CS 580: Algorithm Design and Analysis

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Announcements: Homework 6 and Practice Final Exam Solutions Posted

Course Evaluation Survey: Live until Sunday (4/28/2019) at 11:59PM. Your feedback is valued!

Final Exam Logistics

- **Time:** Monday, April 29th at 8AM
- **Location:** PHYS 223 (adjacent building to FRNY)
- **Duration:** 2 hours
- **Content:** Cumulative, but more heavily focused on recent topics (e.g., PSPACE, Approximation Algorithms, Randomized Algorithms, Local Search etc…)
- No Electronics (calculator/phone/laptop/smartwatch etc…)
- **Index Cards:** We will allow you to bring **two** 3x5 index cards (double sided)
  - **Advice:** Don't expect to rely too heavily on your index card.
  - Effort spent preparing the index cards will likely be most beneficial.
- Practice Exam (Partial Solutions)

Practice Problem 1 (MAX SAT)

- MAX SAT (n variables, k clauses)
  - Each clause can have 1,2 or more literals
- **Part A:** Show that a random assignment satisfies *at least* k/2 clauses in expectation.

Practice Problem 1

- MAX SAT (n variables, k clauses)
  - Each clause can have 1,2 or more literals
- **Part B:** Suppose that we disallow directly contradictory clauses e.g., if $C = \{x\}$ is a clause then we cannot include the clause $C' = \{\bar{x}\}$. Modify the random assignment such that we satisfy at least $0.6k$ clauses in expectation.

- **Tempting Idea:** If (resp. $\{\bar{x}\}$) is a clause then set $x = 1$ (resp. $x = 0$).
- **Counter Example?**

Practice Problem 1

- MAX SAT (n variables, k clauses)
  - Each clause can have 1,2 or 3 literals
- **Part B:** Suppose that we disallow directly contradictory clauses e.g., if $C = \{x\}$ is a clause then we cannot include the clause $C' = \{\bar{x}\}$. Modify the random assignment such that we satisfy at least $0.6k$ clauses in expectation.

- **Tempting Idea: If** $\{x\}$ (resp. $\{\bar{x}\}$) is a clause then set $x = 1$ (resp. $x = 0$).
- **Counter Example?** $\{\bar{x}\}$,$\{\bar{y}\}$,$\{\bar{z}\}$,{x,y},{x,z},{y,z}

Practice Problem 1

- MAX SAT (n variables, k clauses)
  - Each clause can have 1,2 or more literals
- **Part B:** Suppose that we disallow directly contradictory clauses e.g., if $C = \{x\}$ is a clause then we cannot include the clause $C' = \{\bar{x}\}$. Modify the random assignment such that we satisfy at least $0.6k$ clauses in expectation.

- **More Refined Analysis:** Let y_i (denote the number of clauses with exactly i literals).
  - Observe that $\mathrm{k} = \sum_i y_i$
  - **Linearity of Expectation:** We satisfy $\frac{1}{2}y_1 + \frac{3}{4}y_2 + \frac{7}{8}y_3 + \cdots = \sum_i \frac{y_i}{2^i}$ clauses in expectation
    - If $y_1 \le 0.6k$ then this is at least $\frac{1}{2}0.6k + \frac{3}{4}0.4k = 0.6k$ clauses in expectation.
    - What do we do otherwise?

Practice Problem 2

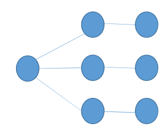
- Greedy Vertex Cover Algorithm

BuildVertexCover(G=(V,E))
  - Initialize $S = \{\}$
  - Find the node v with maximum degree in G
    - Recursively find a vertex cover S' for the graph $G - \{v\}$
    - $S' = \text{BuildVertexCover}(G - \{v\})$
    - Return $S = S' \cup \{v\}$

- **True or False:** Greedy Vertex Cover always returns the optimal vertex cover?

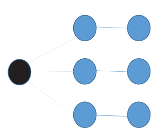
Practice Problem 2

- **True or False:** Greedy Vertex Cover always returns the optimal vertex cover?



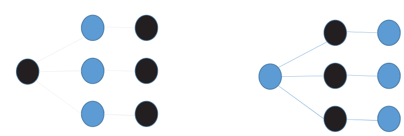
Practice Problem 2

- **True or False:** Greedy Vertex Cover always returns the optimal vertex cover?



Practice Problem 2

- **True or False:** Greedy Vertex Cover always returns the optimal vertex cover?



- Answer: False. Greedy returns vertex cover of size 4. OPT = 3.

Practice Problem 2'

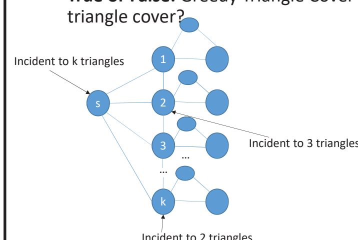
- Greedy **Triangle** Cover Algorithm

BuildTriangleCover(G=(V,E))
  - Initialize $S = \{\}$
  - Find the node v incident to maximum number of **triangles** in G
    - Recursively find a vertex cover S' for the graph $G - \{v\}$
    - $S' = \text{BuildTriangleCover}(G - \{v\})$
    - Return $S = S' \cup \{v\}$

- **True or False:** Greedy Triangle Cover always returns the optimal triangle cover?

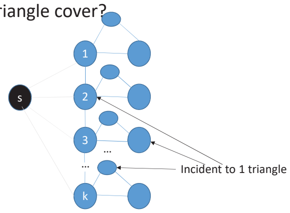
Practice Problem 2

- **True or False:** Greedy Triangle Cover always returns the optimal triangle cover?



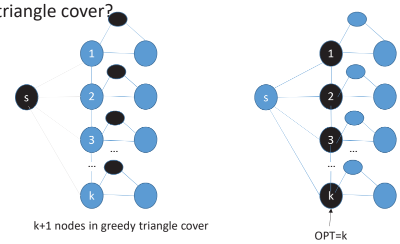
Practice Problem 2

- **True or False:** Greedy Triangle Cover always returns the optimal triangle cover?



Practice Problem 2

- **True or False:** Greedy Triangle Cover always returns the optimal triangle cover?



Practice Problem 3 (Approximate Median)

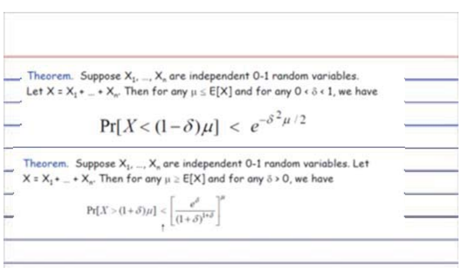
- Suppose we are presented with a very large set S of $n = |S|$ *distinct* real numbers and we want to approximate the median of these numbers by sampling. We say that a real number x is an ε −approximate median of S if *at least* $\left(\frac{1}{2} - \varepsilon\right)$ numbers in S are less than x and *at least* $\left(\frac{1}{2} - \varepsilon\right)$ numbers in S are greater than x.
- Suppose that we sub-sample $k = \frac{40}{\varepsilon^2}$ points $y_1, \dots, y_k \in S$ from S (with replacement) and compute the median y_{med} of these points.
- Given that $0 < \varepsilon < \frac{1}{2}$ **show that** the probability y_{med} is not an ε −approximate median of S is at most $\gamma = 0.0001$.
- **Hint 1:** $\left(\frac{1}{2} + \varepsilon\right)(1 - \varepsilon)k \le \frac{k}{2}$ for positive numbers $k > 0$ when $0 < \varepsilon < \frac{1}{2}$
- **Hint 2:** $e^{-10} \le \frac{\gamma}{2} = 0.00005$ for positive numbers $k > 0$ when $0 < \varepsilon < \frac{1}{2}$

Practice Problem 3 (Approximate Median)

- Suppose that we sub-sample $k = \frac{100}{\varepsilon^2}$ points $y_1, \dots, y_k \in S$ from S (with replacement) and compute the median y_{med} of these points.
- What is the probability that y_{med} is an approximate median of S?
- **Definition:** Let x_{lowest} (resp. $x_{highest}$) denote the smallest (resp. largest) in S which is an ε −approximate median.
- **Introduce Random Variable:** z_i which is 1 if and only if $y_i \ge x_{lowest}$
- **Observation 1:** $\mathrm{E}[z_i] = \Pr[z_i = 1] = \frac{1}{2} + \varepsilon$
- **Observation 2:**

$$\Pr[y_{med} \ge x_{lowest}] = \Pr\left[\sum_{i=1}^{k} z_i \ge \frac{k}{2}\right] = 1 - \Pr\left[\sum_{i=1}^{k} z_i < \frac{k}{2}\right]$$

Practice Problem 2 (Approximate Median)



Theorem. Suppose $X_1, \dots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \dots + X_n$. Then for any $\mu \le E[X]$ and for any $0 < \delta < 1$, we have

$$\Pr[X < (1-\delta)\mu] < e^{-\delta^2 \mu / 2}$$

Theorem. Suppose $X_1, \dots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \dots + X_n$. Then for any $\mu \ge E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > (1+\delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$$

Practice Problem 3 (Approximate Median)

- Suppose that we sub-sample $k = \frac{100}{\varepsilon^2}$ points $y_1, \dots, y_k \in S$ from S (with replacement) and compute the median y_{med} of these points.
- **Introduce Random Variable:** z_i which is 1 if and only if $y_i \ge x_{lowest}$
- **Observation 1:** $E[z_i] = Pr[z_i = 1] = \frac{1}{2} + \varepsilon$
- **Observation 2:** $\Pr[y_{med} \ge x_{lowest}] = 1 - \Pr\left[\sum_{i=1}^{k} z_i < \frac{k}{2}\right]$
- **Chernoff Bound:** $\mu = \left(\frac{1}{2} + \varepsilon\right)k \rightarrow \frac{k}{2} \le \mu(1 - \varepsilon)$

$$\Pr\left[\sum_{i=1}^{k} z_i < \frac{k}{2}\right] \le \Pr\left[\sum_{i=1}^{k} z_i < \mu(1 - \varepsilon)\right] \le \boldsymbol{e^{\frac{-\varepsilon^2 \mu}{2}}} \le \boldsymbol{e^{\frac{-\varepsilon^2 k}{4}}} = \boldsymbol{e^{-10}}$$

Practice Problem 3 (Approximate Median)

- Suppose that we sub-sample $k = \frac{100}{\varepsilon^2}$ points $y_1, \ldots, y_k \in S$ from S (with replacement) and compute the median y_{med} of these points.
- **Introduce Random Variable:** z_i which is 1 if and only if $y_i > x_{lowest}$
- **Observation 1:** $E[z_i] = Pr[z_i = 1] = \frac{1}{2} + \varepsilon$
- **Observation 2:** $\Pr[y_{med} \geq x_{lowest}] = 1 - \Pr\left[\sum_{i=1}^{k} z_i < \frac{k}{2}\right]$
- **Conclusion:**

$$\Pr[y_{med} \geq x_{lowest}] \geq 1 - \boldsymbol{e}^{-10}$$

Symmetric Argument:

$$\Pr[y_{med} \leq x_{highest}] \geq 1 - \boldsymbol{e}^{-10}$$

Practice Problem 3 (Approximate Median)

- Suppose that we sub-sample $k = \frac{100}{\varepsilon^2}$ points $y_1, \ldots, y_k \in S$ from S (with replacement) and compute the median y_{med} of these points.
- **Introduce Random Variable:** z_i which is 1 if and only if $y_i \leq x_{highest}$
- **Observation 1:** $E[z_i] = Pr[z_i = 1] = \frac{1}{2} + \varepsilon$
- **Observation 2:** $\Pr[y_{med} \leq x_{highest}] = 1 - \Pr\left[\sum_{i=1}^{k} z_i < \frac{k}{2}\right]$
- **Chernoff Bound:** $\mu = \left(\frac{1}{2} + \varepsilon\right) k \rightarrow \frac{k}{2} \leq \mu(1 - \varepsilon)$

$$\Pr\left[\sum_{i=1}^{k} z_i < \frac{k}{2}\right] \leq \Pr\left[\sum_{i=1}^{k} z_i < \mu(1 - \varepsilon)\right] \leq \boldsymbol{e}^{\frac{-\varepsilon^2 \mu}{2}} \leq \boldsymbol{e}^{\frac{-\varepsilon^2 k}{4}} = \boldsymbol{e}^{-10}$$

Practice Problem 3 (Approximate Median)

- Conclusion

$$\Pr[y_{med} \leq x_{highest}] = 1 - \Pr\left[\sum_{i=1}^{k} z_i < \frac{k}{2}\right] \leq \mathbf{1} - \boldsymbol{e}^{-10}$$

$$\Pr[y_{med} \geq x_{lowest}] \geq 1 - \boldsymbol{e}^{-10}$$

$$\Pr[y_{med}\ \boldsymbol{is\ not\ \varepsilon - appx}] \leq \mathbf{2}\boldsymbol{e}^{-10} \leq 0.0001$$

Practice with ZPP and RP

- **Notation:** Given a randomized algorithm $\mathcal{A}$ we write $y = \mathcal{A}(x; R)$ to denote the output of $\mathcal{A}$ on input x fixing the random coins to be R.
- **ZPP:** A language X is in ZPP if there is an probabilistic polynomial time algorithm $\mathcal{A}$ such that for all inputs x and all random strings R we have $\Pr[\mathcal{A}(x; R) = 1 \mid x \in X] = 1$ and $\Pr[\mathcal{A}(x; R) = 0 \mid x \notin X] = 1$.
- Show that $ZPP \subseteq NP$
- Certificate for $x \in X$: random string R such that $\mathrm{T}(x; R) \leq p(|x|)$
  - $\mathrm{T}(x; R)$ denotes running time of $\mathcal{A}$ on input x with fixed random coins R
  - Probabilistic polynomial time → $\mathbf{E}[\mathrm{T}(x)] \leq p(|x|)$ → Short R exists

Random Variable: Running time of $\mathcal{A}$ on input x

Practice with ZPP and RP

- Show that $ZPP \subseteq NP$
- Certificate for $x \in X$: random string R such that $\mathrm{T}(x; R) \leq p(|x|)$
  - $\mathrm{T}(x; R)$ denotes running time of $\mathcal{A}$ on input x with fixed random coins R
  - Probabilistic polynomial time → $\mathbf{E}[\mathrm{T}(x)] \leq p(|x|)$ → Short R exists
- Certifier runs $\mathcal{A}(x; R)$ for $p(|x|)$
  - If $\mathcal{A}(x; R)$ returns 1 then output 1 (accept the proof that $x \in X$)
  - If $\mathcal{A}(x; R)$ returns 0 then output 0 (reject the proof that $x \in X$)
  - If $\mathcal{A}(x; R)$ does not halt then output 0 (reject the proof that $x \in X$)

Practice with ZPP and RP

- Show that $ZPP \subseteq coNP$?
- Certificate for $x \notin X$: random string R such that $\mathrm{T}(x; R) \leq p(|x|)$
  - $\mathrm{T}(x; R)$ denotes running time of $\mathcal{A}$ on input x with fixed random coins R
  - Probabilistic polynomial time → $\mathbf{E}[\mathrm{T}(x)] \leq p(|x|)$ → Short R exists
- Certifier runs $\mathcal{A}(x; R)$ for $p(|x|)$
  - If $\mathcal{A}(x; R)$ returns 1 then output 0 (reject the proof that $x \notin X$)
  - If $\mathcal{A}(x; R)$ returns 0 then output 1 (accept the proof that $x \notin X$)
  - If $\mathcal{A}(x; R)$ does not halt then output 0 (reject the proof that $x \notin X$)

Practice with ZPP and RP

- Show that $RP \subseteq NP$?
- A language X is in RP if there is an polynomial time algorithm $\mathcal{A}$ such that for all inputs x and all random strings R we have $\Pr[\mathcal{A}(x;R) = 1 \mid x \in X] \geq \frac{1}{2}$ and $\Pr[\mathcal{A}(x;R) = 0 \mid x \notin X] = 1$.
- Remark 1: For all x, R we have $\mathrm{T}(x;R) \leq p(|x|)$ (polynomial time)
- Remark 2: One sided error. Allowed make mistakes when $x \in X$ (but not when $x \notin X$).
- Certificate: R such that $\mathcal{A}(x;R) = 1$
- **Claim:** If $x \notin X$ there is no valid certificate. Why?

More Complexity Theory

- Suppose that NP=PSPACE. Does it follow that NP=coNP?
- Answer: YES! PSPACE is closed under complementation.
- Though Question: What other complexity classes are closed under complementation?

ZPP and RP

- **Notation:** Given a randomized algorithm $\mathcal{A}$ we write $\mathrm{y} = \mathcal{A}(x;R)$ to denote the output of $\mathcal{A}$ on input x fixing the random coins to be R.
- Running time is $\mathrm{T}(x;R)$ with fixed random coins R
- **Remark:** Once x and R are fixed the output of $\mathcal{A}(x;R)$ is deterministic as is $\mathrm{T}(\mathcal{A};R)$.
- By Contrast, $\mathcal{A}(x)$ and $\mathrm{T}(x)$ are both random variables.
- Probabilistic Polynomial Time: For all inputs x
- $\mathbf{E}[\mathrm{T}(x)] \leq p(|x|)$
- **ZPP:** A language X is in ZPP if there is an probabilistic polynomial time algorithm $\mathcal{A}$ such that for all inputs x and all random strings R we have $\Pr[\mathcal{A}(x;R) = 1 \mid x \in X] = 1$ and $\Pr[\mathcal{A}(x;R) = 0 \mid x \notin X] = 1$

Feasible Subset Sum

- Feasible Subset Sum
  - **Input:** Set $A = \{a_1, \dots, a_n\}$ of positive integers and a positive integer $B > 0$.
  - Feasible Subset: $S \subset A$ is feasible if $\sum_{x \in S} x \leq B$.
  - **Goal:** Find feasible subset $S \subset A$ maximizing $\sum_{x \in S} x$
- **Example:** $A = \{8,2,4\}$ and $B = 11$ ➔ $S = \{8,2\}$ is optimal.
- **Greedy Algorithm:**
  - Initialize: S={}
  - For i=1,…,n
    - If $a_i + \sum_{x \in S} x \leq B$ then update $S = S \cup \{a_i\}$
- **Part 1:** Give an instance where greedy algorithm (above) is *not* a 2-approximation.

Feasible Subset Sum

- **Greedy Algorithm:**
  - Initialize: S={}
  - For i=1,…,n
    - If $a_i + \sum_{x \in S} x \leq B$ then update $S = S \cup \{a_i\}$
- **Part 1:** Give an instance where greedy algorithm (above) is *not* a 2-approximation.
- **Example:** $A = \{2,8\}$ and $B = 9$
  - Greedy Solution: $S = \{2\}$, but optimal is $S^* = \{8\}$ (four times better)

Feasible Subset Sum

- Feasible Subset Sum
  - **Input:** Set $A = \{a_1, \dots, an\}$ of positive integers and a positive integer $B > 0$.
  - Feasible Subset: $S \subset A$ is feasible if $\sum_{x \in S} x \leq B$.
  - **Goal:** Find feasible subset $S \subset A$ maximizing $\sum_{x \in S} x$
- **Example:** $A = \{8,2,4\}$ and $B = 11$ ➔ $S = \{8,2\}$ is optimal.
- **Greedy Algorithm:**
  - Initialize: S={}
  - For i=1,…,n
    - If $a_i + \sum_{x \in S} x \leq B$ then update $S = S \cup \{a_i\}$
- **Part 2:** Modify the above algorithm to yield a 2-approximation.

Feasible Subset Sum

- **Greedy Algorithm:**
  - **Assumption we have eliminated any $a_i > \mathrm{B}$**
  - **Sort $A = \{a_1, \ldots, an\}$ so that $a_1 > a_2 \ldots$**
  - Initialize: S_0={}
  - For i=1,…,n
    - If $a_i + \sum_{x \in S_{i-1}} x \leq B$ then update $S_i = S_{i-1} \cup \{a_i\}$; otherwise $S_i = S_{i-1}$
- **Analysis:** We claim that either $S_n = A$ or $\sum_{x \in S_n} x \geq \frac{B}{2}$
- **Proof:** If $S_n \neq A$ we let j be first index such that $a_j + \sum_{x \in S_{j-1}} x > B$.
  - Case 1: $a_j < \mathrm{B}$ we have $\sum_{x \in S_{j-1}} x > B - a_j > \frac{B}{2} \Rightarrow \sum_{x \in S_n} x > \frac{B}{2}$
  - Case 2: $a_j > \frac{\mathrm{B}}{2}$ then we already added some earlier item $a_i > a_j \Rightarrow$

$$\sum_{x \in S_n} x > a_i > \frac{B}{2}$$

Monotone Satisfiability Problem

- A *monotone* 3-SAT formula φ is specified by clauses $C_1, \ldots, C_k$ over variables $x_1, \ldots, x_n$ with the restriction that for each variable x_j its negation $\bar{x}_j$ is never used in *any* clause.
- **Decision Problem 1:** Does φ have a satisfying assignment?
- **Question 1:** Is there a polynomial time algorithm to solve decision problem 1?

Monotone Satisfiability Problem

- A *monotone* 3-SAT formula φ is specified by clauses $C_1, \ldots, C_k$ over variables $x_1, \ldots, x_n$ with the restriction that for each variable x_j its negation $\bar{x}_j$ is never used in *any* clause.
- **Decision Problem 2:** Does φ have a satisfying assignment in which *at most* k-variables are set to 1?
- **Question 2:** Show that this second decision problem is NP-Complete.

- Step 1?

Monotone Satisfiability Problem

- A *monotone* 3-SAT formula φ is specified by clauses $C_1, \ldots, C_k$ over variables $x_1, \ldots, x_n$ with the restriction that for each variable x_j its negation $\bar{x}_j$ is never used in *any* clause.
- **Decision Problem 2:** Does φ have a satisfying assignment in which *at most* k-variables are set to 1?
- **Question 2:** Show that this second decision problem is NP-Complete.

- **Step 1:** Show that decision problem 2 is in NP.
- **Witness:** satisfying assignment with at most k-variables set to 1.

Monotone Satisfiability Problem

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- **Decision Problem 2:** Does φ have a satisfying assignment in which *at most* k-variables are set to 1?
- **Question 2:** Show that this second decision problem is NP-Complete.

- **Step 2:** Reduction from known NP-Complete Problem
- **Hint:** Try vertex cover

Monotone Satisfiability Problem

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- **Question 2:** Show that this second decision problem is NP-Complete.

- **Step 2:** Reduction from known NP-Complete Problem
- **Vertex Cover Instance:** (G,k)
- **Monotone 3-SAT Formula:** φ_G

Monotone Satisfiability Problem

- A *monotone* 3-SAT formula φ is specified by clauses $C_1, \ldots, C_k$ over variables $x_1, \ldots, x_n$ with the restriction that for each variable x_j its negation $\bar{x}_j$ is never used in *any* clause.
- **Decision Problem 2:** Does φ have a satisfying assignment in which *at most* k-variables are set to 1?
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- **Step 2:** Reduction from known NP-Complete Problem
- **Vertex Cover Instance:** (G,k)
- **Monotone 3-SAT Formula:** φ_G
  - **Add Variable x_v for each node and clause $C_e = \{x_u, x_v\}$ for each edge e={u,v}**

Monotone Satisfiability Problem

- **Step 2:** Reduction from known NP-Complete Problem
- **Vertex Cover Instance:** (G,k)
- **Monotone 3-SAT Instance f(G,k): φ_G and k**
  - **Build φ_G: Add Variable x_v for each node and clause $C_e = \{x_u, x_v\}$ for each edge e={u,v}**
- **Claim 1: If G has a vertex cover of size k then φ_G has a satisfying assignment in which at most k variables are true.**
  - **Proof: Let S be vertex cover and then set $x_u = 1$ for each node $u \in S$. For each clause C_e we either have $u \in S$ or $v \in S$ and hence the clause is satisfied.**

Monotone Satisfiability Problem

- **Step 2:** Reduction from known NP-Complete Problem
- **Vertex Cover Instance:** (G,k)
- **Monotone 3-SAT Instance f(G,k): φ_G and k**
  - **Build φ_G: Add Variable x_v for each node and clause $C_e = \{x_u, x_v\}$ for each edge e={u,v}**
- **Claim 2: If φ_G has a satisfying assignment in which at most k variables are true then G has a vertex cover of size k.**
  - **Proof: Given satisfying assignment we define a vertex cover S in which we add $u \in S$ if and only if $x_u = 1$. For each clause C_e we either have $x_u = 1$ or $x_v = 1$ and hence each edge e is covered by S.**

More Reductions

- Suppose that we have a polynomial time Karp reduction from decision problem X to decision problem Y i.e $X \leq_P Y$.
- Which of the following claims are **necessarily** true?

A. If Y is in PSPACE then X is in PSPACE
B. If Y is PSPACE-Complete then X is in PSPACE
C. If Y is NP-Complete then X is NP-Complete
D. If Y is NP-Complete then X is NP-Hard
E. If X is NP-Complete then Y is NP-Complete
F. If Y is in P then X is in P
G. If Y is in ZPP then X is in ZPP.