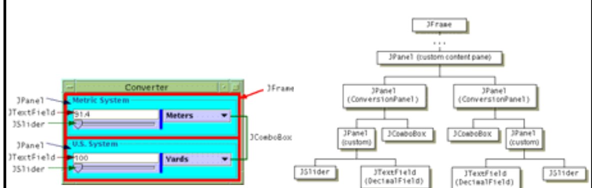
GUI Containment Hierarchy

GUI containment hierarchy. Describe organization of GUI widgets.



Reference: http://java.sun.com/docs/books/tutorial/uiswing/overview/anatomy.html

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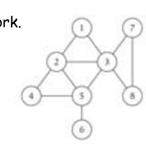
3.2 Graph Traversal

Connectivity

s-t connectivity problem. Given two node s and t, is there a path between s and t?

s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Applications.
- Friendster.
- Maze traversal.
- Kevin Bacon number.
- Fewest number of hops in a communication network.



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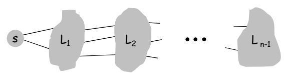
Breadth First Search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



BFS algorithm.

- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

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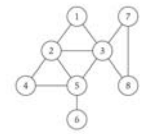
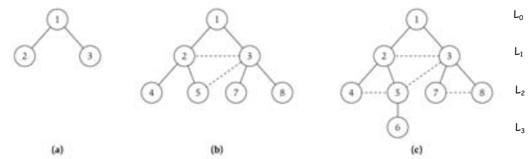
Breadth First Search

Property. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G. Then the level of x and y differ by at most 1.





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Breadth First Search: Analysis

Theorem. The above implementation of BFS runs in O(m + n) time if the graph is given by its adjacency representation.

Pf.

- Easy to prove $O(n^2)$ running time:
  - at most n lists L[i]
  - each node occurs on at most one list; for loop runs $\le$ n times
  - when we consider node u, there are $\le$ n incident edges (u, v), and we spend O(1) processing each edge
- Actually runs in O(m + n) time:
  - when we consider node u, there are deg(u) incident edges (u, v)
  - total time processing edges is $\Sigma_{u \in V} \deg(u) = 2m$ ▪

each edge (u, v) is counted exactly twice in sum: once in deg(u) and once in deg(v)

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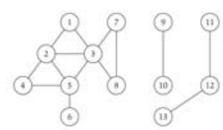
Connected Component

Connected component. Find all nodes reachable from s.



Connected component containing node 1 = { 1, 2, 3, 4, 5, 6, 7, 8 }.

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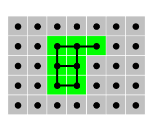
Flood Fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue



83

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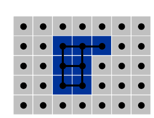

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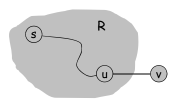
Connected Component

Connected component. Find all nodes reachable from s.

```
R will consist of nodes to which s has a path
Initially R = {s}
While there is an edge (u, v) where u ∈ R and v ∉ R
  Add v to R
Endwhile
```



it's safe to add v

Theorem. Upon termination, R is the connected component containing s.

- BFS = explore in order of distance from s.
- DFS = explore in a different way.

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