CS 580: Algorithm Design and Analysis

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Spring 2019

Announcements: Homework 6 due tonight at 11:59 PM

Practice Final Exam Released on Piazza

Course Evaluation Survey: Live until 4/28/2019 at 11:59PM. Your feedback is valued! (Current Response Rate: 25%)

13.9 Chernoff Bounds

Chernoff Bounds (above mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > \underset{\uparrow}{(1+\delta)}\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

sum of independent 0-1 random variables
is tightly centered on the mean

Pf. We apply a number of simple transformations.

- For any t > 0, $\Pr[X > (1+\delta)\mu] = \Pr\left[e^{tX} \underset{\uparrow}{>} e^{t(1+\delta)\mu}\right]$

  f(x) = e^{tX} is monotone in x

- Let $Y = e^{tX}$ be a random variable

$$\Pr[X > (1+\delta)\mu] = \Pr\left[Y > e^{t(1+\delta)\mu}\right] \underset{\uparrow}{\leq} \frac{\mathbf{E}[Y]}{e^{t(1+\delta)\mu}} = e^{-t(1+\delta)\mu} \times \mathbf{E}[e^{tX}]$$

Markov's inequality: $\Pr[Y > a] \leq \mathbf{E}[Y]/a$

Chernoff Bounds (above mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > (1+\delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

$\uparrow$

sum of independent 0-1 random variables
is tightly centered on the mean

Pf. We apply a number of simple transformations.

- For any t > 0,

$$\Pr[X > (1+\delta)\mu] \leq e^{-t(1+\delta)\mu} \times \mathbf{E}[e^{tX}] = e^{-t(1+\delta)\mu} \prod_{\mathrm{i}} \mathrm{E}[e^{tx_i}]$$

- Now

$$\mathrm{E}[e^{tX}] \underset{\substack{\uparrow \\ \text{definition of X}}}{=} \mathrm{E}\left[e^{t\sum_i x_i}\right] \underset{\substack{\uparrow \\ \text{independence}}}{=} \prod_{\mathrm{i}} \mathrm{E}[e^{tx_i}]$$

Chernoff Bounds (above mean)

Pf. (cont)

- Let $\mathrm{p_i} = \Pr[x_i = 1]$. Then,

$$\mathrm{E}[e^{tx_i}] = \mathrm{p_i} \times e^t + (1 - \mathrm{p_i})e^0 = 1 + \mathrm{p_i}(e^t - 1) \leq e^{\mathrm{p_i}(e^t - 1)}$$

for any $\alpha \geq 0$, $1 + \alpha \leq e^\alpha$

- Combining everything:

previous slide

$$\Pr[X > (1+\delta)\mu] \leq e^{-t(1+\delta)\mu} \prod_{\mathrm{i}} \mathrm{E}[e^{tx_i}]$$

inequality above

$$\leq e^{-t(1+\delta)\mu} \prod_{\mathrm{i}} e^{\mathrm{p_i}(e^t - 1)}$$

$$= e^{-t(1+\delta)\mu} \times e^{(e^t - 1)\sum_i \mathrm{p_i}}$$

$\sum_i \mathrm{p_i} = E[X] \leq \mu$

$$\leq e^{-t(1+\delta)\mu} \times e^{(e^t - 1)\mu}$$

- Finally, choose t = ln(1 + δ).

Chernoff Bounds (above mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > (1 \underset{\uparrow}{+} \delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

sum of independent 0-1 random variables
is tightly centered on the mean

Pf. (cont) We had derived for any $t>0$

$$\Pr[X > (1+\delta)\mu] \leq e^{-t(1+\delta)\mu} \times e^{(e^{t}-1)\mu}$$

Plugging in $t = \ln(1 + \delta)$. We have

$$e^{-t(1+\delta)\mu} \times e^{(e^{t}-1)\mu} = \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

Chernoff Bounds (below mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \leq E[X]$ and for any $0 < \delta < 1$, we have

$$\Pr[X < (1-\delta)\mu] \ < \ e^{-\delta^2 \mu / 2}$$

Pf idea. Similar.

Remark. Not quite symmetric since only makes sense to consider $\delta < 1$.

13.10 Load Balancing

Load Balancing

Load balancing. System in which m jobs arrive in a stream and need to be processed immediately on n identical processors. Find an assignment that balances the workload across processors.

Centralized controller. Assign jobs in round-robin manner. Each processor receives at most $\lceil m/n \rceil$ jobs.

Decentralized controller. Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

Load Balancing

Analysis. (m=n jobs)

- Let X_i = number of jobs assigned to processor i.
- Let Y_{ij} = 1 if job j assigned to processor i, and 0 otherwise.
- We have $E[Y_{ij}] = 1/n$
- Thus, $X_i = \sum_j Y_{ij}$, and $\mu = E[X_i] = 1$.
- Applying Chernoff bounds with $\delta = c - 1$ yields $\Pr[X_i > c] < \frac{e^{c-1}}{c^c}$

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > (1+\delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

Load Balancing

Analysis. (m=n jobs)

- Let X_i = number of jobs assigned to processor i.
- Let Y_{ij} = 1 if job j assigned to processor i, and 0 otherwise.
- We have $E[Y_{ij}] = 1/n$
- Thus, $X_i = \sum_j Y_{ij}$, and $\mu = E[X_i] = 1$.
- Applying Chernoff bounds with $\delta = c - 1$ yields $\Pr[X_i > c] < \frac{e^{c-1}}{c^c}$
- Let $\gamma(n)$ be number x such that $x^x = n$, and choose $c = e\,\gamma(n)$.

$$\Pr[X_i > c] < \frac{e^{c-1}}{c^c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} < \left(\frac{1}{\gamma(n)}\right)^{2\gamma(n)} = \frac{1}{n^2}$$

- Union bound $\Rightarrow$ with probability $\geq 1 - 1/n$ no processor receives more than $e\,\gamma(n) = \Theta(\log n / \log\log n)$ jobs.

Fact: this bound is asymptotically tight: with high probability, some processor receives $\Theta(\log n / \log\log n)$

Load Balancing: Many Jobs

Theorem. Suppose the number of jobs m = 16n ln n. Then on average, each of the n processors handles μ = 16 ln n jobs. With high probability every processor will have between half and twice the average load.

Pf.

- Let X_i , Y_{ij} be as before.
- Applying Chernoff bounds with δ = 1 yields
- $\Pr[X_i > 2\mu] < \left(\frac{e}{4}\right)^{16\ \ln n} \ll \left(\frac{1}{e}\right)^{2\ \ln n} = \frac{1}{n^2}$
- $\Pr\left[X_i < \frac{\mu}{2}\right] < e^{\left(\frac{1}{2}\right)^2 \frac{16 \ln n}{2}} = \left(\frac{1}{e}\right)^{2\ \ln n} = \frac{1}{n^2}$
- Union bound $\Rightarrow$ every processor has load between half and twice the average with probability $\geq$ 1 - 2/n. ▪

Load Balancing with Asymmetry

Centralized controller. Assign jobs in round-robin manner. Each processor receives at most $\lceil m/n \rceil$ jobs.

- Suppose each job has cost between 1 and 4
- Round-Robin assignment may be highly unbalanced
- E.g., (n=4 processors): 1,2,3,4,1,2,3,4,...
  - Processor 1 total cost: m/4
  - Processor 2 total cost: m/2
  - Processor 3 total cost: 3m/4
  - Processor 4 total cost: m
  - Fair: 2.5m/4 per processor

Decentralized controller. Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

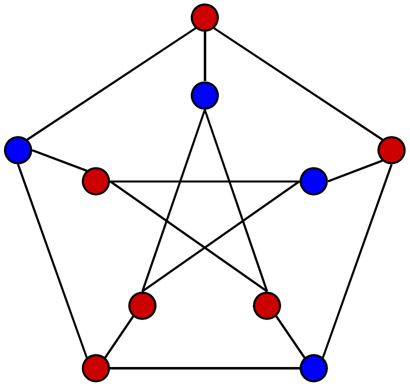
- Still works well in the above scenario (bounded costs)
- Workload on each processor would be $\approx 2.5m/4$ (whp) in above example

12.4 Maximum Cut

Maximum Cut

Maximum cut. Given an undirected graph G = (V, E) with positive integer edge weights w_e, find a node partition (A, B) such that the total weight of edges crossing the cut is maximized.

$$w(A, B) := \sum_{u \in A, v \in B} w_{uv}$$



Toy application.

- n activities, m people.
- Each person wants to participate in two of the activities.
- Schedule each activity in the morning or afternoon to maximize number of people that can enjoy both activities.
  - Nodes: Activities
  - Edge Weights: $w_{uv} =$ #people who want to participate in both activities

Real applications. Circuit layout, statistical physics.

Maximum Cut

Single-flip neighborhood. Given a partition (A, B), move one node from A to B, or one from B to A if it improves the solution.

Greedy algorithm.

```
Max-Cut-Local (G, w) {
   Pick a random node partition (A, B)

   while (∃ improving node v) {
      if (v is in A) move v to B
      else           move v to A
   }

   return (A, B)
}
```

Maximum Cut: Local Search Analysis

Theorem. Let (A, B) be a locally optimal partition and let (A*, B*) be optimal partition. Then $w(A, B) \geq \frac{1}{2} \Sigma_e\, w_e \underset{\substack{\uparrow \\ \text{weights are nonnegative}}}{\geq} \frac{1}{2}\, w(A^*, B^*)$.

Pf.

- Local optimality implies that for all u ∈ A : $\sum_{v \in A} w_{uv} \leq \sum_{v \in B} w_{uv}$
  Adding up all these inequalities yields:

$$2 \sum_{\{u,v\} \subseteq A} w_{uv} \leq \sum_{u \in A, v \in B} w_{uv} = w(A, B)$$

- Similarly $2 \sum_{\{u,v\} \subseteq B} w_{uv} \leq \sum_{u \in A, v \in B} w_{uv} = w(A, B)$

- Now,

$$\sum_{e \in E} \overset{\substack{\text{each edge counted once} \\ \downarrow}}{w_e} = \underbrace{\sum_{\{u,v\} \subseteq A} w_{uv}}_{\leq \frac{1}{2} w(A,B)} + \underbrace{\sum_{u \in A, v \in B} w_{uv}}_{w(A,B)} + \underbrace{\sum_{\{u,v\} \subseteq B} w_{uv}}_{\leq \frac{1}{2} w(A,B)} \leq 2w(A, B)$$

▪

Maximum Cut: Big Improvement Flips

Local search. Within a factor of 2 for MAX-CUT, but not poly-time!

Big-improvement-flip algorithm. Only choose a node which, when flipped, increases the cut value by at least $\frac{2\varepsilon}{n} w(A, B)$

Claim. Upon termination, big-improvement-flip algorithm returns a cut (A, B) with $(2 + \varepsilon)\, w(A, B) \geq w(A^*, B^*)$.

Pf idea. Add $\frac{2\varepsilon}{n} w(A, B)$ to each inequality in original proof.

Claim. Big-improvement-flip algorithm terminates after $O(\varepsilon^{-1} n \log W)$ flips, where $W = \Sigma_e w_e$.

- Each flip improves cut value by at least a factor of $(1 + \varepsilon/n)$.
- After n/ε iterations the cut value improves by a factor of 2.
- Cut value can be doubled at most log W times.

↑

if $x \geq 1$, $(1 + 1/x)^x \geq 2$

Maximum Cut: Context

Theorem. [Sahni-Gonzales 1976] There exists a $\frac{1}{2}$-approximation algorithm for MAX-CUT.

- In fact a random cut will cut $\frac{1}{2}$ of all edges in expectation!

Theorem. [Goemans-Williamson 1995] There exists an 0.878567-approximation algorithm for MAX-CUT.

↑

$$\min_{0 \le \theta \le \pi} \frac{2}{\pi} \frac{\theta}{1 - \cos\theta}$$

Theorem. [Håstad 1997] Unless P = NP, no 16/17 approximation algorithm for MAX-CUT.

↑

0.941176

12.5 Neighbor Relations

Neighbor Relations for Max Cut

1-flip neighborhood. (A, B) and (A', B') differ in exactly one node.

k-flip neighborhood. (A, B) and (A', B') differ in at most k nodes.

- $\Theta(n^k)$ neighbors.

KL-neighborhood. [Kernighan-Lin 1970]

cut value of (A_1, B_1) may be worse than (A, B)

- To form neighborhood of (A, B):
  - Iteration 1: flip node from (A, B) that results in best cut value (A_1, B_1), and mark that node.
  - Iteration i: flip node from (A_{i-1}, B_{i-1}) that results in best cut value (A_i, B_i) among all nodes not yet marked.
- Neighborhood of (A, B) = $(A_1, B_1), \ldots, (A_{n-1}, B_{n-1})$.
- Neighborhood includes some very long sequences of flips, but without the computational overhead of a k-flip neighborhood.
- Practice: powerful and useful framework.
- Theory: explain and understand its success in practice.

12.3 Hopfield Neural Networks

Hopfield Neural Networks

Hopfield networks. Simple model of an associative memory, in which a large collection of units are connected by an underlying network, and neighboring units try to correlate their states.

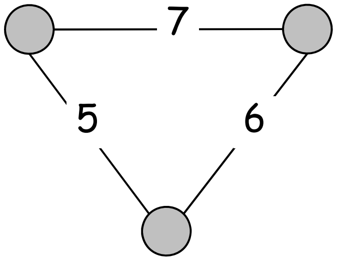
Input: Graph G = (V, E) with integer edge weights w.

positive or negative

Configuration. Node assignment $s_u = \pm 1$.

Intuition. If $w_{uv} < 0$, then u and v want to have the same state; if $w_{uv} > 0$ then u and v want different states.

Note. In general, no configuration respects all constraints.



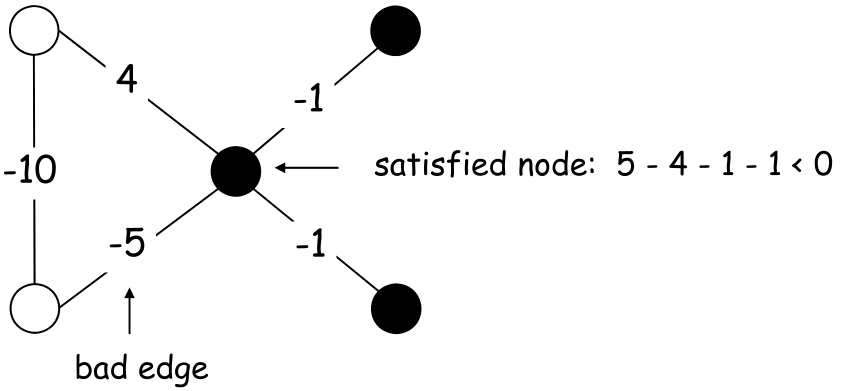
Hopfield Neural Networks

Def. With respect to a configuration S, edge e = (u, v) is good if $w_e s_u s_v < 0$. That is, if $w_e < 0$ then $s_u = s_v$; if $w_e > 0$, $s_u \neq s_v$.

Def. With respect to a configuration S, a node u is satisfied if the weight of incident good edges ≥ weight of incident bad edges.

$$\sum_{v:\ e=(u,v)\in E} w_e\, s_u\, s_v \ \leq\ 0$$

Def. A configuration is stable if all nodes are satisfied.



Goal. Find a stable configuration, if such a configuration exists.

Hopfield Neural Networks

Goal. Find a stable configuration, if such a configuration exists.

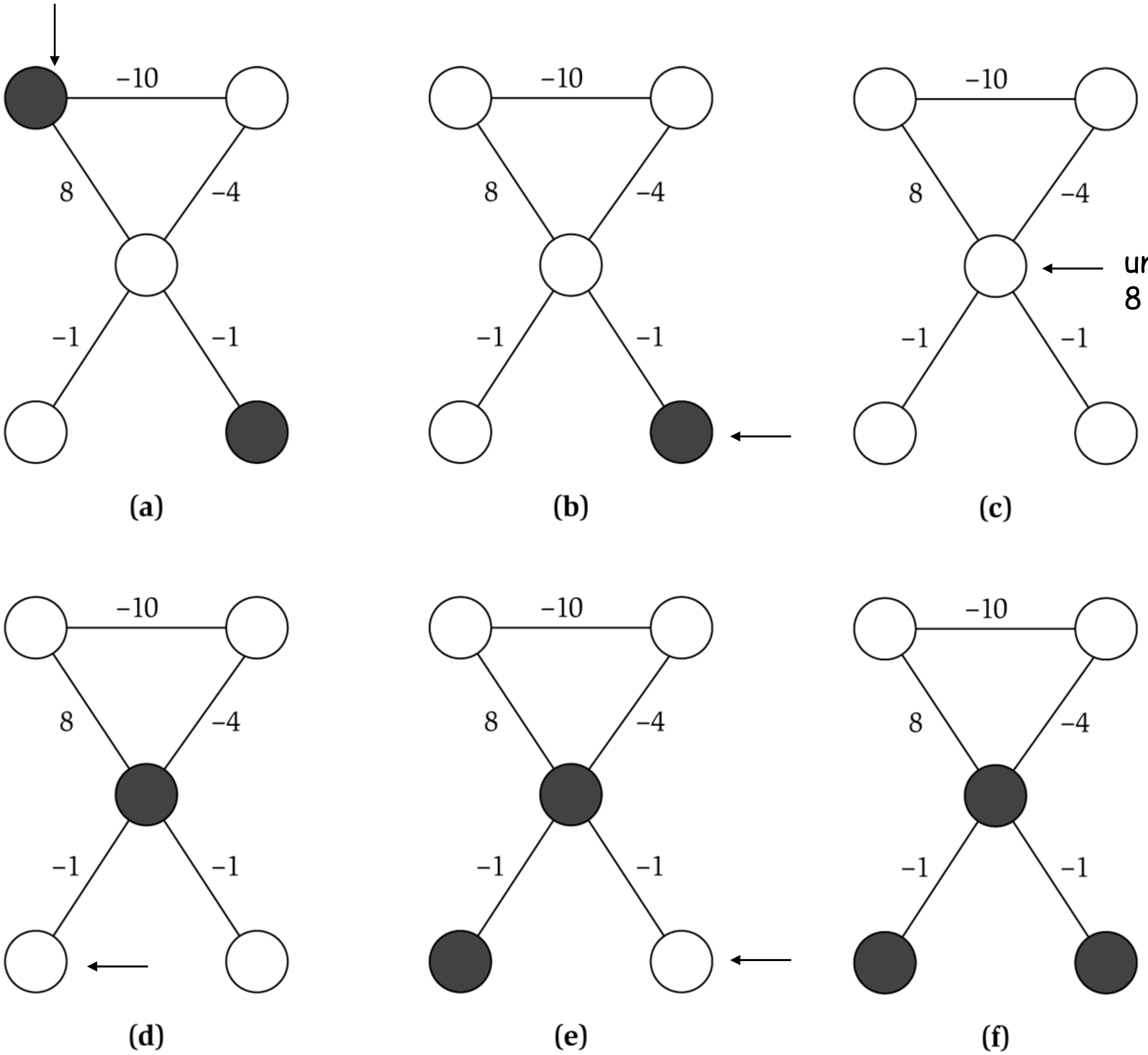
State-flipping algorithm. Repeated flip state of an unsatisfied node.

```
Hopfield-Flip(G, w) {
   S ← arbitrary configuration

   while (current configuration is not stable) {
      u ← unsatisfied node
      $s_u = -s_u$
   }

   return S
}
```

State Flipping Algorithm



unsatisfied node
10 - 8 > 0

unsatisfied node
8 - 4 - 1 - 1 > 0

stable

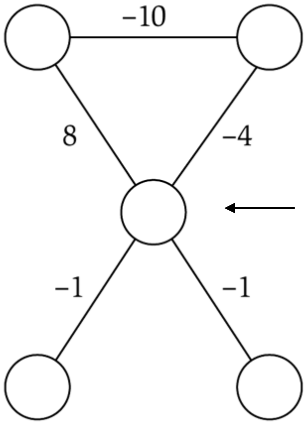
(a) (b) (c)

(d) (e) (f)

Hopfield Neural Networks

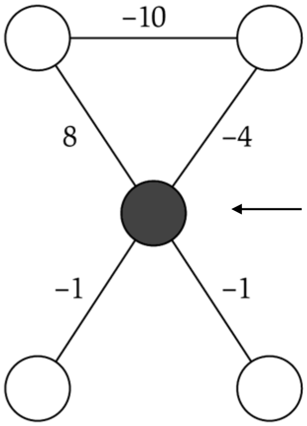
Claim. State-flipping algorithm terminates with a stable configuration after at most $W = \Sigma_e |w_e|$ iterations.

Pf attempt. Consider measure of progress $\Phi(S)$ = # satisfied nodes.



One unsatisfied node
8 - 4 - 1 - 1 > 0

(c)



Two unsatisfied node
8 - 4 - 1 - 1 > 0

(d)

Conclusion: Some local flips actually *decrease* # satisfied nodes.

Hopfield Neural Networks

Claim. State-flipping algorithm terminates with a stable configuration after at most $W = \Sigma_e|w_e|$ iterations.

Pf. Consider measure of progress $\Phi(S) = \Sigma_{e \text{ good}} |w_e|$.

- Clearly $0 \leq \Phi(S) \leq W$.
- We show $\Phi(S)$ increases by at least 1 after each flip.
  When u flips state:
  - all good edges incident to u become bad
  - all bad edges incident to u become good
  - all other edges remain the same

$$\Phi(S') \;=\; \Phi(S) \;-\; \sum_{\substack{e:\ e=(u,v)\in E \\ e \text{ is bad}}} |w_e| \;+\; \sum_{\substack{e:\ e=(u,v)\in E \\ e \text{ is good}}} |w_e| \;\; \underset{\substack{\uparrow \\ \text{u is unsatisfied}}}{\geq} \;\; \Phi(S) + 1$$

Complexity of Hopfield Neural Network

Hopfield network search problem. Given a weighted graph, find a stable configuration if one exists.

Hopfield network decision problem. Given a weighted graph, does there exist a stable configuration?

Remark. The decision problem is trivially solvable (always yes), but there is no known poly-time algorithm for the search problem.

↑

polynomial in n and log W

13.6 Universal Hashing

Dictionary Data Type

Dictionary. Given a universe U of possible elements, maintain a subset S $\subseteq$ U so that inserting, deleting, and searching in S is efficient.

Dictionary interface.

- **`Create()`**: Initialize a dictionary with S = ϕ.
- **`Insert(u)`**: Add element u $\in$ U to S.
- **`Delete(u)`**: Delete u from S, if u is currently in S.
- **`Lookup(u)`**: Determine whether u is in S.

Challenge. Universe U can be extremely large so defining an array of size |U| is infeasible.

Applications. File systems, databases, Google, compilers, checksums P2P networks, associative arrays, cryptography, web caching, etc.

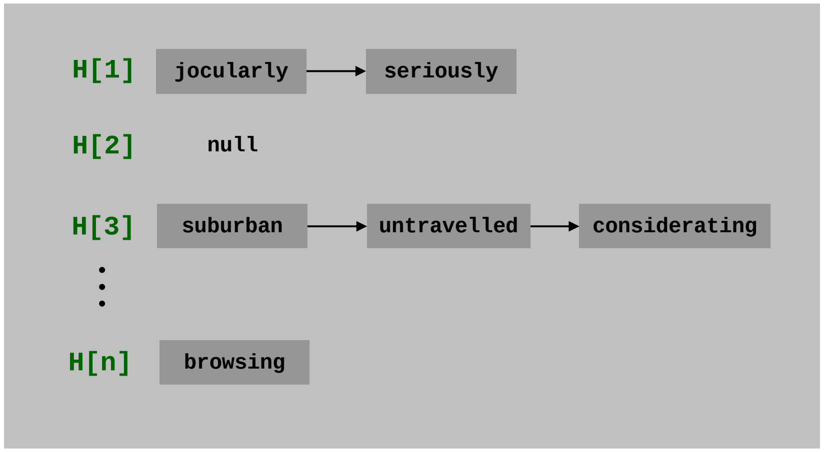
Hashing

Hash function. $h : U \to \{ 0, 1, \ldots, n-1 \}$.

Hashing. Create an array H of size n. When processing element u, access array element H[h(u)].

Collision. When h(u) = h(v) but u ≠ v.

- A collision is expected after $\Theta(\sqrt{n})$ random insertions. This phenomenon is known as the "birthday paradox."
- Separate chaining: H[i] stores linked list of elements u with h(u) = i.



H[1] jocularly → seriously

H[2] null

H[3] suburban → untravelled → considerating

⋮

H[n] browsing

Ad Hoc Hash Function

Ad hoc hash function.

```
int h(String s, int n) {
   int hash = 0;
   for (int i = 0; i < s.length(); i++)
      hash = (31 * hash) + s[i];
   return hash % n;
}
```

hash function ala Java string library

Deterministic hashing. If $|U| \geq n^2$, then for any fixed hash function h, there is a subset $S \subseteq U$ of n elements that all hash to same slot. Thus, $\Theta(n)$ time per search in worst-case.

Q. But isn't ad hoc hash function good enough in practice?

Algorithmic Complexity Attacks

When can't we live with ad hoc hash function?

- Obvious situations: aircraft control, nuclear reactors.
- Surprising situations: denial-of-service attacks.

malicious adversary learns your ad hoc hash function (e.g., by reading Java API) and causes a big pile-up in a single slot that grinds performance to a halt

Real world exploits. [Crosby-Wallach 2003]

- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

Hashing Performance

Idealistic hash function. Maps m elements uniformly at random to n hash slots.

- Running time depends on length of chains.
- Average length of chain = α = m / n.
- Choose n $\approx$ m $\Rightarrow$ on average O(1) per insert, lookup, or delete.

Challenge. Achieve idealized randomized guarantees, but with a hash function where you can easily find items where you put them.

Approach. Use randomization in the choice of h.

↑

adversary knows the randomized algorithm you're using,
but doesn't know random choices that the algorithm makes

Universal Hashing

Universal class of hash functions. [Carter-Wegman 1980s]

- For any pair of elements u, v ∈ U, $\Pr_{h \in H}\left[\, h(u) = h(v) \,\right] \le 1/n$
- Can select random h efficiently.
- Can compute h(u) efficiently.

chosen uniformly at random

Ex. U = { a, b, c, d, e, f }, n = 2.

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1

$H = \{h_1, h_2\}$

$\Pr_{h \in H}[h(a) = h(b)] = 1/2$

$\Pr_{h \in H}[h(a) = h(c)] = 1$

$\Pr_{h \in H}[h(a) = h(d)] = 0$

. . .

not universal

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1
$h_3(x)$	0	0	1	0	1	1
$h_4(x)$	1	0	0	1	1	0

$H = \{h_1, h_2, h_3, h_4\}$

$\Pr_{h \in H}[h(a) = h(b)] = 1/2$

$\Pr_{h \in H}[h(a) = h(c)] = 1/2$

$\Pr_{h \in H}[h(a) = h(d)] = 1/2$

$\Pr_{h \in H}[h(a) = h(e)] = 1/2$

$\Pr_{h \in H}[h(a) = h(f)] = 0$

. . .

universal

Universal Hashing

Universal hashing property. Let H be a universal class of hash functions; let h $\in$ H be chosen uniformly at random from H; and let u $\in$ U. For any subset S $\subseteq$ U of size at most n, the expected number of items in S that collide with u is at most 1.

Pf. For any element s $\in$ S, define indicator random variable X_s = 1 if h(s) = h(u) and 0 otherwise. Let X be a random variable counting the total number of collisions with u.

$$E_{h\in H}[X] \;=\; E[\textstyle\sum_{s\in S} X_s] \;\underset{\substack{\uparrow \\ \text{linearity of expectation}}}{=}\; \textstyle\sum_{s\in S} E[X_s] \;\underset{\substack{\uparrow \\ X_s \text{ is a 0-1 random variable}}}{=}\; \textstyle\sum_{s\in S} \Pr[X_s = 1] \;\underset{\substack{\uparrow \\ \text{universal} \\ \text{(assumes } u \notin S\text{)}}}{\leq}\; \textstyle\sum_{s\in S} \frac{1}{n} \;=\; |S|\,\frac{1}{n} \;\leq\; 1$$

Designing a Universal Family of Hash Functions

Theorem. [Chebyshev 1850] There exists a prime between n and 2n.

Modulus. Choose a prime number $p \approx n$. ← no need for randomness here

Integer encoding. Identify each element $u \in U$ with a base-p integer of r digits: $x = (x_1, x_2, \ldots, x_r)$.

Hash function. Let A = set of all r-digit, base-p integers. For each $a = (a_1, a_2, \ldots, a_r)$ where $0 \leq a_i < p$, define

$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i \right) \bmod p$$

Hash function family. $H = \{ h_a : a \in A \}$.

Designing a Universal Class of Hash Functions

Theorem. H = { h_a : a $\in$ A } is a universal class of hash functions.

Pf. Let x = (x_1, x_2, …, x_r) and y = (y_1, y_2, …, y_r) be two distinct elements of U. We need to show that Pr[$h_a(x) = h_a(y)$] $\leq$ 1/n.

- Since x $\neq$ y, there exists an integer j such that $x_j \neq y_j$.
- We have $h_a(x) = h_a(y)$ iff

$$a_j \underbrace{(y_j - x_j)}_{z} \;=\; \underbrace{\sum_{i \neq j} a_i(x_i - y_i)}_{m} \mod p$$

- Can assume a was chosen uniformly at random by first selecting all coordinates a_i where i $\neq$ j, then selecting a_j at random. Thus, we can assume a_i is fixed for all coordinates i $\neq$ j.
- Since p is prime, a_j z = m mod p has at most one solution among p possibilities. ← see lemma on next slide
- Thus Pr[$h_a(x) = h_a(y)$] = 1/p $\leq$ 1/n. ▪

Number Theory Facts

Fact. Let p be prime, and let $z \neq 0 \bmod p$. Then $\alpha z = m \bmod p$ has at most one solution $0 \leq \alpha < p$.

Pf.

- Suppose α and β are two different solutions.
- Then $(\alpha - \beta)z = 0 \bmod p$; hence $(\alpha - \beta)z$ is divisible by p.
- Since $z \neq 0 \bmod p$, we know that z is not divisible by p; it follows that $(\alpha - \beta)$ is divisible by p.
- This implies $\alpha = \beta$. ▪

Bonus fact. Can replace "at most one" with "exactly one" in above fact.

Pf idea. Euclid's algorithm.

Extra Slides