CS 580: Algorithm Design and Analysis

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Announcements: Homework 6 deadline extended to April 24th at 11:59 PM

Course Evaluation Survey: Live until 4/29/2018 at 11:59PM. Your feedback is valued!

13.4 MAX 3-SAT

Recap: Maximum 3-Satisfiability

exactly 3 distinct literals per clause

MAX-3SAT. Given 3-SAT formula, find a truth assignment that satisfies as many clauses as possible.

$$
\begin{aligned}
C_1 &= x_2 \vee \overline{x_3} \vee \overline{x_4} \\
C_2 &= x_2 \vee x_3 \vee \overline{x_4} \\
C_3 &= \overline{x_1} \vee x_2 \vee x_4 \\
C_4 &= \overline{x_1} \vee \overline{x_2} \vee x_3 \\
C_5 &= x_1 \vee \overline{x_2} \vee \overline{x_4}
\end{aligned}
$$

Simple idea. Flip a coin, and set each variable true with probability $\frac{1}{2}$, independently for each variable.

Observation. Random assignment satisfies $\frac{7k}{8}$ of the k clauses in expectation (**proof**: linearity of expectation)

The Probabilistic Method

Corollary. For any instance of 3-SAT, there exists a truth assignment that satisfies at least a 7/8 fraction of all clauses.

Pf. Random variable is at least its expectation some of the time. ▪

Probabilistic method. We showed the existence of a non-obvious property of 3-SAT by showing that a random construction produces it with positive probability!

Maximum 3-Satisfiability: Analysis

Q. Can we turn this idea into a 7/8-approximation algorithm? In general, a random variable can almost always be below its mean.

Lemma. The probability that a random assignment satisfies $\geq$ 7k/8 clauses is at least 1/(8k).

Pf. Let p_j be probability that exactly j clauses are satisfied; let p be probability that $\geq$ 7k/8 clauses are satisfied.

$$\frac{7}{8}k = E[Z] = \sum_{j \geq 0} j \cdot p_j = \sum_{j < \frac{7}{8}k} j \cdot p_j + \sum_{j \geq \frac{7}{8}k} j \cdot p_j$$

$$\leq \left(\frac{7k}{8} - \frac{1}{8}\right) \sum_{j < \frac{7}{8}k} p_j + k \sum_{j \geq \frac{7}{8}k} p_j \leq \left(\frac{7k}{8} - \frac{1}{8}\right) \cdot 1 + kp$$

Rearranging terms yields $p \geq 1 / (8k)$. ▪

Maximum 3-Satisfiability: Analysis

Johnson's algorithm. Repeatedly generate random truth assignments until one of them satisfies $\geq 7k/8$ clauses.

Theorem. Johnson's algorithm is a 7/8-approximation algorithm.

Pf. By previous lemma, each iteration succeeds with probability at least $1/(8k)$. By the waiting-time bound, the expected number of trials to find the satisfying assignment is at most $8k$. ▪

Maximum Satisfiability

Extensions.

- Allow one, two, or more literals per clause.
- Find max weighted set of satisfied clauses.

Theorem. [Asano-Williamson 2000] There exists a 0.784-approximation algorithm for MAX-SAT.

Theorem. [Karloff-Zwick 1997, Zwick+computer 2002] There exists a 7/8-approximation algorithm for version of MAX-3SAT where each clause has at most 3 literals.

Theorem. [Håstad 1997] Unless P = NP, no ρ-approximation algorithm for MAX-3SAT (and hence MAX-SAT) for any $\rho > 7/8$.

↑ very unlikely to improve over simple randomized algorithm for MAX-3SAT

Monte Carlo vs. Las Vegas Algorithms

Monte Carlo algorithm. Guaranteed to run in poly-time, likely to find correct answer.
Ex: Contraction algorithm for global min cut.

Las Vegas algorithm. Guaranteed to find correct answer, likely to run in poly-time.
Ex: Randomized quicksort, Johnson's MAX-3SAT algorithm.

stop algorithm after a certain point

Remark. Can always convert a Las Vegas algorithm into Monte Carlo, but no known method to convert the other way.

RP and ZPP

RP. [Monte Carlo] Decision problems solvable with one-sided error in poly-time.

Can decrease probability of false negative to 2^{-100} by 100 independent repetitions

One-sided error.

- If the correct answer is no, always return no.
- If the correct answer is yes, return yes with probability $\geq \frac{1}{2}$.

ZPP. [Las Vegas] Decision problems solvable in expected poly-time.

running time can be unbounded, but on average it is fast

Theorem. $P \subseteq ZPP \subseteq RP \subseteq NP$.

Fundamental open questions. To what extent does randomization help? Does P = ZPP? Does ZPP = RP? Does RP = NP?

13.5 Randomized Divide-and-Conquer

Quicksort

Sorting. Given a set of n distinct elements S, rearrange them in ascending order.

```
RandomizedQuicksort(S) {
   if |S| = 0 return

   choose a splitter aᵢ ∈ S uniformly at random
   foreach (a ∈ S) {
      if      (a < aᵢ) put a in S⁻
      else if (a > aᵢ) put a in S⁺
   }
   RandomizedQuicksort(S⁻)
   output aᵢ
   RandomizedQuicksort(S⁺)
}
```

Remark. Can implement in-place.

↑ $O(\log n)$ extra space

Quicksort

Running time.

- [Best case.] Select the median element as the splitter: quicksort makes $\Theta(n \log n)$ comparisons.
- [Worst case.] Select the smallest element as the splitter: quicksort makes $\Theta(n^2)$ comparisons.

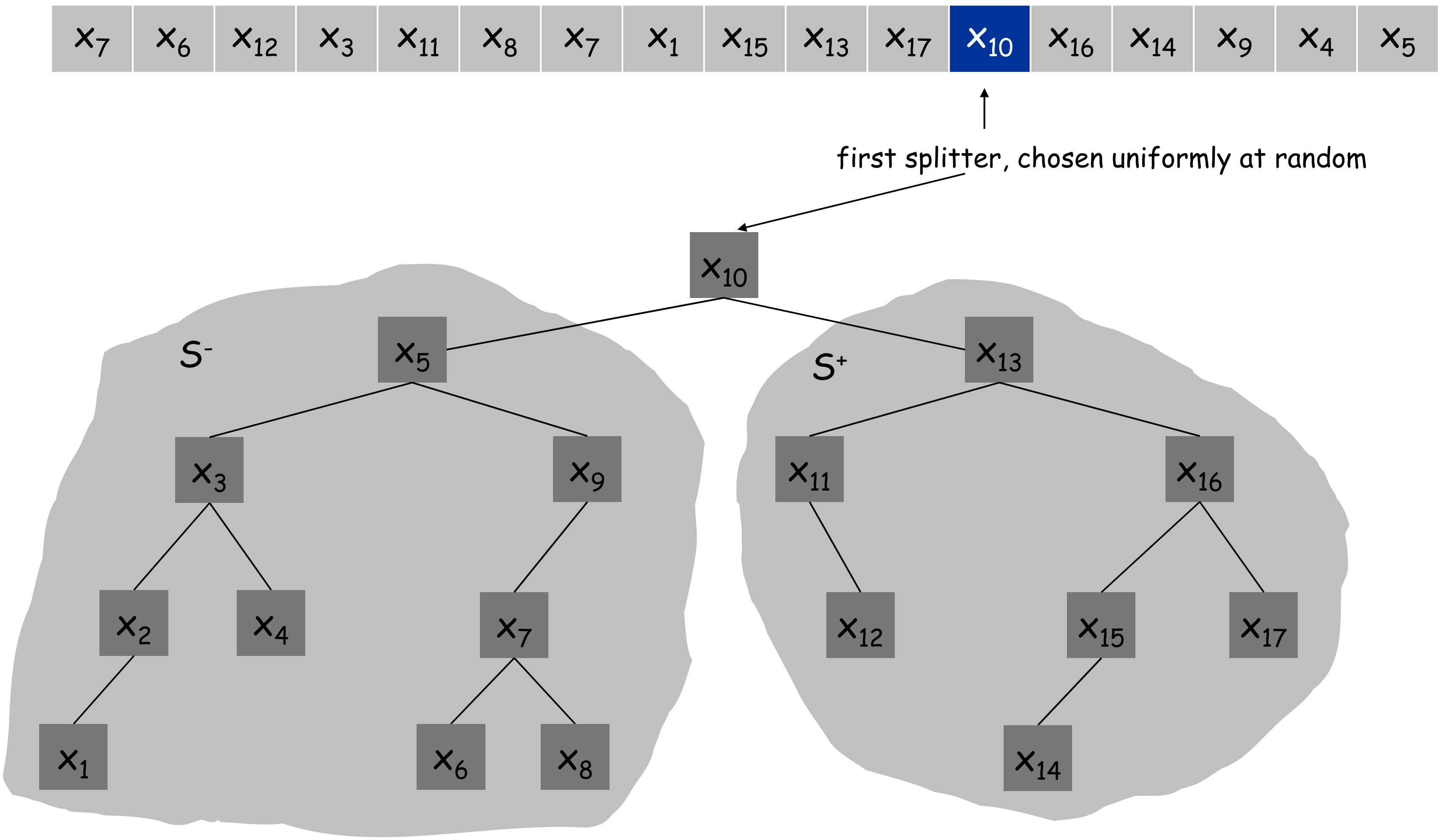
Randomize. Protect against worst case by choosing splitter at random.

Intuition. If we always select an element that is bigger than 25% of the elements and smaller than 25% of the elements, then quicksort makes $\Theta(n \log n)$ comparisons.

Notation. Label elements so that $x_1 < x_2 < \ldots < x_n$.

Quicksort: BST Representation of Splitters

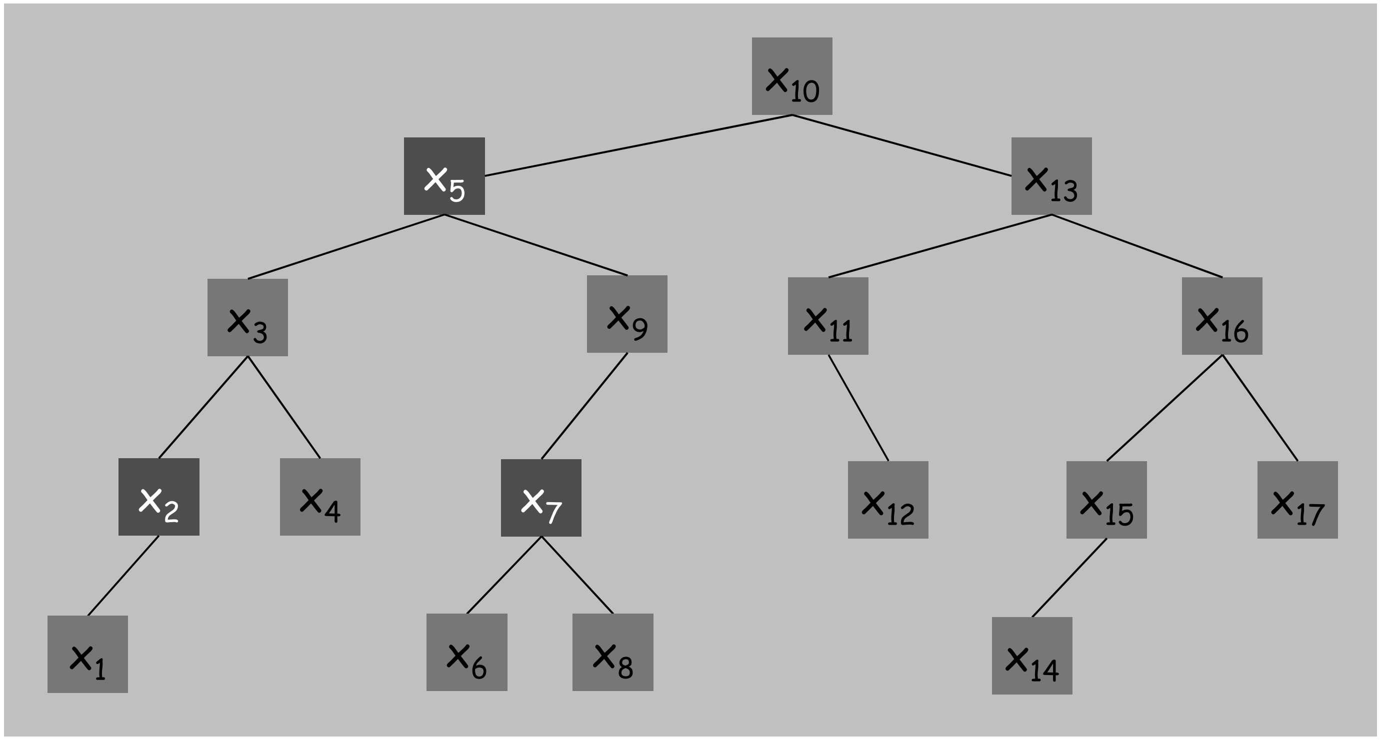
BST representation. Draw recursive BST of splitters.



Observation. Element only compared with its ancestors and descendants.

- x_2 and x_7 are compared if their lca = x_2 or x_7.
- x_2 and x_7 are not compared if their lca = x_3 or x_4 or x_5 or x_6.

Claim. Pr[x_i and x_j are compared] $= \frac{2}{|j-i+1|}$.



Quicksort: Expected Number of Comparisons

Theorem. Expected # of comparisons is O(n log n).
Pf.

$$\sum_{1 \le i < j \le n} \frac{2}{j-i+1} = 2 \sum_{i=1}^{n} \sum_{j=2}^{i} \frac{1}{j} \le 2n \sum_{j=1}^{n} \frac{1}{j} \approx 2n \int_{x=1}^{n} \frac{1}{x} dx = 2n \ln n$$

↑ probability that i and j are compared

Theorem. [Knuth 1973] Stddev of number of comparisons is ~ 0.65N.

Ex. If n = 1 million, the probability that randomized quicksort takes less than 4n ln n comparisons is at least 99.94%.

Chebyshev's inequality. $\Pr[|X - \mu| \ge k\delta] \le 1 / k^2$.

13.9 Chernoff Bounds

Chernoff Bounds (above mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \geq E[X]$ and for any $\delta > 0$, we have

$$\Pr[X > (1+\delta)\mu] < \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right]^{\mu}$$

↑ sum of independent 0-1 random variables is tightly centered on the mean

Pf. We apply a number of simple transformations.

- For any $t > 0$,

$$\Pr[X > (1+\delta)\mu] = \Pr\left[e^{tX} > e^{t(1+\delta)\mu}\right] \leq e^{-t(1+\delta)\mu} \cdot E[e^{tX}]$$

↑ $f(x) = e^{tX}$ is monotone in x

↑ Markov's inequality: $\Pr[X > a] \leq E[X] / a$

- Now $$E[e^{tX}] = E[e^{t \sum_i X_i}] = \prod_i E[e^{t X_i}]$$

↑ definition of X

↑ independence

Chernoff Bounds (above mean)

Pf. (cont)

- Let $p_i = \Pr[X_i = 1]$. Then,

$$E[e^{t X_i}] = p_i e^t + (1 - p_i) e^0 = 1 + p_i (e^t - 1) \le e^{p_i (e^t - 1)}$$

for any $\alpha \ge 0$, $1+\alpha \le e^{\alpha}$

- Combining everything:

$$\Pr[X > (1+\delta)\mu] \le e^{-t(1+\delta)\mu} \prod_i E[e^{tX_i}] \le e^{-t(1+\delta)\mu} \prod_i e^{p_i(e^t - 1)}$$

previous slide

inequality above

$\sum_i p_i = E[X] \le \mu$

$$\le e^{-t(1+\delta)\mu} e^{\mu(e^t - 1)}$$

- Finally, choose $t = \ln(1 + \delta)$. ▪

Chernoff Bounds (below mean)

Theorem. Suppose $X_1, \ldots, X_n$ are independent 0-1 random variables. Let $X = X_1 + \ldots + X_n$. Then for any $\mu \leq E[X]$ and for any $0 < \delta < 1$, we have

$$\Pr[X < (1-\delta)\mu] \; < \; e^{-\delta^2 \mu / 2}$$

Pf idea. Similar.

Remark. Not quite symmetric since only makes sense to consider $\delta < 1$.

13.10 Load Balancing

Load Balancing

Load balancing. System in which m jobs arrive in a stream and need to be processed immediately on n identical processors. Find an assignment that balances the workload across processors.

Centralized controller. Assign jobs in round-robin manner. Each processor receives at most $\lceil m/n \rceil$ jobs.

Decentralized controller. Assign jobs to processors uniformly at random. How likely is it that some processor is assigned "too many" jobs?

Load Balancing

Analysis.

- Let X_i = number of jobs assigned to processor i.
- Let Y_{ij} = 1 if job j assigned to processor i, and 0 otherwise.
- We have $E[Y_{ij}] = 1/n$
- Thus, $X_i = \sum_j Y_{ij}$, and $\mu = E[X_i] = 1$.
- Applying Chernoff bounds with $\delta = c - 1$ yields $\Pr[X_i > c] < \dfrac{e^{c-1}}{c^c}$

- Let $\gamma(n)$ be number x such that $x^x = n$, and choose $c = e\,\gamma(n)$.

$$\Pr[X_i > c] < \frac{e^{c-1}}{c^c} < \left(\frac{e}{c}\right)^c = \left(\frac{1}{\gamma(n)}\right)^{e\gamma(n)} < \left(\frac{1}{\gamma(n)}\right)^{2\gamma(n)} = \frac{1}{n^2}$$

- Union bound $\Rightarrow$ with probability $\geq 1 - 1/n$ no processor receives more than $e\,\gamma(n) = \Theta(\log n / \log\log n)$ jobs.

Fact: this bound is asymptotically tight: with high probability, some processor receives $\Theta(\log n / \log\log n)$

Load Balancing: Many Jobs

Theorem. Suppose the number of jobs $m = 16n \ln n$. Then on average, each of the n processors handles $\mu = 16 \ln n$ jobs. With high probability every processor will have between half and twice the average load.

Pf.

- Let X_i, Y_{ij} be as before.
- Applying Chernoff bounds with $\delta = 1$ yields

$$\Pr[X_i > 2\mu] < \left(\frac{e}{4}\right)^{16n\ln n} < \left(\frac{1}{e}\right)^{\ln n} = \frac{1}{n^2}$$

$$\Pr[X_i < \tfrac{1}{2}\mu] < e^{-\frac{1}{2}\left(\frac{1}{2}\right)^2(16n\ln n)} = \frac{1}{n^2}$$

- Union bound $\Rightarrow$ every processor has load between half and twice the average with probability $\geq 1 - 2/n$. ▪

13.6 Universal Hashing

Dictionary Data Type

Dictionary. Given a universe U of possible elements, maintain a subset $S \subseteq U$ so that inserting, deleting, and searching in S is efficient.

Dictionary interface.

- `Create()`: Initialize a dictionary with $S = \phi$.
- `Insert(u)`: Add element $u \in U$ to S.
- `Delete(u)`: Delete u from S, if u is currently in S.
- `Lookup(u)`: Determine whether u is in S.

Challenge. Universe U can be extremely large so defining an array of size |U| is infeasible.

Applications. File systems, databases, Google, compilers, checksums P2P networks, associative arrays, cryptography, web caching, etc.

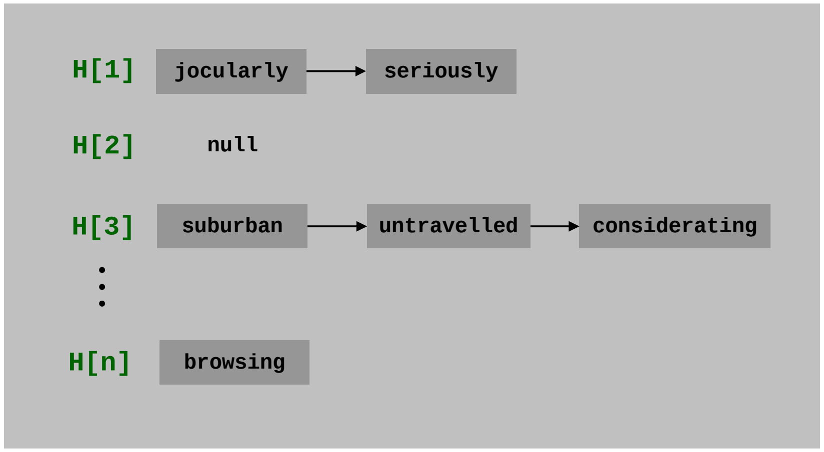
Hashing

Hash function. $h : U \rightarrow \{ 0, 1, \ldots, n-1 \}$.

Hashing. Create an array H of size n. When processing element u, access array element H[h(u)].

Collision. When $h(u) = h(v)$ but $u \neq v$.

- A collision is expected after $\Theta(\sqrt{n})$ random insertions. This phenomenon is known as the "birthday paradox."
- Separate chaining: H[i] stores linked list of elements u with $h(u) = i$.



Ad Hoc Hash Function

Ad hoc hash function.

```
int h(String s, int n) {
   int hash = 0;
   for (int i = 0; i < s.length(); i++)
      hash = (31 * hash) + s[i];
   return hash % n;
}
```

hash function ala Java string library

Deterministic hashing. If $|U| \geq n^2$, then for any fixed hash function h, there is a subset $S \subseteq U$ of n elements that all hash to same slot. Thus, $\Theta(n)$ time per search in worst-case.

Q. But isn't ad hoc hash function good enough in practice?

Algorithmic Complexity Attacks

When can't we live with ad hoc hash function?

- Obvious situations: aircraft control, nuclear reactors.
- Surprising situations: denial-of-service attacks.

malicious adversary learns your ad hoc hash function (e.g., by reading Java API) and causes a big pile-up in a single slot that grinds performance to a halt

Real world exploits. [Crosby-Wallach 2003]

- Bro server: send carefully chosen packets to DOS the server, using less bandwidth than a dial-up modem
- Perl 5.8.0: insert carefully chosen strings into associative array.
- Linux 2.4.20 kernel: save files with carefully chosen names.

Hashing Performance

Idealistic hash function. Maps m elements uniformly at random to n hash slots.

- Running time depends on length of chains.
- Average length of chain = α = m / n.
- Choose $n \approx m \Rightarrow$ on average O(1) per insert, lookup, or delete.

Challenge. Achieve idealized randomized guarantees, but with a hash function where you can easily find items where you put them.

Approach. Use randomization in the choice of h.

↑

adversary knows the randomized algorithm you're using,
but doesn't know random choices that the algorithm makes

Universal Hashing

Universal class of hash functions. [Carter-Wegman 1980s]

- For any pair of elements u, v $\in$ U, $\Pr_{h \in H}\left[h(u) = h(v) \right] \le 1/n$
- Can select random h efficiently. (← chosen uniformly at random)
- Can compute h(u) efficiently.

Ex. U = { a, b, c, d, e, f }, n = 2.

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1

$H = \{h_1, h_2\}$
$\Pr_{h \in H}[h(a) = h(b)] = 1/2$
$\Pr_{h \in H}[h(a) = h(c)] = 1$
$\Pr_{h \in H}[h(a) = h(d)] = 0$
. . .

not universal

	a	b	c	d	e	f
$h_1(x)$	0	1	0	1	0	1
$h_2(x)$	0	0	0	1	1	1
$h_3(x)$	0	0	1	0	1	1
$h_4(x)$	1	0	0	1	1	0

$H = \{h_1, h_2, h_3, h_4\}$
$\Pr_{h \in H}[h(a) = h(b)] = 1/2$
$\Pr_{h \in H}[h(a) = h(c)] = 1/2$
$\Pr_{h \in H}[h(a) = h(d)] = 1/2$
$\Pr_{h \in H}[h(a) = h(e)] = 1/2$
$\Pr_{h \in H}[h(a) = h(f)] = 0$
. . .

universal

Universal Hashing

Universal hashing property. Let H be a universal class of hash functions; let $h \in H$ be chosen uniformly at random from H; and let $u \in U$. For any subset $S \subseteq U$ of size at most n, the expected number of items in S that collide with u is at most 1.

Pf. For any element $s \in S$, define indicator random variable $X_s = 1$ if $h(s) = h(u)$ and 0 otherwise. Let X be a random variable counting the total number of collisions with u.

$$E_{h \in H}[X] = E[\textstyle\sum_{s \in S} X_s] = \sum_{s \in S} E[X_s] = \sum_{s \in S} \Pr[X_s = 1] \le \sum_{s \in S} \frac{1}{n} = |S| \frac{1}{n} \le 1$$

↑ linearity of expectation

↑ X_s is a 0-1 random variable

↑ universal (assumes $u \notin S$)

Designing a Universal Family of Hash Functions

Theorem. [Chebyshev 1850] There exists a prime between n and 2n.

Modulus. Choose a prime number $p \approx n$. ← no need for randomness here

Integer encoding. Identify each element $u \in U$ with a base-p integer of r digits: $x = (x_1, x_2, \ldots, x_r)$.

Hash function. Let A = set of all r-digit, base-p integers. For each $a = (a_1, a_2, \ldots, a_r)$ where $0 \le a_i < p$, define

$$h_a(x) = \left(\sum_{i=1}^{r} a_i x_i \right) \bmod p$$

Hash function family. $H = \{ h_a : a \in A \}$.

Designing a Universal Class of Hash Functions

Theorem. $H = \{ h_a : a \in A \}$ is a universal class of hash functions.

Pf. Let $x = (x_1, x_2, \ldots, x_r)$ and $y = (y_1, y_2, \ldots, y_r)$ be two distinct elements of U. We need to show that $\Pr[h_a(x) = h_a(y)] \leq 1/n$.

- Since $x \neq y$, there exists an integer j such that $x_j \neq y_j$.
- We have $h_a(x) = h_a(y)$ iff

$$a_j \underbrace{(y_j - x_j)}_{z} = \underbrace{\sum_{i \neq j} a_i (x_i - y_i)}_{m} \mod p$$

- Can assume a was chosen uniformly at random by first selecting all coordinates a_i where $i \neq j$, then selecting a_j at random. Thus, we can assume a_i is fixed for all coordinates $i \neq j$.
- Since p is prime, $a_j z = m \mod p$ has at most one solution among p possibilities. ← see lemma on next slide
- Thus $\Pr[h_a(x) = h_a(y)] = 1/p \leq 1/n$. ▪

Number Theory Facts

Fact. Let p be prime, and let $z \neq 0 \bmod p$. Then $\alpha z = m \bmod p$ has at most one solution $0 \leq \alpha < p$.

Pf.

- Suppose α and β are two different solutions.
- Then $(\alpha - \beta)z = 0 \bmod p$; hence $(\alpha - \beta)z$ is divisible by p.
- Since $z \neq 0 \bmod p$, we know that z is not divisible by p; it follows that $(\alpha - \beta)$ is divisible by p.
- This implies $\alpha = \beta$. ▪

Bonus fact. Can replace "at most one" with "exactly one" in above fact.

Pf idea. Euclid's algorithm.

Extra Slides