Announcement: Homework 1 due soon!
Due: January 25th at midnight (Blackboard)
Recap: Graphs

Bipartite Graphs
- Definition (2-Colorable)
- Bipartite $\iff$ No Odd Length Cycles
- Using BFS to check if graph is bipartite

Directed Graphs

Directed Acyclic Graphs
- Topological Ordering
- Algorithm to Compute Topological Order
Chapter 4

Greedy Algorithms

Algorithm Design

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Slides by Kevin Wayne.
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4.1 Interval Scheduling
Interval Scheduling

Interval scheduling.
- Job $j$ starts at $s_j$ and finishes at $f_j$.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.
Interval Scheduling: Greedy Algorithms

Greedy template. Consider jobs in some natural order. Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of $s_j$.

- [Earliest finish time] Consider jobs in ascending order of $f_j$.

- [Shortest interval] Consider jobs in ascending order of $f_j - s_j$.

- [Fewest conflicts] For each job $j$, count the number of conflicting jobs $c_j$. Schedule in ascending order of $c_j$. 
**Interval Scheduling: Greedy Algorithms**

**Greedy template.** Consider jobs in some natural order. Take each job provided it's compatible with the ones already taken.

- Counterexample for earliest start time
- Counterexample for shortest interval
- Counterexample for fewest conflicts
Interval Scheduling: Greedy Algorithm

Greedy algorithm. Consider jobs in increasing order of finish time. Take each job provided it's compatible with the ones already taken.

Sort jobs by finish times so that $f_1 \leq f_2 \leq \ldots \leq f_n$.

\[
\begin{align*}
A & \leftarrow \emptyset \\
\text{for } j = 1 \text{ to } n \{ & \\
\quad \text{if (job } j \text{ compatible with } A) & \\
\quad \quad A & \leftarrow A \cup \{j\} \\
\} & \\
\text{return } A
\end{align*}
\]

Implementation. $O(n \log n)$.
- Remember job $j^*$ that was added last to $A$.
- Job $j$ is compatible with $A$ if $s_j \geq f_{j^*}$. 
Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, ... i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, ... j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, ..., i_r = j_r$ for the largest possible value of $r$.

Why not replace job $j_{r+1}$ with job $i_{r+1}$? Job $i_{r+1}$ finishes before $j_{r+1}$. Why not replace job $j_{r+1}$ with job $i_{r+1}$?
**Theorem.** Greedy algorithm is optimal.

**Pf.** (by contradiction)
- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots, i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots, j_m$ denote set of jobs in the optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of $r$.

![Diagram showing greedy and optimal solutions with job $i_{r+1}$ finishing before $j_{r+1}$, solution still feasible and optimal, but contradicts maximality of $r$.]
4.1 Interval Partitioning
Interval Partitioning

Interval partitioning.

- Lecture $j$ starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

Ex: This schedule uses 4 classrooms to schedule 10 lectures.
Interval partitioning.

- Lecture $j$ starts at $s_j$ and finishes at $f_j$.
- Goal: find minimum number of classrooms to schedule all lectures so that no two occur at the same time in the same room.

**Ex:** This schedule uses only 3.
Interval Partitioning: Lower Bound on Optimal Solution

**Def.** The *depth* of a set of open intervals is the maximum number that contain any given time.

**Key observation.** Number of classrooms needed \( \geq \) depth.

**Ex:** Depth of schedule below = 3 \( \Rightarrow \) schedule below is optimal.

\[ a, b, c \text{ all contain } 9:30 \]

**Q.** Does there always exist a schedule equal to depth of intervals?
Interval Partitioning: Greedy Algorithm

**Greedy algorithm.** Consider lectures in increasing order of start time: assign lecture to any compatible classroom.

```
Sort intervals by starting time so that \( s_1 \leq s_2 \leq \ldots \leq s_n \). 
\[ d \leftarrow 0 \]  
\text{number of allocated classrooms} 
for \( j = 1 \) to \( n \) { 
  if (lecture \( j \) is compatible with some classroom \( k \)) 
    schedule lecture \( j \) in classroom \( k \) 
  else 
    allocate a new classroom \( d + 1 \) 
    schedule lecture \( j \) in classroom \( d + 1 \) 
    \[ d \leftarrow d + 1 \] 
} 
```

Implementation. \( O(n \log n) \).
- For each classroom \( k \), maintain the finish time of the last job added.
- Keep the classrooms in a priority queue.
Observation. Greedy algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Greedy algorithm is optimal.

Pf.

- Let \( d \) = number of classrooms that the greedy algorithm allocates.
- Classroom \( d \) is opened because we needed to schedule a job, say \( j \), that is incompatible with all \( d-1 \) other classrooms.
- These \( d \) jobs each end after \( s_j \).
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than \( s_j \).
- Thus, we have \( d \) lectures overlapping at time \( s_j + \varepsilon \).
- Key observation \( \Rightarrow \) all schedules use \( \geq d \) classrooms.
4.4 Shortest Paths in a Graph

shortest route from Wang Hall to Miami Beach
Shortest Path Problem

Shortest path network.
- Directed graph $G = (V, E)$.
- Source $s$, destination $t$.
- Length $\ell_e = \text{length of edge } e$.

Shortest path problem: find shortest directed path from $s$ to $t$.

Cost of path $s$-2-3-5-$t$

\[
= 9 + 23 + 2 + 16
= 50.
\]
Dijkstra's algorithm.

- Maintain a set of explored nodes $S$ for which we have determined the shortest path distance $d(u)$ from $s$ to $u$.
- Initialize $S = \{ s \}$, $d(s) = 0$.
- Repeatedly choose unexplored node $v$ which minimizes the shortest path to some $u$ in explored part, followed by a single edge $(u, v)$.

$$\pi(v) = \min_{e = (u,v) : u \in S} d(u) + \ell_e,$$

Add $v$ to $S$, and set $d(v) = \pi(v)$. 
Dijkstra's algorithm.

- Maintain a set of explored nodes S for which we have determined the shortest path distance d(u) from s to u.
- Initialize S = {s}, d(s) = 0.
- Repeatedly choose unexplored node v which minimizes
  \[ \pi(v) = \min_{u \in S} \min_{e = (u,v)} d(u) + \ell_e, \]
  add v to S, and set d(v) = \pi(v).

  shortest path to some u in explored part, followed by a single edge (u, v)
Dijkstra's Algorithm: Proof of Correctness

Invariant. For each node \( u \in S \), \( d(u) \) is the length of the shortest \( s-u \) path.

Pf. (by induction on \( |S| \))

Base case: \( |S| = 1 \) is trivial.

Inductive hypothesis: Assume true for \( |S| = k \geq 1 \).

- Let \( v \) be next node added to \( S \), and let \( u-v \) be the chosen edge.
- The shortest \( s-u \) path plus \( (u, v) \) is an \( s-v \) path of length \( \pi(v) \).
- Consider any \( s-v \) path \( P \). We'll see that it's no shorter than \( \pi(v) \).
- Let \( x-y \) be the first edge in \( P \) that leaves \( S \), and let \( P' \) be the subpath to \( x \).
- \( P \) is already too long as soon as it leaves \( S \).

\[
\ell(P) \geq \ell(P') + \ell(x,y) \geq d(x) + \ell(x,y) \geq \pi(y) \geq \pi(v)
\]

- \( \ell(P) \) \( \geq \) \( \ell(P') \) \( + \) \( \ell(x,y) \) \( \geq \) \( d(x) \) \( + \) \( \ell(x,y) \) \( \geq \) \( \pi(y) \) \( \geq \) \( \pi(v) \)

\[\uparrow\] \[\uparrow\] \[\uparrow\] \[\uparrow\]
nonnegative inductive defn of Dijkstra chose v weights hypothesis \( \pi(y) \) instead of \( y \)
Dijkstra's Algorithm: Implementation

For each unexplored node, explicitly maintain $\pi(v) = \min_{e=(u,v) : u \in S} d(u) + \ell_e$.

- Next node to explore = node with minimum $\pi(v)$.
- When exploring $v$, for each incident edge $e = (v, w)$, update

  $$\pi(w) = \min \{ \pi(w), \pi(v) + \ell_e \}.$$

**Efficient implementation.** Maintain a priority queue of unexplored nodes, prioritized by $\pi(v)$.

<table>
<thead>
<tr>
<th>PQ Operation</th>
<th>Dijkstra</th>
<th>Array</th>
<th>Binary heap</th>
<th>d-way Heap</th>
<th>Fib heap†</th>
</tr>
</thead>
<tbody>
<tr>
<td>Insert</td>
<td>$n$</td>
<td>$n$</td>
<td>$\log n$</td>
<td>$d \log_d n$</td>
<td>1</td>
</tr>
<tr>
<td>ExtractMin</td>
<td>$n$</td>
<td>$n$</td>
<td>$\log n$</td>
<td>$d \log_d n$</td>
<td>$\log n$</td>
</tr>
<tr>
<td>ChangeKey</td>
<td>$m$</td>
<td>1</td>
<td>$\log n$</td>
<td>$\log_d n$</td>
<td>1</td>
</tr>
<tr>
<td>isEmpty</td>
<td>$n$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Total</td>
<td>$n^2$</td>
<td>$m \log n$</td>
<td>$m \log_{m/n} n$</td>
<td>$m + n \log n$</td>
<td></td>
</tr>
</tbody>
</table>

† Individual ops are amortized bounds
4.2 Scheduling to Minimize Lateness
Scheduling to Minimizing Lateness

Minimizing lateness problem.
- Single resource processes one job at a time.
- Job \( j \) requires \( t_j \) units of processing time and is due at time \( d_j \).
- If \( j \) starts at time \( s_j \), it finishes at time \( f_j = s_j + t_j \).
- Lateness: \( \ell_j = \max \{ 0, f_j - d_j \} \).
- Goal: schedule all jobs to minimize maximum lateness \( L = \max \ell_j \).

Ex:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_j )</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>( d_j )</td>
<td>6</td>
<td>8</td>
<td>9</td>
<td>9</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

\( d_3 = 9 \) \hspace{1cm} \( d_2 = 8 \) \hspace{1cm} \( d_6 = 15 \) \hspace{1cm} \( d_1 = 6 \) \hspace{1cm} \( d_5 = 14 \) \hspace{1cm} \( d_4 = 9 \)

Lateness:
- \( \ell_3 = 2 \)
- \( \ell_2 = 0 \)
- Max lateness = \( 6 \)
Minimizing Lateness: Greedy Algorithms

**Greedy template.** Consider jobs in some order.

- **[Shortest processing time first]** Consider jobs in ascending order of processing time $t_j$.

- **[Earliest deadline first]** Consider jobs in ascending order of deadline $d_j$.

- **[Smallest slack]** Consider jobs in ascending order of slack $d_j - t_j$. 
Greedy template. Consider jobs in some order.

- [Shortest processing time first] Consider jobs in ascending order of processing time $t_j$.

\[
\begin{array}{|c|c|}
\hline
j & t_j & d_j \\
\hline
1 & 1 & 100 \\
2 & 10 & 10 \\
\hline
\end{array}
\]

- [Smallest slack] Consider jobs in ascending order of slack $d_j - t_j$.

\[
\begin{array}{|c|c|}
\hline
j & t_j & d_j \\
\hline
1 & 1 & 2 \\
2 & 10 & 10 \\
\hline
\end{array}
\]
**Minimizing Lateness: Greedy Algorithm**

**Greedy algorithm.** Earliest deadline first.

Sort $n$ jobs by deadline so that $d_1 \leq d_2 \leq \ldots \leq d_n$

$t \leftarrow 0$

for $j = 1$ to $n$

Assign job $j$ to interval $[t, t + t_j]$  
$s_j \leftarrow t$, $f_j \leftarrow t + t_j$  
$t \leftarrow t + t_j$

output intervals $[s_j, f_j]$  

max lateness $= 1$

<table>
<thead>
<tr>
<th>$d_1 = 6$</th>
<th>$d_2 = 8$</th>
<th>$d_3 = 9$</th>
<th>$d_4 = 9$</th>
<th>$d_5 = 14$</th>
<th>$d_6 = 15$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>
Minimizing Lateness: No Idle Time

**Observation.** There exists an optimal schedule with no idle time.

```
<table>
<thead>
<tr>
<th>d = 4</th>
<th>d = 6</th>
<th>d = 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2</td>
<td>3 4 5</td>
<td>7 8 9 10 11</td>
</tr>
</tbody>
</table>
```

**Observation.** The greedy schedule has no idle time.

```
<table>
<thead>
<tr>
<th>d = 4</th>
<th>d = 6</th>
<th>d = 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2</td>
<td>3 4 5</td>
<td>7 8 9 10 11</td>
</tr>
</tbody>
</table>
```
Minimizing Lateness: Inversions

**Def.** Given a schedule $S$, an inversion is a pair of jobs $i$ and $j$ such that: $i < j$ but $j$ scheduled before $i$.

![Diagram showing an inversion between jobs $j$ and $i$ before and after a swap.]

\[ \text{as before, we assume jobs are numbered so that } d_1 \leq d_2 \leq \ldots \leq d_n \]

**Observation.** Greedy schedule has no inversions.

**Observation.** If a schedule (with no idle time) has an inversion, it has one with a pair of inverted jobs scheduled consecutively.
Minimizing Lateness: Inversions

**Def.** Given a schedule S, an inversion is a pair of jobs i and j such that: i < j but j scheduled before i.

**Claim.** Swapping two consecutive, inverted jobs reduces the number of inversions by one and does not increase the max lateness.

**Pf.** Let \( \ell \) be the lateness before the swap, and let \( \ell' \) be it afterwards.

1. \( \ell'_k = \ell_k \) for all \( k \neq i, j \)
2. \( \ell'_i \leq \ell_i \)
3. If job j is late:

\[
\begin{align*}
\ell'_j &= f'_j - d_j \\
      &= f_i - d_j \\
      &\leq f_i - d_i \\
      &\leq \ell_i
\end{align*}
\]

(definition)
Minimizing Lateness: Analysis of Greedy Algorithm

**Theorem.** Greedy schedule $S$ is optimal.

**Pf.** Define $S^*$ to be an optimal schedule that has the fewest number of inversions, and let’s see what happens.

- Can assume $S^*$ has no idle time.
- If $S^*$ has no inversions, then $S = S^*$.
- If $S^*$ has an inversion, let $i-j$ be an adjacent inversion.
  - swapping $i$ and $j$ does not increase the maximum lateness and strictly decreases the number of inversions
  - this contradicts definition of $S^*$  •
Greedy Analysis Strategies

**Greedy algorithm stays ahead.** Show that after each step of the greedy algorithm, its solution is at least as good as any other algorithm's.

**Structural.** Discover a simple "structural" bound asserting that every possible solution must have a certain value. Then show that your algorithm always achieves this bound.

**Exchange argument.** Gradually transform any solution to the one found by the greedy algorithm without hurting its quality.

**Other greedy algorithms.** Kruskal, Prim, Dijkstra, Huffman, ...
4.3 Optimal Caching
Optimal Offline Caching

**Caching.**
- Cache with capacity to store \( k \) items.
- Sequence of \( m \) item requests \( d_1, d_2, \ldots, d_m \).
- Cache hit: item already in cache when requested.
- Cache miss: item not already in cache when requested: must bring requested item into cache, and evict some existing item, if full.

**Goal.** Eviction schedule that minimizes number of cache misses.

**Ex:** \( k = 2 \), initial cache = ab,
- requests: a, b, c, b, c, a, a, b.
**Optimal eviction schedule:** 2 cache misses.
Optimal Offline Caching: Farthest-In-Future

Farthest-in-future. Evict item in the cache that is not requested until farthest in the future.

current cache:  a b c d e f

future queries:  g a b c e d a b b a c d e a f a d e f g h ...

↑ cache miss

↑ eject this one

Theorem. [Bellady, 1960s] FF is optimal eviction schedule.
Pf. Algorithm and theorem are intuitive; proof is subtle.
Reduced Eviction Schedules

Def. A reduced schedule is a schedule that only inserts an item into the cache in a step in which that item is requested.

Intuition. Can transform an unreduced schedule into a reduced one with no more cache misses.
Reduced Eviction Schedules

Claim. Given any unreduced schedule $S$, can transform it into a reduced schedule $S'$ with no more cache misses.

Pf. (by induction on number of unreduced items)
- Suppose $S$ brings $d$ into the cache at time $t$, without a request.
- Let $c$ be the item $S$ evicts when it brings $d$ into the cache.
  - Case 1: $d$ evicted at time $t'$, before next request for $d$.
  - Case 2: $d$ requested at time $t'$ before $d$ is evicted.

\[\text{Case 1}\]

\[\text{Case 2}\]
Farthest-In-Future: Analysis

Theorem. FF is optimal eviction algorithm.

Pf. (by induction on number or requests \( j \))

Invariant: There exists an optimal reduced schedule \( S \) that makes the same eviction schedule as \( S_{FF} \) through the first \( j+1 \) requests.

Let \( S \) be reduced schedule that satisfies invariant through \( j \) requests. We produce \( S' \) that satisfies invariant after \( j+1 \) requests.

- Consider \((j+1)^{st}\) request \( d = d_{j+1} \).
- Since \( S \) and \( S_{FF} \) have agreed up until now, they have the same cache contents before request \( j+1 \).
- Case 1: (\( d \) is already in the cache). \( S' = S \) satisfies invariant.
- Case 2: (\( d \) is not in the cache and \( S \) and \( S_{FF} \) evict the same element). \( S' = S \) satisfies invariant.
Farthest-In-Future: Analysis

Pf. (continued)

- **Case 3:** (d is not in the cache; $S_{FF}$ evicts e; S evicts $f \neq e$).
  - begin construction of $S'$ from S by evicting e instead of f

\[
\begin{array}{c|c|c|c|c}
 & same & e & f \\
\hline
j & S & & & \\
\hline
j+1 & S & & & \\
\hline
& same & d & f \\
\end{array}
\]

- now $S'$ agrees with $S_{FF}$ on first $j+1$ requests; we show that having element $f$ in cache is no worse than having element e
Farthest-In-Future: Analysis

Let \( j' \) be the first time after \( j+1 \) that \( S \) and \( S' \) take a different action, and let \( g \) be item requested at time \( j' \).

- **Case 3a:** \( g = e \). Can't happen with Farthest-In-Future since there must be a request for \( f \) before \( e \).

- **Case 3b:** \( g = f \). Element \( f \) can't be in cache of \( S \), so let \( e' \) be the element that \( S \) evicts.
  - if \( e' = e \), \( S' \) accesses \( f \) from cache; now \( S \) and \( S' \) have same cache
  - if \( e' \neq e \), \( S' \) evicts \( e' \) and brings \( e \) into the cache; now \( S \) and \( S' \) have the same cache

Note: \( S' \) is no longer reduced, but can be transformed into a reduced schedule that agrees with \( S_{FF} \) through step \( j+1 \)
Let $j'$ be the first time after $j+1$ that $S$ and $S'$ take a different action, and let $g$ be item requested at time $j'$.

```
j'
  same  e
S

  same  f
S'
```

must involve $e$ or $f$ (or both)

otherwise $S'$ would take the same action

• Case 3c: $g \neq e, f$. $S$ must evict $e$.
  Make $S'$ evict $f$; now $S$ and $S'$ have the same cache. •

```
j'
  same  g
S

  same  g
S'
```
Caching Perspective

Online vs. offline algorithms.

- Offline: full sequence of requests is known a priori.
- Online (reality): requests are not known in advance.
- Caching is among most fundamental online problems in CS.

LIFO. Evict page brought in most recently.

LRU. Evict page whose most recent access was earliest.

\[ \text{FF with direction of time reversed!} \]

Theorem. FF is optimal offline eviction algorithm.

- Provides basis for understanding and analyzing online algorithms.
- LRU is k-competitive. [Section 13.8]
- LIFO is arbitrarily bad.
Extra Slides
Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.

- Gordon Gecko (Michael Douglas)
Coin Changing

**Goal.** Given currency denominations: 1, 5, 10, 25, 100, devise a method to pay amount to customer using fewest number of coins.

**Ex:** 34¢.

**Cashier's algorithm.** At each iteration, add coin of the largest value that does not take us past the amount to be paid.

**Ex:** $2.89.
**Coin-Changing: Greedy Algorithm**

**Cashier's algorithm.** At each iteration, add coin of the largest value that does not take us past the amount to be paid.

```
Sort coins denominations by value: c_1 < c_2 < ... < c_n.

coins selected
S ← φ
while (x ≠ 0) {
    let k be largest integer such that c_k ≤ x
    if (k = 0)
        return "no solution found"
    x ← x - c_k
    S ← S ∪ {k}
}
return S
```

**Q. Is cashier's algorithm optimal?**
Coin-Changing: Analysis of Greedy Algorithm

**Theorem.** Greed is optimal for U.S. coinage: 1, 5, 10, 25, 100.

**Pf.** (by induction on \( x \))

- Consider optimal way to change \( c_k \leq x < c_{k+1} \): greedy takes coin \( k \).
- We claim that any optimal solution must also take coin \( k \).
  - if not, it needs enough coins of type \( c_1, \ldots, c_{k-1} \) to add up to \( x \)
  - table below indicates no optimal solution can do this
- Problem reduces to coin-changing \( x - c_k \) cents, which, by induction, is optimally solved by greedy algorithm.

<table>
<thead>
<tr>
<th>( k )</th>
<th>( c_k )</th>
<th>All optimal solutions must satisfy</th>
<th>Max value of coins 1, 2, ..., ( k-1 ) in any OPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( P \leq 4 )</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>( N \leq 1 )</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>( N + D \leq 2 )</td>
<td>4 + 5 = 9</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td>( Q \leq 3 )</td>
<td>20 + 4 = 24</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>no limit</td>
<td>75 + 24 = 99</td>
</tr>
</tbody>
</table>
Coin-Changing: Analysis of Greedy Algorithm

Observation. Greedy algorithm is sub-optimal for US postal denominations: 1, 10, 21, 34, 70, 100, 350, 1225, 1500.

Counterexample. 140¢.

• Greedy: 100, 34, 1, 1, 1, 1, 1, 1.
• Optimal: 70, 70.
Selecting Breakpoints
Selecting breakpoints.

- Road trip from Princeton to Palo Alto along fixed route.
- Refueling stations at certain points along the way.
- Fuel capacity = \( C \).
- Goal: makes as few refueling stops as possible.

**Greedy algorithm.** Go as far as you can before refueling.
Selecting Breakpoints: Greedy Algorithm

Truck driver's algorithm.

```
Sort breakpoints so that: 0 = b_0 < b_1 < b_2 < ... < b_n = L

S ← {0} ← breakpoints selected
x ← 0 ← current location

while (x ≠ b_n)
    let p be largest integer such that b_p ≤ x + C
    if (b_p = x)
        return "no solution"
    x ← b_p
    S ← S ∪ {p}
return S
```

Implementation. $O(n \log n)$

- Use binary search to select each breakpoint $p$. 
Selecting Breakpoints: Correctness

**Theorem.** Greedy algorithm is optimal.

**Pf.** (by contradiction)
- Assume greedy is not optimal, and let’s see what happens.
- Let $0 = g_0 < g_1 < \ldots < g_p = L$ denote set of breakpoints chosen by greedy.
- Let $0 = f_0 < f_1 < \ldots < f_q = L$ denote set of breakpoints in an optimal solution with $f_0 = g_0, f_1 = g_1, \ldots, f_r = g_r$ for largest possible value of $r$.
- Note: $g_{r+1} > f_{r+1}$ by greedy choice of algorithm.
Theorem. Greedy algorithm is optimal.

Pf. (by contradiction)

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Edsger W. Dijkstra

The question of whether computers can think is like the question of whether submarines can swim.

Do only what only you can do.

In their capacity as a tool, computers will be but a ripple on the surface of our culture. In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.

The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offence.

APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.