CS 580: Algorithm Design and Analysis

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No Class March 1
Midterm Exam 1

- **Anticipate having midterm 1 graded at this point**

- **Look for comments on Piazza**
  - **Common Mistakes**
  - **Average, Max, Min, etc...**

- **Grading Mistake?**
  - **Contact course staff to ask for regrade**
  - **Grade could go up or down**
Network Flow Problem

- Directed Graph $G$ with capacities $c(e)$ on each edge
- Source Node: $s$
- Sink Node: $t$
- (Max Flow) How much flow can we push from source to sink?
- (Min Cut) Find a minimum capacity $s-t$ cut
  - An $s-t$ cut is a partition $(A, B)$ of $V$ with $s \in A$ and $t \in B$.
- **Theorem:** The maximum $s-t$ flow is equal to the minimum $s-t$ cut
- Algorithms to compute maximum $s-t$ flow
  - Ford-Fulkerson
    - Residual Graphs and augmenting paths
    - Can run in exponential time.
  - Capacity Scaling Algorithm $O(m^2 \log C)$
  - Dinic’s Algorithm: $O(mn^2)$
- **Integrality:** If all capacities $c(e)$ are integral then we can find a max flow $f(e)$ in which the flow $f(e)$ on every edge is integral.
7.5 Bipartite Matching
Matching

Matching.

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in $M$.
- Max matching: find a max cardinality matching.
Bipartite matching.

- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in $M$.
- Max matching: find a max cardinality matching.

Matching: $1-2', 3-1', 4-5'$
Bipartite Matching

Bipartite matching.

- Input: undirected, \textit{bipartite} graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in $M$.
- Max matching: find a max cardinality matching.

\begin{center}
\begin{tikzpicture}
  \draw[fill=gray!20] (3.5,2.5) rectangle (5.5,4.5);
  \node at (4.5,3.5) {max matching};
  \node at (4.5,3.0) {1-1', 2-2', 3-3' 4-4'};

  \node (1) at (0,5) {1};
  \node (1p) at (5,5) {1'};
  \node (2) at (0,4) {2};
  \node (2p) at (5,4) {2'};
  \node (3) at (0,3) {3};
  \node (3p) at (5,3) {3'};
  \node (4) at (0,2) {4};
  \node (4p) at (5,2) {4'};
  \node (5) at (0,1) {5};
  \node (5p) at (5,1) {5'};

  \draw[thick,blue] (1) -- (1p);
  \draw[thick,blue] (2) -- (2p);
  \draw[thick,blue] (3) -- (3p);
  \draw[thick,blue] (4) -- (4p);
  \draw[thick,blue] (5) -- (5p);

  \draw[thick,blue] (1) -- (2p);
  \draw[thick,blue] (2) -- (3p);
  \draw[thick,blue] (3) -- (4p);
  \draw[thick,blue] (4) -- (5p);
  \draw[thick,blue] (5) -- (1p);

  \node at (-2,2.5) {L};
  \node at (7,2.5) {R};
\end{tikzpicture}
\end{center}
Max flow formulation.

- Create digraph $G' = (L \cup R \cup \{s, t\}, E')$.
- Direct all edges from $L$ to $R$, and assign infinite (or unit) capacity.
- Add source $s$, and unit capacity edges from $s$ to each node in $L$.
- Add sink $t$, and unit capacity edges from each node in $R$ to $t$. 
Theorem. Max cardinality matching in $G =$ value of max flow in $G'$.

Pf. $\leq$

- Given max matching $M$ of cardinality $k$.
- Consider flow $f$ that sends 1 unit along each of $k$ paths.
- $f$ is a flow, and has cardinality $k$. □
Bipartite Matching: Proof of Correctness

**Theorem.** Max cardinality matching in $G$ = value of max flow in $G'$.

**Pf.**

- Let $f$ be a max flow in $G'$ of value $k$.
- Integrality theorem $\Rightarrow$ $k$ is integral and can assume $f$ is 0-1.
- Consider $M =$ set of edges from $L$ to $R$ with $f(e) = 1$.
  - each node in $L$ and $R$ participates in at most one edge in $M$
  - $|M| = k$: consider cut $(L \cup s, R \cup t)$.

![Bipartite Matching Diagram](image)
Perfect Matching

**Def.** A matching $M \subseteq E$ is **perfect** if each node appears in exactly one edge in $M$.

**Q.** When does a bipartite graph have a perfect matching?

**Structure of bipartite graphs with perfect matchings.**
- Clearly we must have $|L| = |R|$.
- What other conditions are necessary?
- What conditions are sufficient?
Notation. Let $S$ be a subset of nodes, and let $N(S)$ be the set of nodes adjacent to nodes in $S$.

Observation. If a bipartite graph $G = (L \cup R, E)$, has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. Each node in $S$ has to be matched to a different node in $N(S)$.

No perfect matching:
$S = \{2, 4, 5\}$
$N(S) = \{2', 5'\}$. 

Perfect Matching
Marriage Theorem. [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, $G$ has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. $\Rightarrow$ This was the previous observation.
Proof of Marriage Theorem

\textbf{Pf.} \iff Suppose \( G \) does not have a perfect matching.
\begin{itemize}
    \item Formulate as a max flow problem and let \((A, B)\) be min cut in \( G' \).
    \item By max-flow min-cut, \( \text{cap}(A, B) < |L| \).
    \item Define \( L_A = L \cap A, \ L_B = L \cap B, \ R_A = R \cap A \).
    \item \( \text{cap}(A, B) = |L_B| + |R_A| \).
    \item Since min cut can't use \( \infty \) edges: \( N(L_A) \subseteq R_A \).
    \item \( |N(L_A)| \leq |R_A| = \text{cap}(A, B) - |L_B| < |L| - |L_B| = |L_A| \).
    \item Choose \( S = L_A \).
\end{itemize}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{proof-marriage-theorem.png}
\end{figure}

\begin{align*}
L_A &= \{2, 4, 5\} \\
L_B &= \{1, 3\} \\
R_A &= \{2', 5'\} \\
N(L_A) &= \{2', 5'\}
\end{align*}
Which max flow algorithm to use for bipartite matching?

- Generic augmenting path: \(O(m \text{val}(f^*)) = O(mn)\).
- Capacity scaling: \(O(m^2 \log C) = O(m^2)\).
- Shortest augmenting path: \(O(m n^{1/2})\).

Non-bipartite matching.

- Structure of non-bipartite graphs is more complicated, but well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm: \(O(n^4)\). [Edmonds 1965]
- Best known: \(O(m n^{1/2})\). [Micali-Vazirani 1980]
7.6 Disjoint Paths
Disjoint path problem. Given a digraph $G = (V, E)$ and two nodes $s$ and $t$, find the max number of edge-disjoint $s$-$t$ paths.

Def. Two paths are edge-disjoint if they have no edge in common.

Ex: communication networks.
Disjoint path problem. Given a digraph $G = (V, E)$ and two nodes $s$ and $t$, find the max number of edge-disjoint $s$-$t$ paths.

**Def.** Two paths are *edge-disjoint* if they have no edge in common.

**Ex:** communication networks.
Max flow formulation: assign unit capacity to every edge.

Theorem. Max number edge-disjoint s-t paths equals max flow value.

Pf. \( \leq \)

- Suppose there are \( k \) edge-disjoint paths \( P_1, \ldots, P_k \).
- Set \( f(e) = 1 \) if \( e \) participates in some path \( P_i \); else set \( f(e) = 0 \).
- Since paths are edge-disjoint, \( f \) is a flow of value \( k \).
Max flow formulation: assign unit capacity to every edge.

Theorem. Max number edge-disjoint s-t paths equals max flow value.

Pf. ≥

- Suppose max flow value is k.
- Integrality theorem ⇒ there exists 0-1 flow f of value k.
- Consider edge (s, u) with f(s, u) = 1.
  - by conservation, there exists an edge (u, v) with f(u, v) = 1
  - continue until reach t, always choosing a new edge
- Produces k (not necessarily simple) edge-disjoint paths.

can eliminate cycles to get simple paths if desired
Network Connectivity

Network connectivity. Given a digraph $G = (V, E)$ and two nodes $s$ and $t$, find min number of edges whose removal disconnects $t$ from $s$.

Def. A set of edges $F \subseteq E$ disconnects $t$ from $s$ if every $s$-$t$ path uses at least one edge in $F$. 
Theorem. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf. ≤
- Suppose the removal of $F \subseteq E$ disconnects $t$ from $s$, and $|F| = k$.
- Every s-t path uses at least one edge in $F$.
  Hence, the number of edge-disjoint paths is at most $k$. □
Theorem. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf. ≥
- Suppose max number of edge-disjoint paths is k.
- Then max flow value is k.
- *Max-flow min-cut* ⇒ cut \((A, B)\) of capacity k.
- Let F be set of edges going from A to B.
- \(|F| = k\) and disconnects t from s.
7.12 Baseball Elimination

"See that thing in the paper last week about Einstein? . . . Some reporter asked him to figure out the mathematics of the pennant race. You know, one team wins so many of their remaining games, the other teams win this number or that number. What are the myriad possibilities? Who's got the edge?"

"The hell does he know?"

"Apparently not much. He picked the Dodgers to eliminate the Giants last Friday."

- Don DeLillo, Underworld
### Baseball Elimination

#### Which teams have a chance of finishing the season with most wins?

- Montreal eliminated since it can finish with at most 80 wins, but Atlanta already has 83.
- \( w_i + r_i < w_j \) \( \Rightarrow \) team \( i \) eliminated.
- Only reason sports writers appear to be aware of.
- Sufficient, but not necessary!

<table>
<thead>
<tr>
<th>Team</th>
<th>Wins ( w_i )</th>
<th>Losses ( l_i )</th>
<th>To play ( r_i )</th>
<th>Against = ( r_{ij} )</th>
</tr>
</thead>
<tbody>
<tr>
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<td></td>
<td></td>
<td>Atl</td>
</tr>
<tr>
<td>Atlanta</td>
<td>83</td>
<td>71</td>
<td>8</td>
<td>-</td>
</tr>
<tr>
<td>Philly</td>
<td>80</td>
<td>79</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>New York</td>
<td>78</td>
<td>78</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>Montreal</td>
<td>77</td>
<td>82</td>
<td>3</td>
<td>1</td>
</tr>
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</table>
Baseball Elimination

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Which teams have a chance of finishing the season with most wins?

- Philly can win 83, but still eliminated . . .
- If Atlanta loses a game, then some other team wins one.

Remark. Answer depends not just on how many games already won and left to play, but also on whom they're against.
Baseball Elimination

49ers, Young Get Big Break
Quarterback may return

By Gary Swan
Chronicle Staff Writer

The bye week has come at a perfect time for the 49ers and quarterback Steve Young. If they had a game next Sunday, there's a good chance Young would not play.

But the pulled groin muscle on his up-

49ers Officially Leave the NL West Race

By Nancy Gay
Chronicle Staff Writer

With the smack of another National League West bat 500 miles away, the Giants' run at the division title ended last night, just as they were handing the visiting St. Louis Cardinals an even bigger lead in the NL Central.

CARDINALS & GIANTS 2

In San Diego, Greg Vaughn's three-run homer in the eighth pushed the Padres over the Pirates and officially shoved the rest of the Giants' season into the background. On the heels of their tedious 6-2 loss before an announced crowd of 10,307 at Candlestick Park, the Giants fell 1½ games off the lead.

As it is, the worst the Padres (80-69) can finish is 80-82. The Giants have fallen to 59-83 with 20 games left; they cannot win 80 games. Coming off a miserable 2-8 mark on a three-city road trip that saw their road record drop to 27-47, the Giants were hoping to get off on the right foot in their longest homestand of the year (15 games, 14 days).

Financing in Place
For Giants' New Stadium
SEE PAGE B1, MAIN NEWS

"Where we are, you're going to be eliminated sooner or later," Baker said quietly. "But it doesn't alter the fact that we've still got to play ball. You've still got to play hard, the fans come out to watch you play. You've got to play for the fact of loving to play, no matter where you are in the standings.

"You've got to play the role of spoiler, to not make it easier on GIIANTS. Page D5 Col. 3"
Baseball elimination problem.

- Set of teams $S$.
- Distinguished team $s \in S$.
- Team $x$ has won $w_x$ games already.
- Teams $x$ and $y$ play each other $r_{xy}$ additional times.
- Is there any outcome of the remaining games in which team $s$ finishes with the most (or tied for the most) wins?
Can team 3 finish with most wins?
- Assume team 3 wins all remaining games $\Rightarrow w_3 + r_3$ wins.
- Divvy remaining games so that all teams have $\leq w_3 + r_3$ wins.
Theorem. Team 3 is not eliminated iff max flow saturates all edges leaving source.

- Integrality theorem $\Rightarrow$ each remaining game between $x$ and $y$ added to number of wins for team $x$ or team $y$.
- Capacity on $(x, t)$ edges ensure no team wins too many games.

Baseball Elimination: Max Flow Formulation

\[ r_{24} = 7 \]

\[ w_3 + r_3 - w_4 \]
Which teams have a chance of finishing the season with most wins?
- Detroit could finish season with 49 + 27 = 76 wins.
Baseball Elimination: Explanation for Sports Writers

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</tr>
</thead>
<tbody>
<tr>
<td>NY</td>
<td>75</td>
<td>59</td>
<td>28</td>
<td>- 3 8 7 7 3</td>
</tr>
<tr>
<td>Baltimore</td>
<td>71</td>
<td>63</td>
<td>28</td>
<td>3 - 2 7 4</td>
</tr>
<tr>
<td>Boston</td>
<td>69</td>
<td>66</td>
<td>27</td>
<td>8 2 - 0 0</td>
</tr>
<tr>
<td>Toronto</td>
<td>63</td>
<td>72</td>
<td>27</td>
<td>7 7 0 - -</td>
</tr>
<tr>
<td>Detroit</td>
<td>49</td>
<td>86</td>
<td>27</td>
<td>3 4 0 0 -</td>
</tr>
</tbody>
</table>

**AL East: August 30, 1996**

Which teams have a chance of finishing the season with most wins?
- Detroit could finish season with $49 + 27 = 76$ wins.

Certificate of elimination. $R = \{NY, Bal, Bos, Tor\}$
- Have already won $w(R) = 278$ games.
- Must win at least $r(R) = 27$ more.
- Average team in $R$ wins at least $305/4 > 76$ games.
Certificate of elimination.

\[ T \subseteq S, \quad w(T) := \sum_{i \in T} w_i, \quad g(T) := \sum_{\{x,y\} \subseteq T} g_{xy}, \]

If \[ \frac{w(T) + g(T)}{|T|} > w_z + g_z \] then z is eliminated (by subset T).

Theorem. [Hoffman-Rivlin 1967] Team z is eliminated iff there exists a subset \( T^* \) that eliminates z.

Proof idea. Let \( T^* = \) team nodes on source side of min cut.
**Pf of theorem.**
- Use max flow formulation, and consider min cut \((A, B)\).
- Define \(T^*\) = team nodes on source side of min cut.
- Observe \(x-y \in A\) iff both \(x \in T^*\) and \(y \in T^*\).
  - infinite capacity edges ensure if \(x-y \in A\) then \(x \in A\) and \(y \in A\)
  - if \(x \in A\) and \(y \in A\) but \(x-y \in T\), then adding \(x-y\) to \(A\) decreases capacity of cut

\[
\begin{align*}
\text{games left} & \quad r_{24} = 7 \\
\text{team x can still win this many more games} & \quad w_z + r_z - w_x
\end{align*}
\]
Baseball Elimination: Explanation for Sports Writers

Pf of theorem.

- Use max flow formulation, and consider min cut \((A, B)\).
- Define \(T^*\) = team nodes on source side of min cut.
- Observe \(x-y \in A\) iff both \(x \in T^*\) and \(y \in T^*\).
- \(g(S - \{z\}) > \text{cap}(A, B)\)

\[
= \frac{g(S - \{z\}) - g(T^*)}{\text{capacity of game edges leaving } s} + \frac{\sum_{x \in T^*} (w_z + g_z - w_x)}{\text{capacity of team edges leaving } s}
\]

\[
= g(S - \{z\}) - g(T^*) - w(T^*) + |T^*|(w_z + g_z)
\]

- Rearranging terms: \(w_z + g_z < \frac{w(T^*) + g(T^*)}{|T^*|}\)
7.7 Extensions to Max Flow
Circulation with Demands

**Circulation with demands.**

- Directed graph $G = (V, E)$.
- Edge capacities $c(e), e \in E$.
- Node supply and demands $d(v), v \in V$.

Demands if $d(v) > 0$; supply if $d(v) < 0$; transshipment if $d(v) = 0$

**Def.** A **circulation** is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

**Circulation problem:** given $(V, E, c, d)$, does there exist a circulation?
Circulation with Demands

Necessary condition: sum of supplies = sum of demands.

\[ \sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v) =: D \]

Pf. Sum conservation constraints for every demand node \( v \).
Circulation with Demands

Max flow formulation.

G:

-7

3

10

-8

6

7

7

4

9

10

0

4

11

demand

supply
Circulation with Demands

Max flow formulation.
- Add new source $s$ and sink $t$.
- For each $v$ with $d(v) < 0$, add edge $(s, v)$ with capacity $-d(v)$.
- For each $v$ with $d(v) > 0$, add edge $(v, t)$ with capacity $d(v)$.
- Claim: $G$ has circulation iff $G'$ has max flow of value $D$.

$G'$:

saturates all edges leaving $s$ and entering $t$
Integrality theorem. If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.

Pf. Follows from max flow formulation and integrality theorem for max flow.

Characterization. Given $(V, E, c, d)$, there does not exists a circulation iff there exists a node partition $(A, B)$ such that $\sum_{v \in B} d_v > \text{cap}(A, B)$

Pf idea. Look at min cut in $G'$. demand by nodes in $B$ exceeds supply of nodes in $B$ plus max capacity of edges going from $A$ to $B$
Circulation with Demands and Lower Bounds

Feasible circulation.

- Directed graph $G = (V, E)$.
- Edge capacities $c(e)$ and lower bounds $\ell(e)$, $e \in E$.
- Node supply and demands $d(v)$, $v \in V$.

Def. A circulation is a function that satisfies:

- For each $e \in E$: $\ell(e) \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

Circulation problem with lower bounds. Given $(V, E, \ell, c, d)$, does there exists a circulation?
Circulation with Demands and Lower Bounds

**Idea.** Model lower bounds with demands.
- Send $\ell(e)$ units of flow along edge $e$.
- Update demands of both endpoints.

**Theorem.** There exists a circulation in $G$ iff there exists a circulation in $G'$. If all demands, capacities, and lower bounds in $G$ are integers, then there is a circulation in $G$ that is integer-valued.

**Pf sketch.** $f(e)$ is a circulation in $G$ iff $f'(e) = f(e) - \ell(e)$ is a circulation in $G'$. 
7.8 Survey Design
Survey Design

Survey design.
- Design survey asking $n_1$ consumers about $n_2$ products.
- Can only survey consumer $i$ about product $j$ if they own it.
- Ask consumer $i$ between $c_i$ and $c_i'$ questions.
- Ask between $p_j$ and $p_j'$ consumers about product $j$.

Goal. Design a survey that meets these specs, if possible.

Bipartite perfect matching. Special case when $c_i = c_i' = p_i = p_i' = 1$. 

one survey question per product
Algorithm. Formulate as a circulation problem with lower bounds.
- Include an edge \((i, j)\) if consumer \(j\) owns product \(i\).
- Integer circulation \(\iff\) feasible survey design.
7.10 Image Segmentation
Image Segmentation

Image segmentation.
- Central problem in image processing.
- Divide image into coherent regions.

Ex: Three people standing in front of complex background scene. Identify each person as a coherent object.
Image Segmentation

Foreground / background segmentation.
- Label each pixel in picture as belonging to foreground or background.
- \( V = \) set of pixels, \( E = \) pairs of neighboring pixels.
- \( a_i \geq 0 \) is likelihood pixel \( i \) in foreground.
- \( b_i \geq 0 \) is likelihood pixel \( i \) in background.
- \( p_{ij} \geq 0 \) is separation penalty for labeling one of \( i \) and \( j \) as foreground, and the other as background.

Goals.
- Accuracy: if \( a_i > b_i \) in isolation, prefer to label \( i \) in foreground.
- Smoothness: if many neighbors of \( i \) are labeled foreground, we should be inclined to label \( i \) as foreground.
- Find partition \((A, B)\) that maximizes:
  \[
  \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i,j) \in E} p_{ij}
  \]
  \begin{align*}
  &\text{foreground} \quad \text{background} \\
  \end{align*}
  \mid A \cap \{i,j\} \mid = 1
Image Segmentation

Formulate as min cut problem.
- Maximization.
- No source or sink.
- Undirected graph.

Turn into minimization problem.

- Maximizing $\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{(i, j) \in E} p_{ij}$
  where $|A \cap \{i, j\}| = 1$

  is equivalent to minimizing $\left(\sum_{i \in V} a_i + \sum_{j \in V} b_j\right) - \sum_{i \in A} a_i - \sum_{j \in B} b_j + \sum_{(i, j) \in E} p_{ij}$
  where $|A \cap \{i, j\}| = 1$

- or alternatively $\sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{(i, j) \in E} p_{ij}$
  where $|A \cap \{i, j\}| = 1$
Image Segmentation

Formulate as min cut problem.
- \( G' = (V', E') \).
- Add source to correspond to foreground; add sink to correspond to background
- Use two anti-parallel edges instead of undirected edge.
Consider min cut \((A, B)\) in \(G'\).

- \(A = \text{foreground}\).

\[

cap(A, B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{(i, j) \in E, i \in A, j \in B} p_{ij}
\]

if \(i\) and \(j\) on different sides, \(p_{ij}\) counted exactly once.

- Precisely the quantity we want to minimize.

\[G'\]

\(A\)
7.11 Project Selection
Project Selection

Projects with prerequisites.

- Set $P$ of possible projects. Project $v$ has associated revenue $p_v$.
  - some projects generate money: create interactive e-commerce interface, redesign web page
  - others cost money: upgrade computers, get site license
- Set of prerequisites $E$. If $(v, w) \in E$, can't do project $v$ and unless also do project $w$.
- A subset of projects $A \subseteq P$ is feasible if the prerequisite of every project in $A$ also belongs to $A$.

Project selection. Choose a feasible subset of projects to maximize revenue.
Prerequisite graph.
- Include an edge from v to w if can't do v without also doing w.
- \{v, w, x\} is feasible subset of projects.
- \{v, x\} is infeasible subset of projects.
Min cut formulation.

- Assign capacity $\infty$ to all prerequisite edge.
- Add edge $(s, v)$ with capacity $p_v$ if $p_v > 0$.
- Add edge $(v, t)$ with capacity $-p_v$ if $p_v < 0$.
- For notational convenience, define $p_s = p_t = 0$. 
Claim. \((A, B)\) is min cut iff \(A - \{s\}\) is optimal set of projects.

- Infinite capacity edges ensure \(A - \{s\}\) is feasible.
- Max revenue because:

\[
\text{cap}(A, B) = \sum_{v \in B: p_v > 0} p_v + \sum_{v \in A: p_v < 0} (-p_v)
\]

\[
= \sum_{v: p_v > 0} p_v - \sum_{v \in A} p_v
\]

constant
Open-pit mining. (studied since early 1960s)

- Blocks of earth are extracted from surface to retrieve ore.
- Each block $v$ has net value $p_v = \text{value of ore} - \text{processing cost}$.
- Can't remove block $v$ before $w$ or $x$. 
k-Regular Bipartite Graphs

**Dancing problem.**
- Exclusive Ivy league party attended by $n$ men and $n$ women.
- Each man knows exactly $k$ women; each woman knows exactly $k$ men.
- Acquaintances are mutual.
- Is it possible to arrange a dance so that each woman dances with a different man that she knows?

**Mathematical reformulation.** Does every $k$-regular bipartite graph have a perfect matching?

**Ex.** Boolean hypercube.
Theorem. [König 1916, Frobenius 1917] Every k-regular bipartite graph has a perfect matching.

Pf. Size of max matching = value of max flow in $G'$. Consider flow:

$$f(u, v) = \begin{cases} 
1/k & \text{if } (u, v) \in E \\
1 & \text{if } u = s \text{ or } v = t \\
0 & \text{otherwise}
\end{cases}$$

- $f$ is a flow and its value = $n$ $\Rightarrow$ perfect matching. □
Feasible matrix rounding.

- Given a p-by-q matrix $D = \{d_{ij}\}$ of real numbers.
- Row i sum = $a_i$, column j sum $b_j$.
- Round each $d_{ij}$, $a_i$, $b_j$ up or down to integer so that sum of rounded elements in each row (column) equals row (column) sum.
- Original application: publishing US Census data.

Goal. Find a feasible rounding, if one exists.

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feasible rounding
Census Tabulation

Feasible matrix rounding.
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Goal. Find a feasible rounding, if one exists.

Remark. "Threshold rounding" can fail.

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feasible rounding
Theorem. Feasible matrix rounding always exists.

Pf. Formulate as a circulation problem with lower bounds.
  - Original data provides circulation (all demands = 0).
  - Integrality theorem ⇒ integral solution ⇒ feasible rounding.