Homework 5. Due Tonight at 11:59 PM (on Gradescope)

Midterm 2. April 3 @ 8PM (EE 170)
Practice Midterm Released
3x5 Index Card (Double Sided)
Schedule Change: Class canceled on April 4 (We will have class on April 25th)
Midterm 2

- **When?**
  - April 3\textsuperscript{rd} from 8PM to 10PM (2 hours)
- **Where?**
  - EE 170
- **What can I bring?**
  - 3x5 inch index card with your notes (double sided)
  - No electronics (phones, computers, calculators etc...)
- **What material should I study?**
  - The midterm will cover recent topics more heavily
    - Network Flow
      - Max-Flow Min-Cut, Augmenting Paths, etc...
      - Ford Fulkerson, Dinic's Algorithm etc...
      - Applications of Network Flow (e.g., Maximum Bipartite Matching)
    - Linear Programming
    - NP-Completeness
      - Polynomial time reductions, P, NP, NP-Hard, NP-Complete, coNP
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      - Polynomial time reductions, P, NP, NP-Hard, NP-Complete, coNP
    - **PSPACE** (only basic questions)
Extending the Limits of Tractability
Coping With NP-Completeness

Q. Suppose I need to solve an NP-complete problem. What should I do?
A. Theory says you're unlikely to find poly-time algorithm.

Must sacrifice one of three desired features.
- Solve problem to optimality.
- Solve problem in polynomial time.
- Solve arbitrary instances of the problem.

This lecture. Solve some special cases of NP-complete problems that arise in practice.
10.1 Finding Small Vertex Covers
**Vertex Cover**

**VERTEX COVER:** Given a graph $G = (V, E)$ and an integer $k$, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge $(u, v)$ either $u \in S$, or $v \in S$, or both.

- $k = 4$
- $S = \{3, 6, 7, 10\}$
Finding Small Vertex Covers

Q. What if $k$ is small?

Brute force. $O(k \cdot n^{k+1})$.
- Try all $C(n, k) = O(n^k)$ subsets of size $k$.
- Takes $O(k \cdot n)$ time to check whether a subset is a vertex cover.

Goal. Limit exponential dependency on $k$, e.g., to $O(2^k \cdot k \cdot n)$.

Ex. $n = 1,000$, $k = 10$.
Brute. $k \cdot n^{k+1} = 10^{34} \Rightarrow$ infeasible.
Better. $2^k \cdot k \cdot n = 10^7 \Rightarrow$ feasible.

Remark. If $k$ is a constant, algorithm is poly-time; if $k$ is a small constant, then it’s also practical.
Claim. Let $u-v$ be an edge of $G$. $G$ has a vertex cover of size $\leq k$ iff at least one of $G - \{u\}$ and $G - \{v\}$ has a vertex cover of size $\leq k-1$.

Pf. $\Rightarrow$
- Suppose $G$ has a vertex cover $S$ of size $\leq k$.
- $S$ contains either $u$ or $v$ (or both). Assume it contains $u$.
- $S - \{u\}$ is a vertex cover of $G - \{u\}$.

Pf. $\Leftarrow$
- Suppose $S$ is a vertex cover of $G - \{u\}$ of size $\leq k-1$.
- Then $S \cup \{u\}$ is a vertex cover of $G$. \[ \square \]

Claim. If $G$ has a vertex cover of size $k$, it has $\leq k(n-1)$ edges.

Pf. Each vertex covers at most $n-1$ edges. \[ \square \]
Claim. The following algorithm determines if $G$ has a vertex cover of size $\leq k$ in $O(2^k kn)$ time.

```java
boolean Vertex-Cover(G, k) {
    if (G contains no edges) return true
    if (G contains $\geq kn$ edges) return false

    let (u, v) be any edge of G
    a = Vertex-Cover(G - {u}, k-1)
    b = Vertex-Cover(G - {v}, k-1)
    return a or b
}
```

Pf.
- Correctness follows from previous two claims.
- There are $\leq 2^{k+1}$ nodes in the recursion tree; each invocation takes $O(kn)$ time. •
Finding Small Vertex Covers: Recursion Tree

\[
T(n, k) \leq \begin{cases} 
  c & \text{if } k = 0 \\
  cn & \text{if } k = 1 \\
  2T(n, k-1) + ckn & \text{if } k > 1
\end{cases} \Rightarrow T(n, k) \leq 2^k c kn
\]

Inductive Step:

\[
T(n, k) \leq 2 \times T(n, k - 1) + c kn \\
\leq 2 \times 2^{k-1} c (k - 1)n + c kn \\
= 2^k c kn - 2^k c n + c kn \\
\leq 2^k c kn
\]
Vertex Cover in Bipartite Graphs
Vertex Cover

**Vertex cover.** Given an undirected graph \( G = (V, E) \), a vertex cover is a subset of vertices \( S \subseteq V \) such that for each edge \( (u, v) \in E \), either \( u \in S \) or \( v \in S \) or both.

\[
S = \{ 3, 4, 5, 1', 2' \} \\
|S| = 5
\]
**Vertex Cover**

**Weak duality.** Let $M$ be a matching, and let $S$ be a vertex cover. Then, $|M| \leq |S|$.

**Pf.** Each vertex can cover at most one edge in any matching.

![Graph showing vertex cover and matching](image)

$M = 1-2', 3-1', 4-5'$

$|M| = 3$
König-Egerváry Theorem. In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

\[ S^* = \{ 3, 1', 2', 5' \} \]
\[ |S^*| = 4 \]

\[ M^* = 1-1', 2-2', 3-3', 5-5' \]
\[ |M^*| = 4 \]
**König-Egerváry Theorem.** In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching \( M \) and cover \( S \) such that \( |M| = |S| \).
- Formulate max flow problem as for bipartite matching.
- Let \( M \) be max cardinality matching and let \((A, B)\) be min cut.
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Impossible! (Min-Cut is finite)
**König-Egerváry Theorem.** In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

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$R_A \cup L_B$

vertex cover of size $|M|=4$
Vertex Cover: Proof of König-Egerváry Theorem

König-Egerváry Theorem. In a bipartite graph, the max cardinality of a matching is equal to the min cardinality of a vertex cover.

- Suffices to find matching \( M \) and cover \( S \) such that \( |M| = |S| \).
- Formulate max flow problem as for bipartite matching.
- Let \( M \) be max cardinality matching and let \( (A, B) \) be min cut.
- Define \( L_A = L \cap A, L_B = L \cap B, R_A = R \cap A, R_B = R \cap B \).

\begin{itemize}
  \item Claim 1. \( S = L_B \cup R_A \) is a vertex cover.
    \begin{itemize}
      \item consider \( (u, v) \in E \)
      \item \( u \in L_A, v \in R_B \) impossible since infinite capacity
      \item thus, either \( u \in L_B \) or \( v \in R_A \) or both
    \end{itemize}
  \item Claim 2. \( |S| = |M| \).
    \begin{itemize}
      \item max-flow min-cut theorem \( \Rightarrow \) \( |M| = \text{cap}(A, B) \)
      \item only edges of form \( (s, u) \) or \( (v, t) \) contribute to \( \text{cap}(A, B) \)
      \item If \( u \in L_A \subset A \) then \( (s,u) \) contributes 0 to \( \text{cap}(A,B) \)
      \item If \( v \in R_B \subset B \) then \( (v,t) \) contributes 0 to \( \text{cap}(A,B) \)
      \item \( |M| = \text{cap}(A, B) = |L_B| + |R_A| = |S| \).
    \end{itemize}
\end{itemize}
10.2 Solving NP-Hard Problems on Trees
Independent set on trees. Given a tree, find a maximum cardinality subset of nodes such that no two share an edge.

Fact. A tree on at least two nodes has at least two leaf nodes.

Key observation. If $v$ is a leaf, there exists a maximum size independent set containing $v$.

Pf. (exchange argument)
- Consider a max cardinality independent set $S$.
- If $v \in S$, we're done.
- If $u \notin S$ and $v \notin S$, then $S \cup \{v\}$ is independent $\Rightarrow S$ not maximum.
- If $u \in S$ and $v \notin S$, then $S \cup \{v\} - \{u\}$ is independent.
Independent Set on Trees: Greedy Algorithm

**Theorem.** The following greedy algorithm finds a maximum cardinality independent set in forests (and hence trees).

\[
\text{Independent-Set-In-A-Forest}(F) \{ \\
\hspace{1em} S \leftarrow \emptyset \\
\hspace{1em} \text{while} \ (F \text{ has at least one edge}) \{ \\
\hspace{2em} \text{Let } e = (u,v) \text{ be an edge such that } v \text{ is a leaf} \\
\hspace{2em} \text{Add } v \text{ to } S \\
\hspace{2em} \text{Delete from } F \text{ nodes } u \text{ and } v, \text{ and all edges incident to them.} \\
\} \\
\hspace{1em} \text{return } S \cup V(F) \quad //S \text{ and all remaining nodes in } F
\}
\]

**Pf.** Correctness follows from the previous key observation. •

**Remark.** Can implement in \(O(n)\) time by considering nodes in postorder.
Weighted Independent Set on Trees

**Weighted independent set on trees.** Given a tree and node weights \( w_v > 0 \), find an independent set \( S \) that maximizes \( \sum_{v \in S} w_v \).

**Observation.** If \( (u, v) \) is an edge such that \( v \) is a leaf node, then either OPT includes \( u \), or it includes all leaf nodes incident to \( u \).

**Dynamic programming solution.** Root tree at some node, say \( r \).
- \( \text{OPT}_{\text{in}}(u) = \text{max weight independent set of subtree rooted at } u, \text{ containing } u \).
- \( \text{OPT}_{\text{out}}(u) = \text{max weight independent set of subtree rooted at } u, \text{ not containing } u \).

\[
\begin{align*}
\text{OPT}_{\text{in}}(u) &= w_u + \sum_{v \in \text{children}(u)} \text{OPT}_{\text{out}}(v) \\
\text{OPT}_{\text{out}}(u) &= \sum_{v \in \text{children}(u)} \max \{ \text{OPT}_{\text{in}}(v), \text{OPT}_{\text{out}}(v) \}
\end{align*}
\]

children(u) = \{ v, w, x \}
Weighted Independent Set on Trees: Dynamic Programming Algorithm

**Theorem.** The dynamic programming algorithm finds a maximum weighted independent set in a tree in $O(n)$ time.

```plaintext
Weighted-Independent-Set-In-A-Tree(T) {
    Root the tree at a node $r$
    foreach (node $u$ of $T$ in postorder) {
        if ($u$ is a leaf) {
            $M_{in}[u] = w_u$
            $M_{out}[u] = 0$
        } else {
            $M_{in}[u] = w_u + \sum_{v \in \text{children}(u)} M_{out}[v]$
            $M_{out}[u] = \sum_{v \in \text{children}(u)} \max(M_{in}[v], M_{out}[v])$
        }
    }
    return $\max(M_{in}[r], M_{out}[r])$
}
```

**Pf.** Takes $O(n)$ time since we visit nodes in postorder and examine each edge exactly once. • can also find independent set itself (not just value)
**Context**

**Independent set on trees.** This structured special case is tractable because we can find a node that breaks the communication among the subproblems in different subtrees.

Graphs of bounded tree width. Elegant generalization of trees that:
- Captures a rich class of graphs that arise in practice.
- Enables decomposition into independent pieces.

see Chapter 10.4, but proceed with caution
Register Allocation
Register Allocation

Register. One of \( k \) of high-speed memory locations in computer's CPU.

Register allocator. Part of an optimizing compiler that controls which variables are saved in the registers as compiled program executes.

Interference graph. Nodes are "live ranges." Edge \( u-v \) if there exists an operation where both \( u \) and \( v \) are "live" at the same time.

Observation. [Chaitin, 1982] Can solve register allocation problem iff interference graph is \( k \)-colorable.

Spilling. If graph is not \( k \)-colorable (or we can't find a \( k \)-coloring), we "spill" certain variables to main memory and swap back as needed.

- Typically infrequently used variables that are not in inner loops
A Useful Property

**Remark.** Register allocation problem is NP-hard.

**Key fact.** If a node $v$ in graph $G$ has fewer than $k$ neighbors, $G$ is $k$-colorable iff $G - \{v\}$ is $k$-colorable.

**Pf.** Delete node $v$ from $G$ and color $G - \{v\}$.

- If $G - \{v\}$ is not $k$-colorable, then neither is $G$.
- If $G - \{v\}$ is $k$-colorable, then there is at least one remaining color left for $v$. ▪

---

![Diagram](image-url)

- $k = 3$
- $k = 2$

$G$ is 2-colorable even though all nodes have degree 2.
Chaitin's Algorithm

Vertex-Color(G, k) {
    while (G is not empty) {
        Pick a node v with fewer than k neighbors
        Push v on stack
        Delete v and all its incident edges
    }
    while (stack is not empty) {
        Pop next node v from the stack
        Assign v a color different from its neighboring
        nodes which have already been colored
    }
}
**Theorem.** [Kempe 1879, Chaitin 1982] Chaitin's algorithm produces a k-coloring of any graph with max degree k-1.

**Pf.** Follows from key fact since each node has fewer than k neighbors.

**Remark.** If algorithm never encounters a graph where all nodes have degree $\geq k$, then it produces a k-coloring.

**Practice.** Chaitin's algorithm (and variants) are extremely effective and widely used in real compilers for register allocation.
8.7 Graph Coloring

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.
3-Colorability

3-COLOR: Given an undirected graph $G$ does there exists a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?

[Diagram of a graph with nodes colored red, green, and blue, illustrating a yes instance.]
Register Allocation

Register allocation. Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR $\leq_p$ k-REGISTER-ALLOCATION for any constant $k \geq 3$. 
3-Colorability

Claim. 3-SAT \leq_p 3-COLOR.

Pf. Given 3-SAT instance \( \Phi \), we construct an instance of 3-COLOR that is 3-colorable iff \( \Phi \) is satisfiable.

Construction.

i. For each literal, create a node.
ii. Create 3 new nodes T, F, B; connect them in a triangle, and connect each literal to B.
iii. Connect each literal to its negation.
iv. For each clause, add gadget of 6 nodes and 13 edges.

↑

to be described next
3-Colorability

Claim. Graph is 3-colorable iff \( \Phi \) is satisfiable.

Pf. \( \Rightarrow \) Suppose graph is 3-colorable.
- Consider assignment that sets all T literals to true.
- (ii) ensures each literal is T or F.
- (iii) ensures a literal and its negation are opposites.
3-Colorability

Claim. Graph is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Suppose graph is 3-colorable.

- Consider assignment that sets all $T$ literals to true.
- (ii) ensures each literal is $T$ or $F$.
- (iii) ensures a literal and its negation are opposites.
- (iv) ensures at least one literal in each clause is $T$.

$$C_i = x_1 \lor \overline{x_2} \lor x_3$$
Claim. Graph is 3-colorable iff \( \Phi \) is satisfiable.

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- Consider assignment that sets all T literals to true.
- (ii) ensures each literal is T or F.
- (iii) ensures a literal and its negation are opposites.
- (iv) ensures at least one literal in each clause is T.

\[
C_i = x_1 \lor \overline{x_2} \lor x_3
\]

not 3-colorable if all are red

contradiction
Claim. Graph is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\iff$ Suppose 3-SAT formula $\Phi$ is satisfiable.
- Color all true literals T.
- Color node below green node F, and node below that B.
- Color remaining middle row nodes B.
- Color remaining bottom nodes T or F as forced.

$$C_i = x_1 \lor \overline{x_2} \lor x_3$$

a literal set to true in 3-SAT assignment
10.3 Circular Arc Coloring
Wavelength-Division Multiplexing

Wavelength-division multiplexing (WDM). Allows m communication streams (arcs) to share a portion of a fiber optic cable, provided they are transmitted using different wavelengths.

Ring topology. Special case is when network is a cycle on n nodes.

Bad news. NP-complete, even on rings.

Brute force. Can determine if k colors suffice in $O(k^m)$ time by trying all k-colorings.

Goal. $O(f(k)) \cdot \text{poly}(m, n)$ on rings.

$n = 4, m = 6$
Wavelength-Division Multiplexing

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**Goal.** $O(f(k)) \cdot \text{poly}(m, n)$ on rings.
**Review: Interval Coloring**

**Interval coloring.** Greedy algorithm finds coloring such that number of colors equals depth of schedule.

- Maximum number of streams at one location

```
c      d      f      j
b      g      i
a      e      h
```

**Circular arc coloring.**
- Weak duality: number of colors \( \geq \) depth.
- Strong duality does not hold.

max depth = 2
min colors = 3
(Almost) Transforming Circular Arc Coloring to Interval Coloring

**Circular arc coloring.** Given a set of n arcs with depth \( d \leq k \), can the arcs be colored with k colors?

**Equivalent problem.** Cut the network between nodes \( v_1 \) and \( v_n \). The arcs can be colored with k colors iff the intervals can be colored with k colors in such a way that "sliced" arcs have the same color.
Circular Arc Coloring: Dynamic Programming Algorithm

Dynamic programming algorithm.

- Assign distinct color to each interval which begins at cut node $v_0$.
- At each node $v_i$, some intervals may finish, and others may begin.
- Enumerate all $k$-colorings of the intervals through $v_i$ that are consistent with the colorings of the intervals through $v_{i-1}$.
- The arcs are $k$-colorable iff some coloring of intervals ending at cut node $v_0$ is consistent with original coloring of the same intervals.
Circular Arc Coloring: Running Time

Running time. \( O(k! \cdot n) \).
- \( n \) phases of the algorithm.
- Bottleneck in each phase is enumerating all consistent colorings.
- There are at most \( k \) intervals through \( v_i \), so there are at most \( k! \) colorings to consider.

Remark. This algorithm is practical for small values of \( k \) (say \( k = 10 \)) even if the number of nodes \( n \) (or paths) is large.
Extra Slides
Extra Slides
8.10 A Partial Taxonomy of Hard Problems
Polynomial-Time Reductions

Dick Karp (1972)
1985 Turing Award

constraint satisfaction

3-SAT

INDEPENDENT SET  3-SAT reduces to INDEPENDENT SET

DIR-HAM-CYCLE

GRAPH 3-COLOR

HAM-CYCLE

SUBSET-SUM

PLANAR 3-COLOR

TSP

SCHEDULING

INDEPENDENT SET

VERTEX COVER

SET COVER

packing and covering

HAM-CYCLE

TSP

sequencing

planarizing

SET COVER

numerical

partitioning
Subset Sum (proof from book)

**Construction.** Let $X \cup Y \cup Z$ be an instance of 3D-MATCHING with triplet set $T$. Let $n = |X| = |Y| = |Z|$ and $m = |T|$.

- Let $X = \{ x_1, x_2, x_3, x_4 \}$, $Y = \{ y_1, y_2, y_3, y_4 \}$, $Z = \{ z_1, z_2, z_3, z_4 \}$
- For each triplet $t= (x_i, y_j, z_k) \in T$, create an integer $w_t$ with $3n$ digits that has a 1 in positions $i$, $n+j$, and $2n+k$.

**Claim.** 3D-matching iff some subset sums to $W = 111, \ldots, 111$.

<table>
<thead>
<tr>
<th>Triplet $t_i$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
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<tr>
<td>$x_3$ $y_1$ $z_1$</td>
<td>0</td>
<td>0</td>
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<td>1,010,001,000</td>
</tr>
<tr>
<td>$x_4$ $y_4$ $z_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
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<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>100,010,001</td>
</tr>
</tbody>
</table>

\[ 111,111,111,111 \]
Partition

SUBSET-SUM. Given natural numbers \( w_1, \ldots, w_n \) and an integer \( W \), is there a subset that adds up to exactly \( W \)?

PARTITION. Given natural numbers \( v_1, \ldots, v_m \), can they be partitioned into two subsets that add up to the same value?

\[
\frac{1}{2} \sum_i v_i
\]

Claim. \( \text{SUBSET-SUM} \leq_p \text{PARTITION} \).

Pf. Let \( W, w_1, \ldots, w_n \) be an instance of \( \text{SUBSET-SUM} \).

- Create instance of \( \text{PARTITION} \) with \( m = n+2 \) elements.
  \[ v_1 = w_1, \; v_2 = w_2, \ldots, v_n = w_n, \; v_{n+1} = 2 \sum_i w_i - W, \; v_{n+2} = \sum_i w_i + W \]

- There exists a subset that sums to \( W \) iff there exists a partition since two new elements cannot be in the same partition.

\begin{align*}
\text{subset A} & \quad v_{n+1} = 2 \sum_i w_i - W & \quad W \\
\text{subset B} & \quad v_{n+2} = \sum_i w_i + W & \quad \sum_i w_i - W
\end{align*}
4 Color Theorem
**PLANAR-3-COLOR.** Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

**YES instance.**
**PLANAR-3-COLOR.** Given a planar map, can it be colored using 3 colors so that no adjacent regions have the same color?

NO instance.
Planarity

Def. A graph is **planar** if it can be embedded in the plane in such a way that no two edges cross.

Applications: VLSI circuit design, computer graphics.

Kuratowski’s Theorem. An undirected graph $G$ is non-planar iff it contains a subgraph homeomorphic to $K_5$ or $K_{3,3}$.
Planarity Testing

Planarity testing. [Hopcroft-Tarjan 1974] $O(n)$.

A simple planar graph can have at most $3n$ edges.

Remark. Many intractable graph problems can be solved in poly-time if the graph is planar; many tractable graph problems can be solved faster if the graph is planar.
Q. Is this planar graph 3-colorable?
Claim. PLANAR-3-COLOR \leq_p PLANAR-GRAPH-3-COLOR.

Pf sketch. Create a vertex for each region, and an edge between regions that share a nontrivial border.
Claim. W is a planar graph such that:
- In any 3-coloring of W, opposite corners have the same color.
- Any assignment of colors to the corners in which opposite corners have the same color extends to a 3-coloring of W.

Pf. Only 3-colorings of W are shown below (or by permuting colors).
Claim. $3\text{-COLOR} \leq_p \text{PLANAR-GRAPH-3-COLOR}.$

Pf. Given instance of $3\text{-COLOR}$, draw graph in plane, letting edges cross.

- Replace each edge crossing with planar gadget $W$.
- In any 3-coloring of $W$, $a \neq a'$ and $b \neq b'$.
- If $a \neq a'$ and $b \neq b'$ then can extend to a 3-coloring of $W$. 

**Diagram:**
- A crossing with nodes $a$, $a'$, $b$, $b'$.
- The gadget $W$ is a planar graph with additional nodes and edges connecting $a$, $a'$, $b$, and $b'$.
Planar Graph 3-Colorability

**Claim.** \( 3\text{-COLOR} \leq_p \text{PLANAR-GRAPH-3-COLOR}. \)

**Pf.** Given instance of \( 3\text{-COLOR} \), draw graph in plane, letting edges cross.

- Replace each edge crossing with planar gadget \( W \).
- In any 3-coloring of \( W \), \( a \neq a' \) and \( b \neq b' \).
- If \( a \neq a' \) and \( b \neq b' \) then can extend to a 3-coloring of \( W \).
Planar k-Colorability

**PLANAR-2-COLOR.** Solvable in linear time.

**PLANAR-3-COLOR.** NP-complete.

**PLANAR-4-COLOR.** Solvable in $O(1)$ time.

**Theorem.** [Appel-Haken, 1976] Every planar map is 4-colorable.
- Resolved century-old open problem.
- Used 50 days of computer time to deal with many special cases.
- First major theorem to be proved using computer.

**False intuition.** If **PLANAR-3-COLOR** is hard, then so is **PLANAR-4-COLOR** and **PLANAR-5-COLOR**.
Graph minor theorem. [Robertson-Seymour 1980s]

Corollary. There exist an $O(n^3)$ algorithm to determine if a graph can be embedded in the torus in such a way that no two edges cross.

Pf of theorem. Tour de force.
Graph minor theorem. [Robertson-Seymour 1980s]

Corollary. There exist an $O(n^3)$ algorithm to determine if a graph can be embedded in the torus in such a way that no two edges cross.

Mind boggling fact 1. The proof is highly non-constructive!

Mind boggling fact 2. The constant of proportionality is enormous!

Unfortunately, for any instance $G = (V, E)$ that one could fit into the known universe, one would easily prefer $n^{70}$ to even constant time, if that constant had to be one of Robertson and Seymour's. - David Johnson

Theorem. There exists an explicit $O(n)$ algorithm.

Practice. LEDA implementation guarantees $O(n^3)$. 