Question 1

Consider the flow graph shown below. Use the Ford-Fulkerson algorithm to determine the maximum flow from s to t using the minimum number of iterations. Also find the minimum cut. Explain why the flow found is a maximum one.

![Flow Graph](image)

Question 2

Suppose that you wish to find, among all minimum cuts in a flow network G with integral capacities, one that contains the smallest number of edges. Show how to modify the capacities of G to create a new flow network G' in which any minimum cut in G' is a minimum cut with the smallest number of edges in G.

Question 3

Consider a group of N students and P courses. Each student s_i takes zero, one or more than one courses, namely c_{i0}, c_{i1}, \ldots, c_{ik}. Design an efficient algorithm to determine whether it is possible to form a committee of exactly P students that satisfies simultaneously the conditions:

- every student in the committee represents a different course (a student can represent a course if he/she takes that course)
- each course has a representative in the committee

Question 4

Write down a linear program to solve the following problems. In each case you should explain why your linear program is correct. Your explanation should include a clear description of each variable and each constraint.
Problem 1 **Network flow with taxation.** Recall that in the standard network flow problem we required that for each vertex $v$ (excluding the source $s$ and sink $t$) the sum of the flow into that vertex is equal to the sum of flow out of that vertex. Suppose that we replace this *conservation constraint* with a *taxation constraint*. In particular, suppose each vertex represents a country and that each country $v \notin \{s, t\}$ has an associated tax-rate $0 < t_v < 1$ meaning that $v$ will keep $t_v$ fraction of the goods flowing through $v$. Given a flow network $G = (V, E)$ with maximum capacities $c(e)$ on each edge $e \in V$ and tax rates $t_v$ for each node $v \notin \{s, t\}$ our goal is to find the maximum amount of goods that can be transported from $s$ to $t$ under taxation. Write down a LP to solve this problem.

Problem 2 **Circulation with multiple goods.** In the standard circulation problem each node $v$ had a supply $s_v$ and a demand $d_v$ and the goal is to find a feasible flow which satisfies all of the demands (e.g., for each $v$ the flow into node $v$ minus the flow out of node $v$ exceeds $s_v - d_v$). Suppose now that we have $m$ goods and that each edge $e$ has a capacity $c_{e,i}$ for each good $i \leq m$ i.e. each directed edge $e$ can transport at most $c_{e,i}$ units of good $i$. Furthermore, each edge has a maximum capacity $c_e$ on the total number of goods that can be transported over that edge. Additionally, each vertex $v$ has a maximum capacity $c_v$ on the total amount of goods flowing through that node. Suppose now that each node $v$ has a supply $s_{v,i}$ of (resp. demand $d_{v,i}$ for) each each good $i \leq m$. We want to ensure that each of the demands constraints are satisfied (e.g., for each node $v$ and each good $i$ node $v$ ends up with at least $d_{v,i}$ units of good $i$). Write down a linear program to find a solution to this multiple goods circulation problem (or determine if no solution exists).

**Bonus (10 Points)**

Consider the following bipartite matching problem. $n$ students are selected to be TAs at Purdue. Each student selects the courses he/she wants to TA and the instructor for each course selects the students s/he wants to have as a TA. We draw an edge between a student and a course if the interest is mutual.

Suppose that evaluating mutual interests shows that (i) every student has edges to exactly three courses and (ii) every course has edges to exactly three students. Show that in this scenario a perfect matching exists.