CS 580: Algorithm Design and Analysis

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Purdue University
Spring 2019

Administrative Stuff

Lectures: Jeremiah Blocki
- Tuesday/Thursday 3PM - 4:15, FNY B124
- Office Hours: Wed/Fri 11am-noon
  - Lawson 1165
  - Google Hangouts (EPE Students)

TAs:
- Akash Kumar
- Hamidreza Amini Khorasgani

Prereq: Mathematical maturity. Undergraduate algorithms (e.g., CS 381).

Textbook: Algorithm Design by Jon Kleinberg and Éva Tardos.

Course web site:
https://www.cs.purdue.edu/homes/jblocki/courses/580_Spring19/

Grading

- 20% for homework
- 20% for the midterm 1 (Feb 21, Evening Exam)
- 20% for the midterm 2 (April 4, Evening Exam)
- 35% for the final
- 5% for class participation
  - EPE Students: Piazza Discussion
  - On Campus: Participation during lecture + piazza discussion

Homework

- 20% for homework
- Weekly/Biweekly problem sets
- Turn in online on Gradescope
  - Entry Code: 97D2ER

Homework Policy

- Difficult Question?
  - Partial Credit (15%) for admitting "I don't know the answer."
- No credit for bad answer
- Collaboration
  - You may discuss homeworks with classmates and course staff
  - Office Hours
  - Discussion on piazza
  - However...
    - You must write down your own solutions
    - You must completely understand your solution!
    - You must acknowledge your collaborators!
    - You may not search online for solutions
      (Course Hero, Stack Overflow etc...)

Course Readings

Schedule (subject to change)

Jan 8: Lecture 1 (Slides) Introduction. The stable matching problem. [Read Chapter 1 of Algorithm Design, Homework 0]
Jan 10: Lecture 2 (Slides), Asymmetric complexity. [Read Chapter 2]
Jan 15: Lecture 3 (Slides), Basic graph algorithms [Read Chapter 3, HW 1 Released]
Jan 17: Lecture 4 (Slides), Graph Algorithms Continued [Read Chapter 4, HW 1 Released]
Jan 22: Lecture 5 (Slides), Greedy Algorithms [Read Chapter 4.1]
Jan 24: Lecture 6 (Slides), Cache Policies, Minimum Spanning Tree [Read Chapter 4.5-6.4] Homework 1 Due, HW 2 Released, HW 1 Released
Homework Policy

• Late Work?
  • 0.01 to 24 hours late (10% penalty)
  • 24 to 48 hours late (25% penalty)
  • > 48 minutes late (no credit)
• Exceptions for extreme circumstances
  • Serious Illness
  • Bereavement
• Re-grading
  • Must be done within two weeks of the day the work is returned to class.
  • Entire assignment/exam may be re-graded

Exams

• Most Significant Part of Your Grade
  • 20% for the midterm 1 (Feb 20. Evening Exam)
  • 20% for the midterm 2 (April 3. Evening Exam)
  • 35% for the final (TBD. April 29-May 4th)
  • Plan to remain on campus through May 4th until final exam has been scheduled
• Allowed to bring double sided index card (3x5 inches) with your own notes
  • No electronics
• Disabilities Requiring Special Accommodations
  • Speak with me within the first three (3) weeks of the semester.
  • Note: We cannot arrange special accommodations without confirmation from the Disability Resource Center here at Purdue (http://www.purdue.edu/drc)

Chapter 1

Introduction: Some Representative Problems

Algorithms

• [webster.com] A procedure for solving a mathematical problem (as of finding the greatest common divisor) in a finite number of steps that frequently involves repetition of an operation.
• [Knuth, TAOCP] An algorithm is a finite, definite, effective procedure, with some input and some output.

Great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing.  - Francis Sullivan
Theory of Algorithms

"As soon as an Analytic Engine exists, it will necessarily
guide the future course of the science. Whenever any
result is sought by its aid, the question will arise - By what
course of calculation can these results be arrived at by the
machine in the shortest time? - Charles Babbage

Algorithmic Paradigms

- Greedy.
- Divide-and-conquer.
- Dynamic programming.
- Network flow.
- Randomized algorithms.
- Intractability.
- Coping with intractability.

Critical thinking and problem-solving.

Applications

- Artistic intelligence.
- Machine Learning.
- Computational.
- Biology.
- Physics.
- Chemistry.
- National Kidney Registry.
- National Resident Matching Program.

We focus on algorithms and techniques that are useful in practice.

1.1 A First Problem: Stable Matching

Matching Problem

Goal: Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

Candidate Assignment 1: X-A, Y-C, Z-B
Candidate Assignment 2: X-B, Y-C, Z-A

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Unstable pair: Man X and Woman A are unstable in matching M
if:

- X prefers A to his assigned partner in matching M.
- A prefers X to her assigned partner in matching M.
- Unstable pair X-A could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

- No incentive for some pair of participants to undermine assignment by joint action.

Stable matching problem. Given the preference lists of n men
and n women, find a stable matching if one exists.
Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a self-reinforcing admissions process.

Unstable pair: applicant $x$ and hospital $y$ are unstable if:
- $x$ prefers $y$ to its assigned hospital.
- $y$ prefers $x$ to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.
- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?
A. Yes.

Propose-And-Reject Algorithm


```
Initialize each person to be free.
while (some man is free and hasn’t proposed to every woman)
{
  Choose such a man $m$.
  $w = 1$st woman on $m$’s list to whom $m$ has not yet proposed
  if ($w$ is free)
    assign $m$ and $w$ to be engaged
  else if ($w$ prefers $m$ to her fiancé $m’$)
    assign $m$ and $w$ to be engaged, and $m’$ to be free
  else
    $w$ rejects $m$.
}
```
Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most $n^2$ iterations of while loop. 

Pf. Each time through the while loop a man proposes to a new woman. There are only $n^2$ possible proposals. ▪

Wyatt Victor
1st A B
2nd C D
3rd C B
A

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Zeus is not matched upon termination of algorithm.
- Then some woman, say Amy, is not matched upon termination.
- By Observation 2, Amy was never proposed to.
- But, Zeus proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching $S^*$.

Case 1: Z never proposed to A.

$\implies Z$ prefers his GS partner to A.

$\implies A-Z$ is stable.

Case 2: Z previously proposed to A.

$\implies A$ rejected Z (right away or later)

$\implies A$ prefers her GS partner to Z.

$\implies A-Z$ is stable.

In either case A-Z is stable, a contradiction. ▪

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In either case A-Z is stable, a contradiction. ▪

Efficient Implementation

Representing men and women.

- Assume men are named 1, ..., n.
- Assume women are named 1', ..., n'.

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays $\text{wife}[m]$, and $\text{husband}[w]$.
- set entry to 0 if unmatched
- if $m$ matched to $w$ then $\text{wife}[m]=w$ and $\text{husband}[w]=m$

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array $\text{count}[m]$ that counts the number of proposals made by man $m$.

Women rejecting/accepting.

- Does woman $w$ prefer man $m$ to man $m'$?
- For each woman, create inverse of preference list of men.
- Constant time access for each query after $O(n)$ preprocessing.

for i = 1 to n
inverse[Pref[i]] = i

for i = 1 to n
if i + 1 to n
Pref[i] = inverse[i]

Efficient Implementation

Women preferring man 3 to 6
SPACE $\text{Pref[3]} = \text{Pref[6]}$

for i = 1 to 7
inverse[Pref[i]] = i
Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings:
- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a stable matching.

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds woman-pessimal stable matching $S^*$. 

Ex: Man = hospitals, Women = med school residents.

Variant 1. Some participants declare others as unacceptable.

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

Def. Matching S unstable if there is a hospital h and resident r such that:
- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.
Application: Matching Residents to Hospitals

NRMP. (National Resident Matching Program)
- Original use just after WWII
- Idea of March, 23,000 residents.

Rural hospital dilemma.
- Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
- Rural hospitals were under-subscribed in NRMP matching.
- How can we find stable matching that benefits "rural hospitals"?

Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

Lessons Learned

Powerful ideas learned in course.
- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]

1.2 Five Representative Problems

Interval Scheduling
Input. Set of jobs with start times and finish times.
Goal. Find maximum cardinality subset of mutually compatible jobs.

Weighted Interval Scheduling
Input. Set of jobs with start times, finish times, and weights.
Goal. Find maximum weight subset of mutually compatible jobs.

Bipartite Matching
Input. Bipartite graph.
Goal. Find maximum cardinality matching.
Independent Set

Input: Graph.
Goal: Find maximum cardinality independent set.

A subset of nodes such that no two joined by an edge.

Competitive Facility Location

Input: Graph with weight on each node.
Game: Two competing players alternate in selecting nodes.
Not allowed to select a node if any of its neighbors have been selected.

Goal: Select a maximum weight subset of nodes.

Second player can guarantee 20, but not 25.
Five Representative Problems

Variations on a theme: independent set.

Interval scheduling: \( n \log n \) greedy algorithm.
Weighted interval scheduling: \( n \log n \) dynamic programming algorithm.
Bipartite matching: \( n^2 \) max-flow based algorithm.
Independent set: \( \text{NP-complete} \).
Competitive facility location: \( \text{PSPACE-complete} \).

Extra Slides

Stable Matching Problem

Goal: Given \( n \) men and \( n \) women, find a "suitable" matching.
- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

Understanding the Solution

Claim. The man-optimal stable matching is weakly Pareto optimal.

Pf.
- Let \( A \) be last woman in some execution of GS algorithm to receive a proposal.
- No man is rejected by \( A \) since algorithm terminates when last woman receives first proposal.
- No man matched to \( A \) will be strictly better off than in man-optimal stable matching.

Deceit: Machiavelli Meets Gale-Shapley

Q. Can there be an incentive to misrepresent your preference profile?
- Assume you know men's propose-and-reject algorithm will be run.
- Assume that you know the preference profiles of all other participants.

Fact. No, for any man yes, for some woman. No mechanism can guarantee a stable matching and be cheatproof.
Lessons Learned

Powerful ideas learned in course.
- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

Potentially deep social ramifications. [legal disclaimer]
- Historically, men propose to women. Why not vice versa?
  - Men: propose early and often.
  - Men: be more honest.
- Women: ask out the guys.
- Theory can be socially enriching and fun
- CS majors get the best partners!