CS 580: Algorithm Design and Analysis

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Recap: Stable Matching Problem

- Definition of a Stable Matching

- Stable Roomate Matching Problem
  - Stable matching does not always exist!

- Gale-Shapley Algorithm (Propose-And-Reject)
  - Proof that Algorithm Terminates in $O(n^2)$ steps
  - Proof that Algorithm Outputs Stable Matching
  - Matching is male-optimal
    - If there are multiple different stable matchings each man gets his best valid partner
  - Matching is female-pessimal
    - If there are multiple different stable matchings each man gets her worst valid partner
Extensions: Matching Residents to Hospitals

**Ex:** Men ≈ hospitals, Women ≈ med school residents.

**Variant 1.** Some participants declare others as unacceptable.

**Variant 2.** Unequal number of men and women.

**Variant 3.** Limited polygamy.

_Gale-Shapley Algorithm Still Works._ Minor modifications to code to handle variations!
Extensions: Matching Residents to Hospitals

Ex: Men ≈ hospitals, Women ≈ med school residents.

Variant 1. Some participants declare others as unacceptable.

Variant 2. Unequal number of men and women.

Variant 3. Limited polygamy.

Def. Matching $S$ unstable if there is a hospital $h$ and resident $r$ such that:

- $h$ and $r$ are acceptable to each other; and
- either $r$ is unmatched, or $r$ prefers $h$ to her assigned hospital; and
- either $h$ does not have all its places filled, or $h$ prefers $r$ to at least one of its assigned residents.
1.2 Five Representative Problems
Interval Scheduling

**Input.** Set of jobs with start times and finish times.

**Goal.** Find maximum cardinality subset of mutually compatible jobs.

Jobs don't overlap
Interval Scheduling

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jobs don't overlap

**Greedy Choice.** Select job with earliest finish time and eliminate incompatible jobs.
Input. Set of jobs with start times and finish times.

Goal. Find maximum cardinality subset of mutually compatible jobs.

Greedy Choice. Select job with earliest finish time and eliminate incompatible jobs.
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**Interval Scheduling**

**Input.** Set of jobs with start times and finish times.

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**Greedy Choice.** Select job with **earliest finish time** and eliminate **incompatible jobs**.
Interval Scheduling

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Chapter 4: We will prove that this greedy algorithm always finds the optimal solution!
Weighted Interval Scheduling

**Input.** Set of jobs with start times, finish times, and weights.  
**Goal.** Find maximum weight subset of mutually compatible jobs.

**Greedy Algorithm No Longer Works!**
Weighted Interval Scheduling

**Input.** Set of jobs with start times, finish times, and weights.

**Goal.** Find maximum weight subset of mutually compatible jobs.

Greedy Algorithm No Longer Works!
Weighted Interval Scheduling

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**Goal.** Find maximum weight subset of mutually compatible jobs.

Problem can be solved using technique called Dynamic Programming.
Bipartite Matching

**Input.** Bipartite graph.

**Goal.** Find **maximum cardinality** matching.

Different from Stable Matching Problem! How?
Bipartite Matching

**Input.** Bipartite graph.

**Goal.** Find **maximum cardinality** matching.

Problem can be solved using Network Flow Algorithms
**Independent Set**

**Input.** Graph.

**Goal.** Find **maximum cardinality** independent set.

\[
\text{subset of nodes such that no two joined by an edge}
\]

**Brute-Force Algorithm:** Check every possible subset.  
**Running Time:** \( \geq 2^n \) steps

**NP-Complete:** Unlikely that efficient algorithm exists!

**Positive:** Can easily check that there is an independent set of size \( k \)
Competitive Facility Location

Input. Graph with weight on each node.

Game. Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

Goal. Select a maximum weight subset of nodes.

Second player can guarantee 20, but not 25.
**Competitive Facility Location**

**Input.** Graph with weight on each node.

**Game.** Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected.

**Goal.** Select a *maximum weight* subset of nodes.

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PSPACE-Complete: Even harder than NP-Complete!

No short proof that player can guarantee value B. (Unlike previous problem)

Second player can guarantee 20, but not 25.
Five Representative Problems

Variations on a theme: independent set.

Interval scheduling: \( n \log n \) greedy algorithm.
Weighted interval scheduling: \( n \log n \) dynamic programming algorithm.
Bipartite matching: \( n^k \) max-flow based algorithm.
Independent set: NP-complete.
Competitive facility location: PSPACE-complete.
Chapter 2
Basics of Algorithm Analysis
2.1 Computational Tractability

"For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing." - Francis Sullivan
As soon as an Analytic Engine exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will arise - By what course of calculation can these results be arrived at by the machine in the shortest time? - Charles Babbage
Polynomial-Time

**Brute force.** For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution.

- Typically takes $2^N$ time or worse for inputs of size $N$.
- Unacceptable in practice.

**Desirable scaling property.** When the input size doubles, the algorithm should only slow down by some constant factor $C$.

There exists constants $c > 0$ and $d > 0$ such that on every input of size $N$, its running time is bounded by $c N^d$ steps.

**Def.** An algorithm is poly-time if the above scaling property holds.

choose $C = 2^d$
Worst-Case Analysis

Worst case running time. Obtain bound on largest possible running time of algorithm on input of a given size N.
- Generally captures efficiency in practice.
- Draconian view, but hard to find effective alternative.

Average case running time. Obtain bound on running time of algorithm on random input as a function of input size N.
- Hard (or impossible) to accurately model real instances by random distributions.
- Algorithm tuned for a certain distribution may perform poorly on other inputs.
Worst-Case Polynomial-Time

**Def.** An algorithm is *efficient* if its running time is polynomial.

**Justification:** *It really works in practice!*
- Although $6.02 \times 10^{23} \times N^{20}$ is technically poly-time, it would be useless in practice.
- In practice, the poly-time algorithms that people develop almost always have low constants and low exponents.
- Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem.

**Exceptions.**
- Some poly-time algorithms do have high constants and/or exponents, and are useless in practice.
- Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare.

- simplex method
- Unix grep
Why It Matters

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds $10^{25}$ years, we simply record the algorithm as taking a very long time.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$n$</th>
<th>$n \log_2 n$</th>
<th>$n^2$</th>
<th>$n^3$</th>
<th>$1.5^n$</th>
<th>$2^n$</th>
<th>$n!$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 10$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>4 sec</td>
</tr>
<tr>
<td>$n = 30$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>18 min</td>
<td>$10^{25}$ years</td>
</tr>
<tr>
<td>$n = 50$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>11 min</td>
<td>36 years</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>1 sec</td>
<td>12,892 years</td>
<td>$10^{17}$ years</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>1 sec</td>
<td>18 min</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 10,000$</td>
<td>$&lt; 1$ sec</td>
<td>$&lt; 1$ sec</td>
<td>2 min</td>
<td>12 days</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 100,000$</td>
<td>$&lt; 1$ sec</td>
<td>2 sec</td>
<td>3 hours</td>
<td>32 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
<tr>
<td>$n = 1,000,000$</td>
<td>1 sec</td>
<td>20 sec</td>
<td>12 days</td>
<td>31,710 years</td>
<td>very long</td>
<td>very long</td>
<td>very long</td>
</tr>
</tbody>
</table>
2.2 Asymptotic Order of Growth
Asymptotic Order of Growth

**Upper bounds.** $T(n)$ is $O(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $T(n) \leq c \cdot f(n)$.

**Lower bounds.** $T(n)$ is $\Omega(f(n))$ if there exist constants $c > 0$ and $n_0 \geq 0$ such that for all $n \geq n_0$ we have $T(n) \geq c \cdot f(n)$.

**Tight bounds.** $T(n)$ is $\Theta(f(n))$ if $T(n)$ is both $O(f(n))$ and $\Omega(f(n))$.

**Ex:** $T(n) = 32n^2 + 17n + 32$.
  - $T(n)$ is $O(n^2)$, $O(n^3)$, $\Omega(n^2)$, $\Omega(n)$, and $\Theta(n^2)$.
  - $T(n)$ is not $O(n)$, $\Omega(n^3)$, $\Theta(n)$, or $\Theta(n^3)$. 
Slight abuse of notation.  $T(n) = O(f(n))$.

- Not transitive:
  - $f(n) = 5n^3; \ g(n) = 3n^2$
  - $f(n) = O(n^3) = g(n)$
  - but $f(n) \neq g(n)$.
- Better notation:  $T(n) \in O(f(n))$.

Meaningless statement.  Any comparison-based sorting algorithm requires at least $O(n \log n)$ comparisons.

- Statement doesn't "type-check."
- Use $\Omega$ for lower bounds.
Properties

Transitivity.

- If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.
- If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.
- If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

- If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.
- If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.
- If $f = \Theta(h)$ and $g = O(h)$ then $f + g = \Theta(h)$. 
Asymptotic Bounds for Some Common Functions

**Polynomials.** \( a_0 + a_1 n + \ldots + a_d n^d \) is \( \Theta(n^d) \) if \( a_d > 0 \).

**Polynomial time.** Running time is \( O(n^d) \) for some constant \( d \) independent of the input size \( n \).

**Logarithms.** \( O(\log_a n) = O(\log_b n) \) for any constants \( a, b > 0 \).

**Logarithms.** For every \( x > 0 \), \( \log n = O(n^x) \).

**Exponentials.** For every \( r > 1 \) and every \( d > 0 \), \( n^d = O(r^n) \).
2.4 A Survey of Common Running Times
Linear Time: $O(n)$

Linear time. Running time is proportional to input size.

Computing the maximum. Compute maximum of $n$ numbers $a_1, \ldots, a_n$. 

```
max ← a_1
for i = 2 to n {
  if (a_i > max)
    max ← a_i
}
```
**Linear Time: \( O(n) \)**

**Merge.** Combine two sorted lists \( A = a_1,a_2,\ldots,a_n \) with \( B = b_1,b_2,\ldots,b_n \) into sorted whole.

**Claim.** Merging two lists of size \( n \) takes \( O(n) \) time.

**Pf.** After each comparison, the length of output list increases by 1.
\textbf{O(n log n) Time}

\textit{O(n log n) time.} Arises in divide-and-conquer algorithms.

\textit{also referred to as linearithmic time}

\textbf{Sorting.} Mergesort and heapsort are sorting algorithms that perform \textit{O(n log n)} comparisons.

\textbf{Largest empty interval.} Given \( n \) time-stamps \( x_1, \ldots, x_n \) on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?

\textbf{O(n log n) solution.} Sort the time-stamps. Scan the sorted list in order, identifying the maximum gap between successive time-stamps.
Quadratic time. Enumerate all pairs of elements.

Closest pair of points. Given a list of $n$ points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest.

$O(n^2)$ solution. Try all pairs of points.

\[
\begin{align*}
\text{min} & \leftarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 \\
\text{for } i = 1 \text{ to } n \{ \\
& \quad \text{for } j = i+1 \text{ to } n \{ \\
& \quad \quad \text{d} \leftarrow (x_i - x_j)^2 + (y_i - y_j)^2 \\
& \quad \quad \text{if } (d < \text{min}) \\
& \quad \quad \quad \text{min} \leftarrow d \\
& \quad \} \\
& \} 
\end{align*}
\]

Remark. $\Omega(n^2)$ seems inevitable, but this is just an illusion.

see chapter 5
Cubic Time: $O(n^3)$

Cubic time. Enumerate all triples of elements.

Set disjointness. Given $n$ sets $S_1, \ldots, S_n$ each of which is a subset of $1, 2, \ldots, n$, is there some pair of these which are disjoint?

$O(n^3)$ solution. For each pairs of sets, determine if they are disjoint.

```plaintext
foreach set $S_i$ {
    foreach other set $S_j$ {
        foreach element $p$ of $S_i$ {
            determine whether $p$ also belongs to $S_j$
        }
        if (no element of $S_i$ belongs to $S_j$)
            report that $S_i$ and $S_j$ are disjoint
    }
}
```
Polynomial Time: $O(n^k)$ Time

**Independent set of size $k$.** Given a graph, are there $k$ nodes such that no two are joined by an edge? \( \text{k is a constant} \)

$O(n^k)$ solution. Enumerate all subsets of $k$ nodes.

```plaintext
foreach subset $S$ of $k$ nodes {
    check whether $S$ is an independent set
    if (S is an independent set)
        report S is an independent set
}
```

- Check whether $S$ is an independent set $= O(k^2)$.
- Number of $k$ element subsets $= \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(2)(1)} \leq \frac{n^k}{k!}$
- $O(k^2 n^k / k!) = O(n^k)$.

poly-time for $k=17$, but not practical
Independent set. Given a graph, what is maximum size of an independent set?

$O(n^2 2^n)$ solution. Enumerate all subsets.

```plaintext
S* ← φ
foreach subset S of nodes {
    check whether S in an independent set
    if (S is largest independent set seen so far)
        update S* ← S
}
```
Heap Data Structure

Min Heap Order: For each node $v$ in the tree
\[ \text{Parent}(v).\text{Value} \leq v.\text{Value} \]

Max Heap Order: For each node $v$ in the tree
\[ \text{Parent}(v).\text{Value} \geq v.\text{Value} \]
Heap Insertion

Min Heap Order: For each node $v$ in the tree

\[ \text{Parent}(v).\text{Value} \leq v.\text{Value} \]
Heap Insertion

Min Heap Order: For each node $v$ in the tree

\[
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\]
Heap Insertion

Min Heap Order: For each node $v$ in the tree

$$\text{Parent}(v).\text{Value} \leq v.\text{Value}$$
Heap Insertion

Min Heap Order: For each node $v$ in the tree
Parent($v$).Value $\leq$ $v$.Value

Theorem 2.12 [KT]: The procedure Heapify-up fixes the heap property and allows us to insert a new element into a heap of $n$ elements in $O(\log n)$ time.
Heap Extract Minimum

Min Heap Order: For each node $v$ in the tree

$$\text{Parent}(v).\text{Value} \leq v.\text{Value}$$

**Theorem 2.13 [KT]:** The procedure Heapify-down fixes the heap property and allows us to delete an element in a heap of $n$ elements in $O(\log n)$ time.
Heap Extract Minimum

Min Heap Order: For each node \( v \) in the tree

\[
\text{Parent}(v).\text{Value} \leq v.\text{Value}
\]

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Heap Summary

Insert: $O(\log n)$
FindMin: $O(1)$
Delete: $O(\log n)$ time
ExtractMin: $O(\log n)$ time

Thought Question: $O(n \log n)$ time sorting algorithm using heaps?