Cryptography
CS 555

Week 9:

- Number Theory + Public Key Crypto

Readings: Katz and Lindell Chapter 8, B.1, B.2

Limits of Symmetric Crypto

- **Key-Exchange Problem:**
  - Obi-Wan and Yoda want to communicate securely
  - Suppose that
    - Obi-Wan and Yoda don't have time to meet privately and generate a shared secret key
    - ~~Use AES-GCM~~ (requires shared secret key!)
    - Trusted Intermediary: If Obi-Wan and Yoda both have secret keys with Anakin they can exchange a secret key via the trusted party.

Limits of Symmetric Crypto

- Key-Exchange Problem:
  - Obi-Wan and Yoda want to communicate securely
  - Suppose that
    - Obi-Wan and Yoda don't have time to meet privately and generate a shared secret key
    - ~~Use AES-GCM~~ (requires shared secret key!)
    - **Trusted Intermediary:** If Obi-Wan and Yoda both have secret keys with Anakin ($K_{Y,A}$ and $K_{O,A}$) they can exchange a secret key via the trusted party.
      - Obi-Wan picks a key K, computes $c = \mathrm{Enc}_{K_{O,A}}(K)$ and sends c to Anakin with instructions to re-encrypt and forward to Yoda.
      - Anakin computes $\mathrm{K} = \mathrm{Dec}_{K_{O,A}}(c)$ and $c' = \mathrm{Enc}_{K_{Y,A}}(K)$ and forwards to Yoda.
      - Yoda recovers $K = \mathrm{Dec}_{K_{Y,A}}(c')$
      - Anakin also learns the secret key
    - **Remark**: Obi-Wan and Yoda both trust Anakin, but would prefer to keep the key private just in case.

Limits of Symmetric Crypto

- Key-Exchange Problem:
  - Obi-Wan and Yoda want to communicate securely
  - **Trusted Intermediary:** If Obi-Wan and Yoda both have secret keys with Anakin ($K_{Y,A}$ and $K_{O,A}$) they can exchange a secret key via the trusted party.
  - **Remark**: Obi-Wan and Yoda both trust Anakin, but would prefer to keep the key private just in case.

Limits of Symmetric Crypto

- Key-Exchange Problem:
  - Obi-Wan and Yoda want to communicate securely
  - Suppose that
    - Obi-Wan and Yoda don't have time to meet privately and generate one
    - Obi-Wan and Yoda share an asymmetric key with Anakin
    - Can they use Anakin to exchange a secret key?
    - **Remark**: Obi-Wan and Yoda both trust Anakin, but would prefer to keep the key private just in case.
- Need for Public-Key Crypto
  - We can solve the key-exchange problem using public-key cryptography.
  - No solution is known using symmetric key cryptography alone

Symmetric Key Explosion Problem

- Suppose we have n people and each pair of people want to be able to maintain a secure communication channel.
  - How many private keys per person?
  - **Answer**: n-1

- Key Explosion Problem
  - n can get very big if you are Google or Amazon!

Public Key Encryption: Basic Terminology

- Plaintext/Plaintext Space
  - A message $\mathrm{m} \in \mathcal{M}$
- Ciphertext $\mathrm{c} \in \mathcal{C}$
- **Public/Private Key Pair $(\boldsymbol{pk}, \boldsymbol{sk}) \in \boldsymbol{\mathcal{K}}$**

Public Key Encryption Syntax

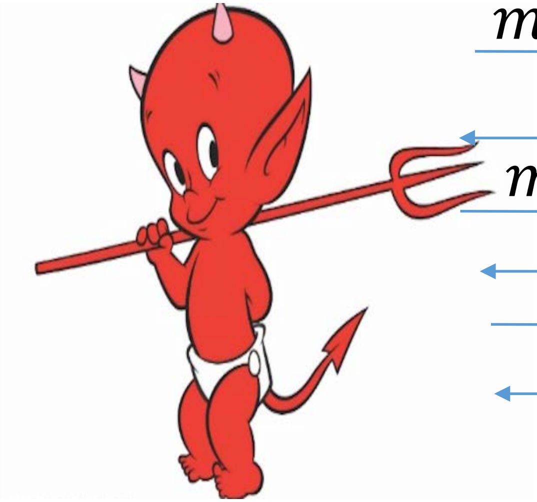
- Three Algorithms
  - $\mathrm{Gen}(1^n, R)$ (Key-generation algorithm)
    - Input: Random Bits R
    - Output: $(\boldsymbol{pk}, \boldsymbol{sk}) \in \boldsymbol{\mathcal{K}}$
  - $\mathrm{Enc}_{\mathrm{pk}}(m) \in \mathcal{C}$ (Encryption algorithm)
  - $\mathrm{Dec}_{\mathrm{sk}}(c)$ (Decryption algorithm)
    - Input: Secret key sk and a ciphertex c
    - Output: a plaintext message $\mathrm{m} \in \mathcal{M}$

Alice must run key generation algorithm in advance an publishes the public key: pk

Assumption: Adversary only gets to see pk (not sk)

- **Invariant**: $\mathrm{Dec}_{sk}(\mathrm{Enc}_{pk}(m))=m$

CPA-Security ($\mathrm{PubK}_{A,\Pi}^{\mathrm{LR-cpa}}(\mathrm{n})$)



Public Key: pk

m_0^1, m_1^1

$\boldsymbol{c_1 = \mathrm{Enc}_{pk}(m_b^1)}$

m_0^2, m_1^2

$\boldsymbol{c_2 = \mathrm{Enc}_{pk}(m_b^2)}$

m_0^3, m_1^3

$\boldsymbol{c_3 = \mathrm{Enc}_{pk}(m_b^3)}$

⋮

b'

Random bit b

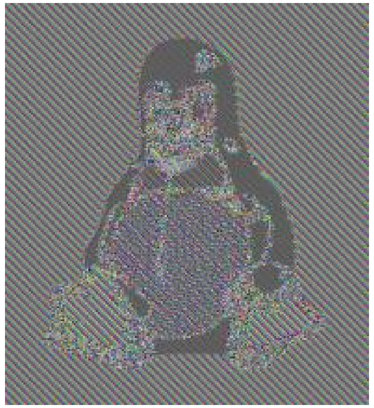
(pk,sk) = Gen(.)

$$\forall PPT\ A\ \exists \mu\ (\mathrm{negligible})\ \mathrm{s.t}$$

$$\Pr\left[\mathrm{PubK}_{A,\Pi}^{\mathrm{LR-cpa}}(\mathrm{n}) = 1\right] \leq \frac{1}{2} + \mu(n)$$

Public Key Crypto

- **Fact 1:** CPA Security and Eavesdropping Security are Equivalent
  - **Key Insight:** The attacker has the public key so he doesn't gain anything from being able to query the encryption oracle!
- **Fact 2:** Any deterministic encryption scheme is not CPA-Secure
  - Historically overlooked in many real world public key crypto systems
- **Fact 3:** No Public Key Cryptosystem can achieve Perfect Secrecy!
  - Exercise 11.1
  - **Hint:** Unbounded attacker can keep encrypting the message m using the public key to recover all possible encryptions of m.



- **Key Question:** How do we achieve CPA/CCA-Secure Public Key Encryption?

Number Theory

- Key tool behind (most) public key-crypto
  - RSA, El-Gamal, Diffie-Hellman Key Exchange

- Aside: don’t worry we will still use symmetric key crypto
  - It is more efficient in practice
  - First step in many public key-crypto protocols is to generate symmetric key
    - Then communicate using authenticated encryption e.g., AES-GCM

Polynomial Time Factoring Algorithm?

FindPrimeFactor

Input: N

For i=1,…,N

  if N/i is an integer then

    Output i

Did we just break RSA?

Running time: O(N) steps

Correctness: Always returns a factor

Polynomial Time Factoring Algorithm?

FindPrimeFactor

Input: N

For i=1,…,N

  if N/i is an integer then

    Output i

We measure running time of an arithmetic algorithm (multiply, divide, GCD, remainder) in terms of the number of bits necessary to encode the inputs.

How many bits $||N||$ to encode N?
Answer: $||N||= \log_2(N)$

Running time: O(N) steps

Correctness: Always returns a factor

Polynomial Time Operations on Integers

Polynomial time in $||a||$and $||b||$

- Addition
- Multiplication
- Division with Remainder
  - **Input: a** and divisor **b**
  - **Output**: quotient q and remainder r < **b** such that

$$\boldsymbol{a} = q\boldsymbol{b} + r$$

Convenient Notation: r = **a** mod **b**
Note 1: We require that quotient q and remainder r are both integers
Note 2: If remainder is $r = 0$ (i.e., $\boldsymbol{a} = q\boldsymbol{b} + 0$) we say that **b** divides **a** (Notation: b|a)

- Greatest Common Divisor
  - **Example:** gcd(9,15) = 3
- Extended GCD(**a**,**b**)
  - Output integers X,Y such that

$$X\boldsymbol{a} + Y\boldsymbol{b} = \gcd(\boldsymbol{a}, \boldsymbol{b})$$

Polynomial Time Operations on Integers

- Division with Remainder
  - **Input:** a and b
  - **Output**: quotient q and remainder r < b such that

$$\boldsymbol{a} = q\boldsymbol{b} + r$$

- Greatest Common Divisor
  - **Key Observation:** if $\boldsymbol{a} = q\boldsymbol{b} + r$

  Then gcd(**a,b**) = gcd(r, **b**)=gcd(**a** mod **b**, **b**)

  Proof:

  - Let d = gcd(**a**,**b**). Then d divides both a and b. Thus, d also divides r=a-qb.

    →d=gcd(**a**,**b**) $\leq$ gcd(r, **b**)
  - Let d' = gcd(r, **b**). Then d' divides both b and r. Thus, d' also divides a = qb+r.

    →gcd(**a**,**b**) $\geq$ gcd(r, **b**)=d'
  - Conclusion: d=d'.

More Polynomial Time Operations on Integers

- **(Modular Arithmetic)** The following operations are polynomial time in $\|a\|$ and $\|b\|$ and $\|N\|$.

1. Compute [**a** mod **N**]
2. Compute sum [(**a**+**b**) mod **N**], difference [(**a**-**b**) mod **N**] or product [**ab** mod **N**]
3. Determine whether **a** has an inverse $\mathbf{a}^{-1}$ such that 1=[$\mathbf{a}\mathbf{a}^{-1}$ mod **N**]
4. Find $\mathbf{a}^{-1}$ if it exists
5. Compute the exponentiation [$\mathbf{a}^{\mathbf{b}}$ mod **N**]

More Polynomial Time Operations on Integers

Remark: Part 3 and 4 use extended GCD algorithm

- (Modular Arithmetic) The
  in |

1. Compute [**a** mod **N**]
2. Compute sum [(
   [**ab** mod **N**]
3. Determine whether **a** has an inverse $\mathbf{a}^{-1}$ such that 1=[$\mathbf{aa}^{-1}$ mod **N**]
4. Find $\mathbf{a}^{-1}$ if it exists
5. Compute the exponentiation [$\mathbf{a}^{\mathbf{b}}$ mod **N**]

More Polynomial Time Operations on Integers

- (Modular Arithmetic) The following operations are polynomial time in in $||a||$ and $||b||$ and $||N||$.

1. Compute [**a** mod **N**]
2. Compute sum [(**a**+**b**) mod **N**], difference [(**a**-**b**) mod **N**] or product [**ab** mod **N**]
3. Determine whether **a** has an inverse $\mathbf{a}^{-1}$ such that 1=[$\mathbf{aa}^{-1}$ mod **N**]
4. Find $\mathbf{a}^{-1}$ if it exists
   - **Note:** $\mathbf{a}^{-1}$ exists if and only if GCD(a,N) = 1.
   - **Extended Euclidean Algorithm:** Finds integers x,y s.t. ax+Ny =GCD(a,N)=1.
   - **Define:** $\mathbf{a}^{-1}$ **=[x** mod **N**] and observe [$\mathbf{aa}^{-1}$ mod **N**]=[ax-Ny mod **N**] = GCD(a,N)=1.
5. Compute the exponentiation [$\mathbf{a}^{\mathbf{b}}$ mod **N**]

More Polynomial Time Operations on Integers

- (Modular Arithmetic) The following operations are polynomial time in in $||a||$ and $||b||$ and $||N||$.

1. Compute the exponentiation [$\mathbf{a}^b$ mod **N**]

Attempt 1:

X =1

For i=1,…,b

    X = X*a

What is wrong?

More Polynomial Time Operations on Integers

(Modular Arithmetic) The following operations are polynomial time in $\|a\|$, $\|b\|$ and $\|N\|$.

1. Compute the exponentiation [$\mathbf{a}^{\mathbf{b}}$ mod **N**]

Attempt 2:

If (b=0) return 1

X[0]=a;

For i=1,…,$\log_2$(b)+1

X[i] = X[i-1]*X[i-1] // Invariant: X[i] = $\boldsymbol{a}^{2^i}$

$$[\mathbf{a}^{\mathbf{b}} \bmod \mathbf{N}]=\boldsymbol{a}^{\sum_i \boldsymbol{b}[i]2^i} \bmod \mathbf{N}$$

$$= \prod_i \mathrm{X[i]}^{\mathbf{b}[\mathrm{i}]} \bmod \mathbf{N}$$

What is wrong?

The number of bits in $a^{2^{\|b\|+1}}$ is O($2^{\|b\|+1}$).

More Polynomial Time Operations on Integers

(Modular Arithmetic) The following operations are polynomial time in $\|a\|$, $\|b\|$ and $\|N\|$.

1. Compute the exponentiation [$\mathbf{a}^{\mathbf{b}}$ mod **N**]

Fixed Algorithm:

If (b=0) return 1

X[0]=a;

For i=1,…,$\log_2$(b)+1

  X[i] = X[i-1]*X[i-1] mod **N**   // Invariant: X[i] = $\boldsymbol{a}^{2^i}$ mod **N**

[$\mathbf{a}^{\mathbf{b}}$ mod **N**]=$\boldsymbol{a}^{\sum_i \boldsymbol{b}[i]2^i} \text{mod } \mathbf{N}$

$$= \prod_i \mathrm{X[i]}^{\mathbf{b}[\mathrm{i}]} \text{ mod } \mathbf{N}$$

More Polynomial Time Operations on Integers

(Sampling) Let

$$\mathbb{Z}_N = \{1, \ldots, N\}$$
$$\mathbb{Z}_N^* = \{x \in \mathbb{Z}_N \mid \gcd(N, x) = 1\}$$

Examples:

$$\mathbb{Z}_6^* = \{1,5\}$$

$$\mathbb{Z}_7^* = \{1,2,3,4,5,6\}$$

More Polynomial Time Operations on Integers

(Sampling) Let

$$\mathbb{Z}_N = \{1, \ldots, N\}$$
$$\mathbb{Z}_N^* = \{x \in \mathbb{Z}_N \mid \gcd(N, x) = 1\}$$

- There is a probabilistic polynomial time algorithm (in |N|) to sample from $\mathbb{Z}_N^*$ and $\mathbb{Z}_N$
- Algorithm to sample from $\mathbb{Z}_N^*$ is allowed to output "fail" with negligible probability in $||N||$.
- Conditioned on not failing sample must be uniform.

Useful Facts

Fact: $x, y \in \mathbb{Z}_{\mathrm{N}}^{*} \rightarrow [xy \bmod \mathrm{N}] \in \mathbb{Z}_{\mathrm{N}}^{*}$ where $\mathbb{Z}_{N}^{*} = \{x \in \mathbb{Z}_N \mid \gcd(N, x) = 1\}$

Example 1: $\mathbb{Z}_8^* = \{1,3,5,7\}$

$$[3 \times 7 \bmod 8] = [21 \bmod 8] = [5 \bmod 8] \in \mathbb{Z}_{\mathrm{N}}^{*}$$

Proof (by contradiction): Let d:=gcd(xy,N)

Suppose d>1 then for some prime p and integer q we have d=pq.

Now p must divide N and xy (by definition) and hence p must divide either x or y.

(WLOG) say p divides x. In this case gcd(x,N)=p > 1, which means $x \notin \mathbb{Z}_{\mathrm{N}}^{*}$

More Useful Facts

$$x, y \in \mathbb{Z}^*_{\mathrm{N}} \rightarrow [xy \bmod \mathrm{N}] \in \mathbb{Z}^*_{\mathrm{N}}$$

Fact 1: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}^*_{\mathrm{N}}|$ then for any $x \in \mathbb{Z}^*_{\mathrm{N}}$ we have

$$\left[x^{\boldsymbol{\phi}(\boldsymbol{N})} \bmod \mathrm{N}\right] = 1$$

Example: $\mathbb{Z}^*_8 = \{1,3,5,7\}$, $\phi(8) = 4$

$$\left[3^{\mathbf{4}} \bmod 8\right] = [9 \times 9 \bmod 8] = 1$$
$$\left[5^{\mathbf{4}} \bmod 8\right] = [25 \times 25 \bmod 8] = 1$$
$$\left[7^{\mathbf{4}} \bmod 8\right] = [49 \times 49 \bmod 8] = 1$$

More Useful Facts

$$x, y \in \mathbb{Z}_{\mathrm{N}}^{*} \to [xy \bmod \mathrm{N}] \in \mathbb{Z}_{\mathrm{N}}^{*}$$

Fact 1: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}_{\mathrm{N}}^{*}|$ then for any $x \in \mathbb{Z}_{\mathrm{N}}^{*}$ we have $\left[x^{\boldsymbol{\phi}(\boldsymbol{N})} \bmod \mathrm{N}\right] = 1$

Fact 2: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}_{\mathrm{N}}^{*}|$ and let $N = \prod_{i=1}^{m} p_i^{e_i}$, where each p_i is a distinct prime number and $e_i > 0$ then

$$\boldsymbol{\phi}(\boldsymbol{N}) = \prod_{i=1}^{m} (p_i - 1) p_i^{e_i - 1} = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)$$

Recap

- Polynomial time algorithms (in bit lengths $||\boldsymbol{a}||$, $||\boldsymbol{b}||$ and $||\mathbf{N}||$) to do important stuff
  - GCD(**a**,**b**)
  - Find inverse $\mathbf{a}^{-1}$ of **a** such that 1=[$\mathbf{a}\mathbf{a}^{-1}$ mod **N**] (if it exists)
  - PowerMod: [$\mathbf{a}^{\mathbf{b}}$ mod **N**]
  - Draw uniform sample from $\mathbb{Z}_N^* = \{x \in \mathbb{Z}_N | \gcd(N, x) = 1\}$
    - Randomized PPT algorithm

More Useful Facts

$$x, y \in \mathbb{Z}^*_{\mathrm{N}} \rightarrow [xy \bmod \mathrm{N}] \ \in \mathbb{Z}^*_{\mathrm{N}}$$

Fact 1: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}^*_{\mathrm{N}}|$ then for any $x \in \mathbb{Z}^*_{\mathrm{N}}$ we have

$$\left[x^{\boldsymbol{\phi}(\boldsymbol{N})} \bmod \mathrm{N}\right] = 1$$

Example: $\mathbb{Z}^*_8 = \{1,3,5,7\}$, $\phi(8) = 4$

$$\left[3^{\mathbf{4}} \bmod 8\right] = [9 \times 9 \bmod 8] = 1$$

$$\left[5^{\mathbf{4}} \bmod 8\right] = [25 \times 25 \bmod 8] = 1$$

$$\left[7^{\mathbf{4}} \bmod 8\right] = [49 \times 49 \bmod 8] = 1$$

More Useful Facts

$$x, y \in \mathbb{Z}^*_{\mathrm{N}} \rightarrow [xy \bmod \mathrm{N}] \in \mathbb{Z}^*_{\mathrm{N}}$$

Fact 1: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}^*_{\mathrm{N}}|$ then for any $x \in \mathbb{Z}^*_{\mathrm{N}}$ we have $\left[x^{\boldsymbol{\phi}(\boldsymbol{N})} \bmod \mathrm{N}\right] = 1$

Fact 2: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}^*_{\mathrm{N}}|$ and let $N = \prod_{i=1}^{m} p_i^{e_i}$, where each p_i is a distinct prime number and $e_i > 0$ then

$$\boldsymbol{\phi}(\boldsymbol{N}) = \prod_{i=1}^{m} (p_i - 1) p_i^{e_i - 1} = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)$$

More Useful Facts

Fact 2: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}_N^*|$ and let $N = \prod_{i=1}^{m} p_i^{e_i}$, where each p_i is a distinct prime number and $e_i > 0$ then

$$\boldsymbol{\phi}(\boldsymbol{N}) = \prod_{i=1}^{m} (p_i - 1) p_i^{e_i - 1} = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)$$

Example 0: Let p be a prime so that $\mathbb{Z}_p^* = \{1, \dots, p-1\}$

$$\boldsymbol{\phi}(\boldsymbol{p}) = p\left(1 - \frac{1}{p}\right) = p - 1$$

More Useful Facts

Fact 2: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}_{\mathrm{N}}^*|$ and let $N = \prod_{i=1}^{m} p_i^{e_i}$, where each p_i is a distinct prime number and $e_i > 0$ then

$$\boldsymbol{\phi}(\boldsymbol{N}) = \prod_{i=1}^{m} (p_i - 1) p_i^{e_i - 1} = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)$$

Example 1: N = 9 = 3^2 (m=1, e_1=2)

$$\boldsymbol{\phi}(\boldsymbol{9}) = \prod_{i=1}^{1} (p_i - 1) p_i^{2-1} = 2 \times 3$$

More Useful Facts

Example 1: N = 9 = 3^2 (m=1, e_1=2)

$$\boldsymbol{\phi}(\mathbf{9}) = \prod_{i=1}^{1} (p_i - 1) p_i^{2-1} = 2 \times 3$$

Double Check: $\mathbb{Z}_9^* = \{1,2,4,5,7,8\}$

More Useful Facts

Fact 2: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}^*_{\mathrm{N}}|$ and let $N = \prod_{i=1}^{m} p_i^{e_i}$, where each p_i is a distinct prime number and $e_i > 0$ then

$$\boldsymbol{\phi}(\boldsymbol{N}) = \prod_{i=1}^{m} (p_i - 1) p_i^{e_i - 1} = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)$$

Example 2: N = 15 = 5×3 (m=2, e_1=e_2=1)

$$\boldsymbol{\phi}(\mathbf{15}) = \prod_{i=1}^{2} (p_i - 1) p_i^{1-1} = (5-1)(3-1) = 8$$

More Useful Facts

Example 2: N = 15 = 5×3 (m=2, e_1=e_2=1)

$$\boldsymbol{\phi}(\mathbf{15}) = \prod_{i=1}^{2} (p_i - 1) p_i^{1-1} = (5-1)(3-1) = 8$$

Double Check: $\mathbb{Z}_{15}^{*} = \{1,2,4,7,8,11,13,14\}$

I count 8 elements in $\mathbb{Z}_{15}^{*}$

More Useful Facts

Fact 2: Let $\boldsymbol{\phi}(\boldsymbol{N}) = |\mathbb{Z}_N^*|$ and let $N = \prod_{i=1}^{m} p_i^{e_i}$, where each p_i is a distinct prime number and $e_i > 0$ then

$$\boldsymbol{\phi}(\boldsymbol{N}) = \prod_{i=1}^{m} (p_i - 1) p_i^{e_i - 1} = n \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right)$$

Special Case: N = pq   (p and q are distinct primes)

$$\boldsymbol{\phi}(\boldsymbol{N}) = (p-1)(q-1)$$

More Useful Facts

Special Case: N = pq (p and q are distinct primes)

$$\boldsymbol{\phi}(\boldsymbol{N}) = (p-1)(q-1)$$

Proof Sketch: If $x \in \mathbb{Z}_N$ is not divisible by p or q then $x \in \mathbb{Z}_N^*$**.** How many elements are not in $\mathbb{Z}_N^*$?

- **Multiples of p:** p, 2p, 3p,…,pq (q multiples of p)
- **Multiples of q:** q, 2q,…,pq (p multiples of q)
- **Double Counting?** N=pq is in both lists. Any other duplicates?
- No! cq = dp → q divides d (since, gcd(p,q)=1) and consequently $\mathrm{d} \geq q$
  - Hence, $\mathrm{dp} \geq pq = N$

More Useful Facts

Special Case: N = pq (p and q are distinct primes)

$$\boldsymbol{\phi}(\boldsymbol{N}) = (p-1)(q-1)$$

Proof Sketch: If $x \in \mathbb{Z}_N$ is not divisible by p or q then $x \in \mathbb{Z}_N^*$**.** How many elements are not in $\mathbb{Z}_N^*$?

- **Multiples of p:** p, 2p, 3p,…,pq (q multiples of p)
- **Multiples of q:** q, 2q,…,pq (p multiples of q)
- **Answer:** p+q-1 elements are not in $\mathbb{Z}_N^*$

$$\boldsymbol{\phi}(\boldsymbol{N}) = \boldsymbol{N} - (\boldsymbol{p} + \boldsymbol{q} - \mathbf{1})$$
$$= \mathbf{pq} - \mathbf{p} - \mathbf{q} + \mathbf{1} = (\mathbf{p} - \mathbf{1})(\mathbf{q} - \mathbf{1})$$

Groups

Definition: A (finite) group is a (finite) set $\mathbb{G}$ with a binary operation $\circ$ (over G) for which we have

- (**Closure**:) For all $g, h \in \mathbb{G}$ we have $g \circ h \in \mathbb{G}$
- **(Identity**:) There is an element $e \in \mathbb{G}$ such that for all $g \in \mathbb{G}$ we have
$$g \circ e = g = e \circ g$$
- **(Inverses**:) For each element $g \in \mathbb{G}$ we can find $h \in \mathbb{G}$ such that $g \circ h = e = h \circ g$. We say that h is the inverse of g.
- (**Associativity:**) For all $g_1, g_2, g_3 \in \mathbb{G}$ we have
$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3)$$

We say that the group is **abelian** if

- (**Commutativity:**) For all $g, h \in \mathbb{G}$ we have $g \circ h = h \circ g$

Groups

Definition: A (finite) group is a (finite) set $\mathbb{G}$ with a binary operation $\circ$ (over G) for which we have

- (**Closure**:) For all $\mathrm{g}, \mathrm{h} \in \mathbb{G}$ we have $\mathrm{g} \circ \mathrm{h} \in \mathbb{G}$
- **(Identity**:) There is an element $\mathrm{e} \in \mathbb{G}$ such that for all $\mathrm{g} \in \mathbb{G}$ we have
$$\mathrm{g} \circ \mathrm{e} = \mathrm{g} = \mathrm{e} \circ \mathrm{g}$$
- **(Inverses**:) For each element $\mathrm{g} \in \mathbb{G}$ we can find $\mathrm{h} \in \mathbb{G}$ such that $\mathrm{g} \circ \mathrm{h} = \mathrm{e} = \mathrm{h} \circ \mathrm{g}$. We say that h is the inverse of g.
- (**Associativity:**) For all $\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3 \in \mathbb{G}$ we have
$$(\mathrm{g}_1 \circ \mathrm{g}_2) \circ \mathrm{g}_3 = \mathrm{g}_1 \circ (\mathrm{g}_2 \circ \mathrm{g}_3)$$

Fact: The identity is unique + inverses must be unique

Proof: If e and e' are both identity then $\mathrm{e} = \mathrm{e} \circ \mathrm{e}' = \mathrm{e}'$

If h and h' are both inverses of g then $\mathrm{h} = \mathrm{h} \circ \mathrm{e} = \mathrm{h} \circ (\mathrm{g} \circ \mathrm{h}') = (\mathrm{g} \circ \mathrm{h}) \circ h' = h'$.

Associativity

Abelian Groups (Examples)

- **Example 1:** $\mathbb{Z}_N$ when ∘ denotes addition modulo N
- Identity: 0, since 0 ∘ x =[0+x mod N] = [x mod N].
- Inverse of x? Set x^{-1}=N-x so that [x^{-1}+x mod N] = [N-x+x mod N] = 0.

- **Example 2:** $\mathbb{Z}_N^*$ when ∘ denotes multiplication modulo N
- Identity: 1, since 1∘ x =[1(x) mod N] = [x mod N].
- Inverse of x? Run extended GCD to obtain integers a and b such that

$$ax + bN = \gcd(x, N) = 1$$

Observe that: x^{-1} = a. Why?

Abelian Groups (Examples)

- **Example 1:** $\mathbb{Z}_N$ when $\circ$ denotes addition modulo N
- Identity: 0, since 0 $\circ$ x =[0+x mod N] = [x mod N].
- Inverse of x? Set x^{-1}=N-x so that [x^{-1}+x mod N] = [N-x+x mod N] = 0.

- **Example 2:** $\mathbb{Z}_N^*$ when $\circ$ denotes multiplication modulo N
- Identity: 1, since 1$\circ$ x =[1(x) mod N] = [x mod N].
- Inverse of x? Run extended GCD to obtain integers a and b such that

$$ax + bN = \gcd(x, N) = 1$$

Observe that: x^{-1} = a, since [ax mod N] = [1-bN mod N] = 1

Groups

Lemma 8.13: Let $\mathbb{G}$ be a group with a binary operation $\circ$ (over G) and let $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathbb{G}$. If $\mathrm{a} \circ \mathrm{c} = \mathrm{b} \circ \mathrm{c}$ then $\mathrm{a} = \mathrm{b}$.

Proof Sketch: Apply the unique inverse to c^{-1} both sides.

$\mathrm{a} \circ \mathrm{c} = \mathrm{b} \circ \mathrm{c}$ → $(\mathrm{a} \circ \mathrm{c}) \circ c^{-1} = (\mathrm{b} \circ \mathrm{c}) \circ c^{-1}$

→ $\mathrm{a} \circ (\mathrm{c} \circ c^{-1}) = \mathrm{b} \circ (\mathrm{c} \circ c^{-1})$

→ $\mathrm{a} \circ (e) = \mathrm{b} \circ (e)$

→ $\mathrm{a} = \mathrm{b}$

(**Remark**: it is not to difficult to show that a group has a *unique* identity and that inverses are *unique*).

Groups

Lemma 8.13: Let $\mathbb{G}$ be a group with a binary operation $\circ$ (over G) and let $\mathrm{a}, \mathrm{b}, \mathrm{c} \in \mathbb{G}$. If $\mathrm{a} \circ \mathrm{c} = \mathrm{b} \circ \mathrm{c}$ then $\mathrm{a} = \mathrm{b}$.

Proof Sketch: Apply the unique inverse to c^{-1} both sides.

$\mathrm{a} \circ \mathrm{c} = \mathrm{b} \circ \mathrm{c}$ → $(\mathrm{a} \circ \mathrm{c}) \circ c^{-1} = (\mathrm{b} \circ \mathrm{c}) \circ c^{-1}$

→ $\mathrm{a} \circ (\mathrm{c} \circ c^{-1}) = \mathrm{b} \circ (\mathrm{c} \circ c^{-1})$

→ $\mathrm{a} \circ (e) = \mathrm{b} \circ (e)$

→ $\mathrm{a} = \mathrm{b}$

(**Remark**: it is not to difficult to show that a group has a *unique* identity and that inverses are *unique*).

Group Exponentiation

Definition: Let $\mathbb{G}$ be a group with a binary operation $\circ$ (over G) let m be a positive integer and let $\mathrm{g} \in \mathbb{G}$ be a group element then we define

$$g^m := \underbrace{\mathrm{g} \circ \cdots \circ \mathrm{g}}_{\text{m times}}$$

Theorem: Let $\mathbb{G}$ be finite group with size $\mathrm{m} = |\mathbb{G}|$ and let $\mathrm{g} \in \mathbb{G}$ be a group element then g^m=1 (where 1 denotes the unique identity of $\mathbb{G}$).

Group Exponentiation

Theorem 8.14: Let $\mathbb{G}$ be finite group with size $\mathrm{m} = |\mathbb{G}|$ and let $\mathrm{g} \in \mathbb{G}$ be a group element then g^m=1 (where 1 denotes the unique identity of $\mathbb{G}$).

Proof: (for abelian group) Let $\mathbb{G} = \{g_1, \dots, g_m\}$ then we claim

$$g_1 \circ \cdots \circ g_m = (g \circ g_1) \circ \cdots \circ (g \circ g_m)$$

Why? If $(g_i \circ g) = \left(g_j \circ g\right)$ then $g_j = g_i$ (by Lemma 8.13)

Group Exponentiation

Theorem 8.14: Let $\mathbb{G}$ be finite group with size $\mathrm{m} = |\mathbb{G}|$ and let $\mathrm{g} \in \mathbb{G}$ be a group element then g^m=1 (where 1 denotes the unique identity of $\mathbb{G}$).

Proof: (for abelian group) Let $\mathbb{G} = \{g_1, \ldots, g_m\}$ then we claim

$$g_1 \circ \cdots \circ g_m = (g \circ g_1) \circ \cdots \circ (g \circ g_m)$$

Because $\mathbb{G}$ is abelian we can re-arrange terms

$$1 \circ (g_1 \circ \cdots \circ g_m) = (g^m) \circ (g_1 \circ \cdots \circ g_m)$$

By Lemma 8.13 we have $1 = g^m$. QED

Group Exponentiation

Theorem 8.14: Let $\mathbb{G}$ be finite group with size $\mathrm{m} = |\mathbb{G}|$ and let g $\in \mathbb{G}$ be a group element then g^m=1 (where 1 denotes the unique identity of $\mathbb{G}$).

Corollary 8.15: Let $\mathbb{G}$ be finite group with size $\mathrm{m} = |\mathbb{G}| > 1$ and let g $\in \mathbb{G}$ be a group element then for any integer x we have $g^x = g^{[x \: mod \: m]}$.

Proof: $g^x = g^{qm+[x \: mod \: m]} = 1 \times g^{[x \: mod \: m]}$, where q is unique integer such that x=qm+ $[x \: mod \: m]$

Group Exponentiation

Special Case: $\mathbb{Z}_N^*$ is a group of size $\boldsymbol{\phi(N)}$ so we have now proved

Corollary 8.22: For any $g \in \mathbb{Z}_{\mathrm{N}}^*$ and integer x we have

$$[g^x \bmod \mathrm{N}] = \left[g^{[x \bmod \boldsymbol{\phi(N)}]} \bmod \mathrm{N}\right]$$

Chinese Remainder Theorem

Theorem: Let N = pq (where gcd(p,q)=1) be given and let $f: \mathbb{Z}_{\mathrm{N}} \rightarrow \mathbb{Z}_p \times \mathbb{Z}_q$ be defined as follows

$$f(x) = ([x \bmod p], [x \bmod q])$$

then

- f is a bijective mapping (invertible)
- f and its inverse $f^{-1}: \mathbb{Z}_p \times \mathbb{Z}_q \rightarrow \mathbb{Z}_{\mathrm{N}}$ can be computed efficiently
- $f(x+y) = f(x) + f(y) = ([x+y \bmod p], [x+y \bmod q])$
- The restriction of f to $\mathbb{Z}_N^*$ yields a bijective mapping to $\mathbb{Z}_p^* \times \mathbb{Z}_q^*$
- For inputs $x, y \in \mathbb{Z}_N^*$ we have $f(x)f(y) = f(xy)$

Chinese Remainder Theorem

Application of CRT: Faster computation modulo N=pq.

Example: Compute [11^{53} mod 15]
f(11)=([-1 mod 3],[1 mod 5])
f(11^{53}) =([$(-1)^{53}$ mod 3],[1^{53} mod 5])= (-1,1)

f^{-1}(-1,1)=11

Thus, 11=[11^{53} mod 15]

CS 555: Week 10: Topic 1
Finding Prime Numbers, RSA

RSA Key-Generation

KeyGeneration(1^n)

Step 1: Pick two random n-bit primes p and q

Step 2: Let N=pq, $\phi(N) = (p-1)(q-1)$

Step 3: …

Question: How do we accomplish step one?

Bertrand's Postulate

Theorem 8.32. For any n > 1 the fraction of n-bit integers that are prime is at least $^{1}/_{3n}$.

GenerateRandomPrime(1^n)
For i=1 to $3n^2$:
  p'← $\{0,1\}^{n-1}$
  p← $1||p'$
  if isPrime(p) **then**
    return p
return fail

Can we do this in polynomial time?

Bertrand's Postulate

Theorem 8.32. For any n > 1 the fraction of n-bit integers that are prime is at least $^1/_{3n}$.

GenerateRandomPrime(1^n)

For i=1 to $3n^2$:

p' ← $\{0,1\}^{n-1}$

p ← $1||p'$

if isPrime(p) **then**

return p

return fail

Assume for now that we can run isPrime(p). What are the odds that the algorithm fails?

On each iteration the probability that p is not a prime is $\left(1-\frac{1}{3n}\right)$

We fail if we pick a non-prime in all $3n^2$ iterations. The probability of failure is at most

$$\left(1-\frac{1}{3n}\right)^{3n^2} = \left(\left(1-\frac{1}{3n}\right)^{3n}\right)^{n} \leq e^{-n}$$

isPrime(p): Miller-Rabin Test

- We can check for primality of p in polynomial time in $||p||$.

Theory: Deterministic algorithm to test for primality.

- See breakthrough paper “Primes is in P”

Practice: Miller-Rabin Test (randomized algorithm)

- **Guarantee 1:** If p is prime then the test outputs YES
- **Guarantee 2:** If p is not prime then the test outputs NO except with negligible probability.

https://www.cse.iitk.ac.in/users/manindra/algebra/primality_v6.pdf

The “Almost” Miller-Rabin Test

Input: Integer N and parameter 1^t

Output: “prime” or “composite”

for i=1 to t:

  a $\leftarrow$ {1,…,N-1}

  if $a^{N-1} \neq 1 \text{ mod N}$ then return “composite”

Return “prime”

Claim: If N is prime then algorithm always outputs “prime”

Proof: For any a ∈ {1,…,N−1} we have $a^{N-1} = a^{\phi(N)} = 1 \; mod \; N$

The "Almost" Miller-Rabin Test

Input: Integer N and parameter 1^t

Output: "prime" or "composite"

for i=1 to t:

 a ← {1,…,N-1}

 if $a^{N-1} \neq 1 \bmod \mathrm{N}$ then return "composit

Return "prime"

Need a bit of extra work to handle Carmichael numbers (see textbook).

Fact: If N is composite and not a Carmichael number then the algorithm outputs "composite" with probability

$$1 - 2^{-t}$$

Miller-Rabin Primality Test

Input: Integer N and parameter 1^t

Output: “prime” or “composite”

If Even(N) or **PerfectPower(N)** return “composite”

Else find u (odd) and $r \geq 1$ s.t. $\mathrm{N} - 1 = 2^r u$

for j=1 to t:

  if $a^u \neq \pm 1 \bmod \mathrm{N}$ and $a^{2^i u} \neq -1 \bmod \mathrm{N}$ for all $1 \leq \ i \leq \ r - 1$

    return “composite”

Return “prime”

Miller-Rabin Primality Test

Lemma: If p is prime and $x^2 = 1 \bmod \mathrm{p}$ then
$$x = \pm 1 \bmod \mathrm{p}$$

Input: Integer N and parameter 1^t

Output: “prime” or “composite”

If Even(N) or **PerfectPower(N)** return “compo

Else find u (odd) and $r \geq 1$ s.t. $\mathrm{N} - 1 =$

for j=1 to t:

  if $a^u \neq \pm 1 \bmod \mathrm{N}$ and $a^{2^i u} \neq -1 \bmod \mathrm{N}$ for all $1 \leq i \leq r - 1$

    return “composite”

Return “prime”

Miller-Rabin Primality Test

Input: Integer N and parameter 1^t

Output: "prime" or "composite"

If Even(N) or **PerfectPower(N**) return "composite"

Else find u (odd) and $r \geq 1$ s.t. $\mathrm{N} - 1 = 2^r u$

for j=1 to t:

if $a^u \neq \pm 1 \bmod \mathrm{N}$ and $a^{2^i u} \neq -1 \bmod \mathrm{N}$ for all $1 \leq i \leq r - 1$

return "composite"

Return "prime"

Observe:

$$\left(a^{2^{r-1}u}\right)^2 = a^{N-1} \bmod \mathrm{N}$$
$$= 1 \quad \bmod \mathrm{N}$$

$$\left(a^{2^i u}\right)^2 - 1$$
$$= \left(a^{2^{i-1}u} - 1\right)\left(a^{2^{i-1}u} + 1\right) + 1$$

Miller-Rabin Primality Test

Input: Integer N and parameter 1^t

Output: "prime" or "composite"

If Even(N) or **PerfectPower(N**) return "composite"

Else find u (odd) and $r \geq 1$ s.t. $\mathrm{N} - 1 = 2^r u$

for j=1 to t:

  if $a^u \neq \pm 1 \bmod \mathrm{N}$ and $a^{2^i u} \neq -1 \bmod \mathrm{N}$ for all $1 \leq i \leq r-1$

    return "composite"

Return "prime"

Observe:

$$\left(a^{2^{r-1}u}\right)^2 = a^{N-1} \bmod \mathrm{N}$$
$$= 1 \bmod \mathrm{N}$$

If N is prime we won't return composite

$$\left(a^{2^r u}\right) - 1 = \left(a^{2^{r-1}u} - 1\right)\left(a^{2^{r-1}u} + 1\right)$$
$$= \cdots = \left(a^{2^{r-2}u} - 1\right)\left(a^{2^{r-2}u} + 1\right)\left(a^{2^{r-1}u} + 1\right)$$

Miller-Rabin Primality Test

Input: Integer N and parameter 1^t

Output: "prime" or "composite"

If Even(N) or **PerfectPower(N)** return "cor

Else find u (odd) and $r \geq 1$ s.t. $\mathrm{N} - 1 = 2$

for j=1 to t:

if $a^u \neq \pm 1 \bmod \mathrm{N}$ and $a^{2^i u} \neq -1 \bmod \mathrm{N}$ for all $1 \leq i \leq r - 1$

return "composite"

Return "prime"

Observe:

$$\left(a^{2^{r-1}u}\right)^2 = a^{N-1} \bmod \mathrm{N}$$
$$= 1 \quad \bmod \mathrm{N}$$

One of the factors must be 0 (mod N)

If N is prime we won't return composite

$$\mathbf{0} = \left(a^{2^r u}\right) - \mathbf{1} = \cdots = (a^u - \mathbf{1}) \prod_{i=0}^{r-1} \left(a^{2^i u} + \mathbf{1}\right)$$

Back to RSA Key-Generation

KeyGeneration(1^n)

Step 1: Pick two random n-bit primes p and q

Step 2: Let N=pq, $\phi(N) = (p-1)(q-1)$

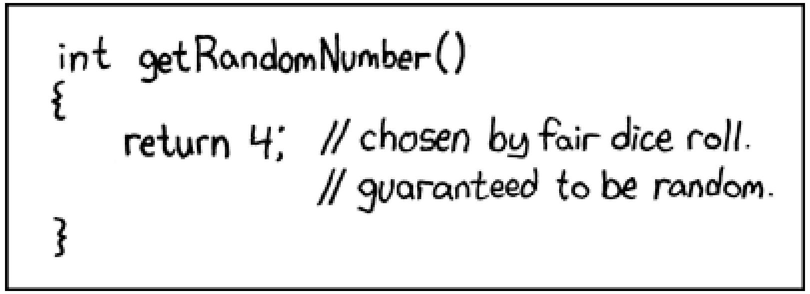
Step 3: Pick e > 1 such that gcd(e, $\phi(N)$)=1

Step 4: Set d=[e^{-1} mod $\phi(N)$]   (secret key)

Return: N, e, d

- How do we find d?
- **Answer:** Use extended gcd algorithm to find e^{-1}mod $\phi(N)$.

Be Careful Where You Get Your “Random Bits!”



- RSA Keys Generated with weak PRG
  - Implementation Flaw
  - Unfortunately Commonplace
- Resulting Keys are Vulnerable
  - Sophisticated Attack
  - Coppersmith’s Method

The Return of Coppersmith's Attack: Practical Factorization of Widely Used RSA Moduli (CCS 2017)

(Plain) RSA Encryption

- Public Key: PK=(N,e)
- Message $\mathrm{m} \in \mathbb{Z}_{\mathrm{N}}$

$$\mathbf{Enc_{PK}}(\mathrm{m}) = [m^e \bmod \mathrm{N}]$$

- **Remark:** Encryption is efficient if we use the power mod algorithm.

(Plain) RSA Decryption

- Secret Key: SK=(N,d)
- Ciphertext $\mathrm{c} \in \mathbb{Z}_{\mathrm{N}}$

$$\mathbf{Dec_{SK}(c)} = [c^d \bmod \mathrm{N}]$$

- **Remark 1:** Decryption is efficient if we use the power mod algorithm.
- **Remark 2:** Suppose that $\mathrm{m} \in \mathbb{Z}_{\mathrm{N}}^*$ and let c=**Enc$_{\mathbf{PK}}$(**m**)** $= [m^e \bmod \mathrm{N}]$

$$\begin{aligned} \mathbf{Dec_{SK}(c)} &= [(m^e)^d \bmod \mathrm{N}] = [m^{ed} \bmod \mathrm{N}] \\ &= \left[m^{[ed\ mod\ \boldsymbol{\phi}(\boldsymbol{N})]} \bmod \mathrm{N}\right] \\ &= [m^1 \bmod \mathrm{N}] = m \end{aligned}$$

RSA Decryption

- Secret Key: SK=(N,d)
- Ciphertext $c \in \mathbb{Z}_N$

$$\textbf{Dec}_{\textbf{SK}}\textbf{(c)} = [c^d \bmod \mathrm{N}]$$

- **Remark 1:** Decryption is efficient if we use the power mod algorithm.
- **Remark 2:** Suppose that $\mathrm{m} \in \mathbb{Z}_N^*$ and let c=**Enc$_{\textbf{PK}}$(m)** $= [m^e \bmod \mathrm{N}]$ **then**

$$\textbf{Dec}_{\textbf{SK}}\textbf{(c)} = m$$

- **Remark 3:** Even if $\mathrm{m} \in \mathbb{Z}_N - \mathbb{Z}_N^*$ and let $\mathrm{c} =$ **Enc$_{\textbf{PK}}$(m)** $= [m^e \bmod \mathrm{N}]$ **then**

$$\textbf{Dec}_{\textbf{SK}}\textbf{(c)} = m$$

  - Use Chinese Remainder Theorem to show this

$$ed = 1 + k(p-1)(q-1)$$

$$\rightarrow \mathrm{f}\left(c^d\right) = \left([m^{ed} \bmod \mathrm{p}], [m^{ed} \bmod \mathrm{q}]\right) = ([m^1 \bmod \mathrm{p}], [m^1 \bmod \mathrm{q}])$$

$$\rightarrow f^{-1}\left(\mathrm{f}\left(c^d\right)\right) = f^{-1}([m^1 \bmod \mathrm{p}], [m^1 \bmod \mathrm{q}]) = m$$

Plain RSA (Summary)

- Public Key (pk): $\mathrm{N} = \mathrm{pq}$, e such that $\mathrm{GCD}\big(\mathrm{e}, \phi(N)\big) = 1$
  - $\phi(N) = (p-1)(q-1)$ for distinct primes p and q
- Secret Key (sk): N, d such that ed=1 mod $\phi(N)$
- **Encrypt(pk=(N,e),m)** $= m^{\boldsymbol{e}} \; mod \; N$
- **Decrypt(sk=(N,d),c)** $= c^{\boldsymbol{d}} \; mod \; N$

- Decryption Works because
  $[c^d \mathrm{mod\ N}] = [m^{ed} \mathrm{mod\ N}] = \left[m^{[ed\ mod\ \boldsymbol{\phi}(\boldsymbol{N})]} \mathrm{mod\ N}\right] = [m \ \mathrm{mod\ N}]$

Factoring Assumption

Let **GenModulus**(1^n) be a randomized algorithm that outputs ($N=pq,p,q$) where p and q are n-bit primes (except with negligible probability **negl**(n)).

Experiment $\text{FACTOR}_{A,n}$

1. $(N=pq,p,q) \leftarrow$ **GenModulus**(1^n)
2. Attacker A is given N as input
3. Attacker A outputs $p' > 1$ and $q' > 1$
4. Attacker A wins if $N=p'q'$.

Factoring Assumption

- Necessary for security of RSA.
- Not known to be sufficient.

Experiment $\text{FACTOR}_{A,n}$

1. $(N=pq,p,q) \leftarrow$ **GenModulus**(1^n)
2. Attacker A is given N as input
3. Attacker A outputs $p' > 1$ and $q' > 1$
4. Attacker A wins ($\text{FACTOR}_{A,n} = 1$) if and only if $N=p'q'$.

$$\forall PPT\ A\ \exists \mu\ (\text{negligible})\ \text{s.t}\ \Pr[\text{FACTOR}_{A,n} = 1] \leq \mu(n)$$

RSA-Assumption

RSA-Experiment: RSA-INV$_{A,n}$

1. **Run KeyGeneration**(1^n) **to obtain (N,e,d)**
2. **Pick uniform** $\mathrm{y} \in \mathbb{Z}_{\mathrm{N}}^{*}$
3. Attacker A is given N, e, y and outputs $\mathrm{x} \in \mathbb{Z}_{\mathrm{N}}^{*}$
4. Attacker wins (RSA-INV$_{A,n}$=1) if $x^e = \mathrm{y} \bmod \mathrm{N}$

$$\forall PPT\ A\ \exists \mu\ (\text{negligible})\ \text{s.t}\ \Pr[\text{RSA–INVA}_{,n} = 1] \leq \mu(n)$$

RSA-Assumption

RSA-Experiment: RSA-INV$_{A,n}$

1. **Run KeyGeneration**(1^n) **to obtain (N,e,d)**
2. **Pick uniform** $y \in \mathbb{Z}_N^*$
3. Attacker A is given N, e, y and outputs $x \in \mathbb{Z}_N^*$
4. Attacker wins (RSA-INV$_{A,n}$=1) if $x^e = y \bmod N$

$$\forall PPT\ A\ \exists \mu\ (\text{negligible})\ \text{s.t}\ \Pr[\text{RSA–INVA}_{,n} = 1] \leq \mu(n)$$

- Plain RSA Encryption behaves like a one-way function
- Attacker cannot invert encryption of random message

Discussion of RSA-Assumption

- Plain RSA Encryption behaves like a one-way-function

- Decryption key is a “trapdoor” which allows us to invert the OWF

- RSA-Assumption $\rightarrow$ OWFs exist

Recap

- Plain RSA
- Public Key (pk): $\mathrm{N} = \mathrm{pq}$, e such that $\mathrm{GCD}\big(\mathrm{e}, \phi(N)\big) = 1$
  - $\phi(N) = (p-1)(q-1)$ for distinct primes p and q
- Secret Key (sk): N, d such that ed=1 mod $\phi(N)$
- **Encrypt(pk=(N,e),m)** $= m^{\boldsymbol{e}}\ mod\ N$
- **Decrypt(sk=(N,d),c)** $= c^{\boldsymbol{d}}\ mod\ N$

- Decryption Works because
  $[c^d \text{mod N}] = [m^{ed} \text{mod N}] = \left[m^{[ed\ mod\ \boldsymbol{\phi}(\boldsymbol{N})]} \text{mod N}\right] = [m \text{ mod N}]$

Mathematica Demo

https://www.cs.purdue.edu/homes/jblocki/courses/555_Spring17/slides/Lecture24Demo.nb

http://develop.wolframcloud.com/app/

Note: Online version of mathematica available at https://sandbox.open.wolframcloud.com (reduced functionality, but can be used to solve homework bonus problems)

(Toy) RSA Implementation in Mathematica

```
(* Random Seed 123456 is not secure, but it allows us to repeat the experiment *)
    SeedRandom[123456]
(* Step 1: Generate primes for an RSA key *)
    p = RandomPrime[{10^1000, 10^1050}];
    q = RandomPrime[{10^1000, 10^1050}];
    NN = p q;   (*Symbol N is protected in mathematica *)
    phi = (p - 1) (q - 1);
```

https://www.cs.purdue.edu/homes/jblocki/courses/555_Spring17/slides/Lecture24Demo.nb

(Toy) RSA Implementation in Mathematica

(* Step 1.A: Find e *)

GCD[phi,7]

Output: 7

(* GCD[phi,7] is not 1, so he have to try a different value of e *)

GCD[phi,3]

Output: 1

(* We can set e=3 *)

e=3;

https://www.cs.purdue.edu/homes/jblocki/courses/555_Spring17/slides/Lecture24Demo.nb

(Toy) RSA Implementation in Mathematica

(* Step 1.B find d s.t. ed = 1 mod N by using the extended GCD algorithm *)

(* Mathematica is clever enough to do this automatically *)

Solve[e x == 1, Modulus->phi]

Output:

{{x->364696805906630283017006261328838672727187289052050 88...

....

3940694217786102094256244409800844813981 31}}

(* We can now set d = x *)

d=364696805…. 8131;

(Toy) RSA Implementation in Mathematica

(* Double Check 1 = [ed mod $\phi(N)$] *)

Mod [e d, (p-1)(q-1)]

Output: 1

(* Encrypt the message 200, c= m^e mod N *)

m = 200;

PowerMod[m,e,NN]

Output: 8 000 000

(Toy) RSA Implementation in Mathematica

(* Encrypt the message 200, c= m^e mod N *)

```
m = 200;
PowerMod[m,e,NN]
```

Output: 8 000 000

(* Hm...That doesn't seem too secure *)

```
CubeRoot[PowerMod[m,e,NN]]
```

Output: 200

(* Moral: if $m^e < N$ then Plain RSA does not hide the message m. *)

RSA Implementation in Mathematica

(* Encrypt a larger message, c= m^e mod N *)

```
SeedRandom[1234567];
m2= RandomInteger[{10^1500,10^1501}];
c=PowerMod[m2,e,NN]
```

Output: 405215834903772786……… 388068292685976133

(* Does it Decrypt Properly? *)

```
PowerMod[c,d, NN]-m2
```

Output: 0

(* Yes! *)

CS 555: Week 10: Topic 2
Attacks on Plain RSA

(Plain) RSA Discussion

- We have not introduced security models like CPA-Security or CCA-security for Public Key Cryptosystems

- However, notice that (Plain) RSA Encryption is stateless and deterministic.

→Plain RSA is not secure against chosen-plaintext attacks

- As we will see Plain RSA is also highly vulnerable to chosen-ciphertext attacks

(Plain) RSA Discussion

- However, notice that (Plain) RSA Encryption is stateless and deterministic.

→Plain RSA is not secure against chosen-plaintext attacks

- **Remark:** In a public key setting the attacker who knows the public key *always* has access to an encryption oracle

- Encrypted messages with low entropy are particularly vulnerable to brute-force attacks
  - **Example:** If $m < B$ then attacker can recover m from $\mathrm{c} = \mathrm{Enc}_{\mathrm{pk}}(m)$ after at most B queries to encryption oracle (using public key)

Chosen Ciphertext Attack on Plain RSA

1. Attacker intercepts ciphertext $c = [m^e \bmod \mathrm{N}]$
2. Attacker generates ciphertext c’ for secret message 2m as follows
3. c’ $= [(c2^e) \bmod \mathrm{N}]$
4. $= [(m^e 2^e) \bmod \mathrm{N}]$
5. $= [(2m)^e \bmod \mathrm{N}]$
6. Attacker asks for decryption of $[c2^e \bmod \mathrm{N}]$ and receives 2m.
7. Divide by two to recover message

Above Example: Shows plain RSA is highly vulnerable to ciphertext-tampering attacks

More Weaknesses: Plain RSA with small e

- (Small Messages) If $m^e < N$ then we can decrypt $c = m^e \bmod N$ directly e.g., $m=c^{(1/e)}$

- (Partially Known Messages) If an attacker knows first 1-(1/e) bits of secret message $\mathrm{m} = \mathrm{m}_1 \| ??$ then he can recover m given
$$\mathbf{Encrypt}(\mathrm{pk}, \mathrm{m}) = m^e \bmod \mathrm{N}$$

Theorem[Coppersmith]: If p(x) is a polynomial of degree e then in polynomial time (in log(N), 2^e) we can find all m such that p(m) = 0 mod N and $|m|<N^{(1/e)}$

More Weaknesses: Plain RSA with small e

Theorem[Coppersmith]: If p(x) is a polynomial of degree e then in polynomial time (in log(N), e) we can find all m such that p(m) = 0 mod N and $|m|<N^{(1/e)}$

Example: e = 3, $m = m_1 \| m_2$ and attacker knows $m_1(2k\ bits)$ and $\boldsymbol{c} = (m_1 \| m_2)^e \bmod \mathrm{N}$, but not $m_2(k\ bits)$

$$p(x) = \left(2^k m_1 + x\right)^3 - c$$

Polynomial has a small root mod N at x= m_2 and coppersmith's method will find it!

D. Coppersmith (1996). "Finding a Small Root of a Univariate Modular Equation".

More Weaknesses: Plain RSA with small e

Theorem[Coppersmith]: Can also find small roots of bivariate polynomial $\mathrm{p}(\boldsymbol{x_1}, \boldsymbol{x_2})$

- **Similar Approach used to factor weak RSA secret keys N=$\mathbf{q_1 q_2}$**
- Weak PRG → Can guess many of the bits of prime factors
  - Obtain $\widetilde{q_1} \approx q_1$ and $\widetilde{q_2} \approx q_2$
- Coppersmith Attack: Define polynomial p(.,.) as follows

$$\mathrm{p}(\boldsymbol{x_1}, \boldsymbol{x_2}) = (\boldsymbol{x_1} + \widetilde{\boldsymbol{q_1}})(\boldsymbol{x_2} + \widetilde{\boldsymbol{q_2}}) - \boldsymbol{N}$$

- **Small Roots of** $\mathrm{p}(\boldsymbol{x_1}, \boldsymbol{x_2})$**:** $x_1 = q_1 - \widetilde{q_1}$ and $x_2 = q_2 - \widetilde{q_2}$

D. Coppersmith (1996). "Finding a Small Root of a Bivariate Integer Equation; Factoring with high bits known" ars TECHNICA BIZ & IT TECH SCIENCE POLICY CARS GAMING & CULTURE FORUMS

COMPLETELY BROKEN —

Millions of high-security crypto keys crippled by newly discovered flaw

Factorization weakness lets attackers impersonate key holders and decrypt their data.

DAN GOODIN - 10/16/2017, 7:00 AM

Steve Jurvetson

Enlarge / 750,000 Estonian cards that look like this use a 2048-bit RSA key that can be factored in a matter of days.

*The Return of **Coppersmith's Attack**: Practical Factorization of Widely Used RSA Moduli (CCS 2017)*

Fixes for Plain RSA

- Approach 1: RSA-OAEP
  - Incorporates random nonce r
  - CCA-Secure (in random oracle model)

- Approach 2: Use RSA to exchange symmetric key for Authenticated Encryption scheme (e.g., AES)
  - Key Encapsulation Mechanism (KEM)

- More details in future lectures…stay tuned!
  - For now we will focus on attacks on Plain RSA

Chinese Remainder Theorem

Theorem: Let N = pq (where gcd(p,q)=1) be given and let $f: \mathbb{Z}_N \to \mathbb{Z}_p \times \mathbb{Z}_q$ be defined as follows

$$f(x) = ([x \bmod p], [x \bmod q])$$

then

- f is a bijective mapping (invertible)
- f and its inverse $f^{-1}: \mathbb{Z}_p \times \mathbb{Z}_q \to \mathbb{Z}_N$ can be computed efficiently
- $f(x+y) = f(x) + f(y)$
- The restriction of f to $\mathbb{Z}_N^*$ yields a bijective mapping to $\mathbb{Z}_p^* \times \mathbb{Z}_q^*$
- For inputs $x, y \in \mathbb{Z}_N^*$ we have $f(x)f(y) = f(xy)$

Chinese Remainder Theorem

Application of CRT: Faster computation

Example: Compute [11^{53} mod 15]

f(11)=([-1 mod 3],[1 mod 5])

f(11^{53}) =([$(-1)^{53}$ mod 3],[1^{53} mod 5])= (-1,1)

f^{-1}(-1,1)=11

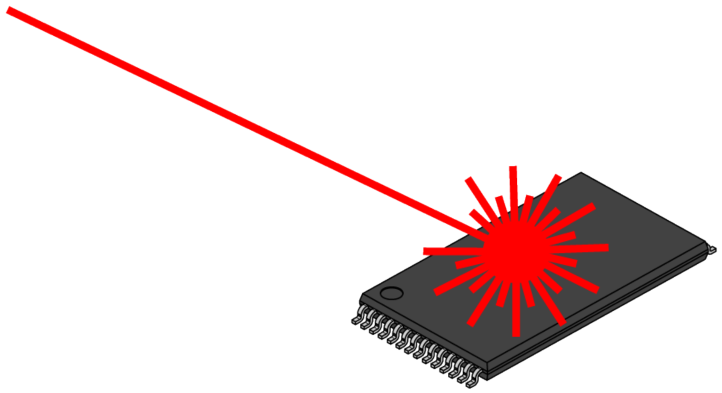
Thus, 11=[11^{53} mod 15]

A Side Channel Attack on RSA with CRT

- Suppose that decryption is done via Chinese Remainder Theorem for speed.

$$\mathbf{Dec}_{sk}(c) = c^d \bmod N \leftrightarrow \left(c^d \bmod p, c^d \bmod q\right)$$

- Attacker has physical access to smartcard
  - Can mess up computation of $c^d \bmod p$
  - Response is $\mathrm{R} \leftrightarrow \left(r, c^d \bmod q\right)$
  - $\mathrm{R} - \mathrm{m} \leftrightarrow (r - m \bmod p, 0 \bmod q)$
  - GCD(R-m,N)=q



Recovering Encrypted Message faster than Brute-Force

Claim: Let m < 2^n be a secret message. For some constant $\alpha = \frac{1}{2} + \varepsilon$.
We can recover m in in time $T = 2^{\alpha n}$ with high probability.

For r=1,…,T
 let $\mathrm{x_r} = [cr^{-e} mod\ N]$, where $r^{-e} = (r^{-1})^e mod\ N$
Sort $\mathbf{L} = \{(\boldsymbol{r}, \boldsymbol{x_r})\}_{\boldsymbol{r=1}}^{\boldsymbol{T}}$ **(by the $\mathbf{x_r}$ values)**
For s=1,…,T
 if $[s^e mod\ N] = \boldsymbol{x_r}$ for some r **then**
  return $[rs\ mod\ N]$

Recovering Encrypted Message faster than Brute-Force

For r=1,…,T

  let $\mathrm{x_r} = [cr^{-e} mod\ N]$, where $r^{-e} = (r^{-1})^e mod\ N$

Sort $\mathbf{L} = \{(\boldsymbol{r}, \boldsymbol{x_r})\}_{\boldsymbol{r=1}}^{\boldsymbol{T}}$ **(by the $\mathbf{x_r}$ values)**

For s=1,…,T

  if $[s^e mod\ N] = x_r$ for some r **then**

    return $[rs\ mod\ N]$

Analysis: $[rs\ mod\ N] = [r(s^e)^d\ mod\ N] = [r(x_r)^d\ mod\ N]$
$= [r(cr^{-e})^d\ mod\ N] = [rr^{-ed}(c)^d\ mod\ N]$
$= [rr^{-1}m\ mod\ N] = \mathrm{m}$

Recovering Encrypted Message faster than Brute-Force

For r=1,…,T

let $\mathrm{x_r} = [cr^{-e} mod\ N]$, where $r^{-e} = (r^{-1})^e mod\ N$

Sort $\mathbf{L} = \{(\boldsymbol{r}, \boldsymbol{x_r})\}_{\boldsymbol{r=1}}^{\boldsymbol{T}}$ **(by the $\mathbf{x_r}$ values)**

For s=1,…,T

if $[s^e mod\ N] = x_r$ for some r **then**

return $[rs\ mod\ N]$

Fact: some constant $\alpha = \frac{1}{2} + \varepsilon$ setting $T = 2^{\alpha n}$ with high probability we will find a pair **s** and $\mathbf{x_r}$ with $[s^e mod\ N] = xr$.

Recovering Encrypted Message faster than Brute-Force

Claim: Let m < 2^n be a secret message. For some constant $\alpha = \frac{1}{2} + \varepsilon$.
We can recover m in in time $T = 2^{\alpha n}$ with high probability.

Roughly $\sqrt{B}$ steps to find a secret message m < B

CS 555: Week 10: Topic 3
Discrete Log + DDH Assumption

(Recap) Finite Groups

Definition: A (finite) group is a (finite) set $\mathbb{G}$ with a binary operation $\circ$ (over G) for which we have

- (**Closure**:) For all $\mathrm{g}, \mathrm{h} \in \mathbb{G}$ we have $\mathrm{g} \circ \mathrm{h} \in \mathbb{G}$
- **(Identity**:) There is an element $\mathrm{e} \in \mathbb{G}$ such that for all $\mathrm{g} \in \mathbb{G}$ we have
$$\mathrm{g} \circ \mathrm{e} = \mathrm{g} = \mathrm{e} \circ \mathrm{g}$$
- **(Inverses**:) For each element $\mathrm{g} \in \mathbb{G}$ we can find $\mathrm{h} \in \mathbb{G}$ such that $\mathrm{g} \circ \mathrm{h} = \mathrm{e}$. We say that h is the inverse of g.
- (**Associativity:**) For all $\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3 \in \mathbb{G}$ we have
$$(\mathrm{g}_1 \circ \mathrm{g}_2) \circ \mathrm{g}_3 = \mathrm{g}_1 \circ (\mathrm{g}_2 \circ \mathrm{g}_3)$$

We say that the group is **abelian** if

- (**Commutativity:**) For all $\mathrm{g}, \mathrm{h} \in \mathbb{G}$ we have $\mathrm{g} \circ \mathrm{h} = \mathrm{h} \circ \mathrm{g}$

Finite Abelian Groups (Examples)

- **Example 1:** $\mathbb{Z}_N$ when $\circ$ denotes addition modulo N
- Identity: 0, since 0 $\circ$ x =[0+x mod N] = [x mod N].
- Inverse of x? Set x^{-1}=N-x so that [x^{-1}+x mod N] = [N-x+x mod N] = 0.

- **Example 2:** $\mathbb{Z}_N^*$ when $\circ$ denotes multiplication modulo N
- Identity: 1, since 1$\circ$ x =[1(x) mod N] = [x mod N].
- Inverse of x? Run extended GCD to obtain integers a and b such that

$$ax + bN = \gcd(x, N) = 1$$

Observe that: x^{-1} = a. Why?

Cyclic Group

- Let $\mathbb{G}$ be a group with order $\mathrm{m} = |\mathbb{G}|$ with a binary operation $\circ$ (over G) and let g $\in$ $\mathbb{G}$ be given consider the set

$$\langle g \rangle = \{g^0, g^1, g^2, \dots\}$$

Fact: $\langle g \rangle$ defines a subgroup of $\mathbb{G}$.

- Identity: g^0
- Closure: $g^i \circ g^j = g^{i+j} \in \langle g \rangle$
- g is called a "generator" of the subgroup.

Fact: Let $\mathrm{r} = |\langle g \rangle|$ then $g^i = g^j$ if and only if $i = j \ mod \ r$. Also m is divisible by r.

Finite Abelian Groups (Examples)

Fact: Let p be a prime then $\mathbb{Z}_p^*$ is a cyclic group of order p-1.

- **Note**: Number of generators g s.t. of $\langle g \rangle = \mathbb{Z}_p^*$ is $\phi(p-1)$

Example (non-generator): p=7, g=2

<2>={1,2,4}

Example (generator): p=7, g=5

<2>={1,5,4,6,2,3}

Discrete Log Experiment $DLog_{A,G}(n)$

1. Run $G(1^n)$ to obtain a cyclic group $\mathbb{G}$ of order q (with $\|q\| = n$) and a generator g such that $< \mathrm{g} >= \mathbb{G}$.
2. Select $\mathrm{h} \in \mathbb{G}$ uniformly at random.
3. Attacker A is given $\mathbb{G}$, q, g, h and outputs integer x.
4. Attacker wins ($DLog_{A,G}(n)=1$) if and only if $g^x=h$.

We say that the discrete log problem is hard relative to generator G if

$$\forall PPT\ A\ \exists \mu\ (\text{negligible})\ \mathrm{s.t}\ \Pr[DLog_{A,n} = 1] \leq \mu(n)$$

Diffie-Hellman Problems

Computational Diffie-Hellman Problem (CDH)

- Attacker is given $\mathrm{h}_1 = g^{x_1} \in \mathbb{G}$ and $\mathrm{h}_2 = g^{x_2} \in \mathbb{G}$.
- Attackers goal is to find $g^{x_1 x_2} = (\mathrm{h}_1)^{x_2} = (\mathrm{h}_2)^{x_1}$
- **CDH Assumption**: For all PPT A there is a negligible function negl upper bounding the probability that A succeeds with probability at most negl(n).

Decisional Diffie-Hellman Problem (DDH)

- Let $\mathrm{z}_0 = g^{x_1 x_2}$ and let $\mathrm{z}_1 = g^{r}$, where x_1,x_2 and r are random
- Attacker is given g^{x_1}, g^{x_2} and z_b (for a random bit b)
- Attackers goal is to guess b
- **DDH Assumption**: For all PPT A there is a negligible function negl such that A succeeds with probability at most ½ + negl(n).

Secure key-agreement with DDH

1. Alice publishes g^{x_A} and Bob publishes g^{x_B}
2. Alice and Bob can both compute $K_{A,B} = g^{x_B\,x_A}$ but to Eve this key is indistinguishable from a random group element (by DDH)

Remark: Protocol is vulnerable to Man-In-The-Middle Attacks if Bob cannot validate g^{x_A}.

Can we find a cyclic group where DDH holds?

- **Example 1:** $\mathbb{Z}_p^*$ where p is a random n-bit prime.
  - CDH is believed to be hard
  - DDH is *not* hard (Exercise 13.15)

- **Theorem:** Let p=rq+1 be a random n-bit prime where q is a large λ-bit prime then the set of rth residues modulo p is a cyclic subgroup of order q. Then $\mathbb{G}_r = \{[h^r mod\ p] \mid h \in \mathbb{Z}_p^*\}$ is a cyclic subgroup of $\mathbb{Z}_p^*$ of order q.
  - **Remark 1:** DDH is believed to hold for such a group
  - **Remark 2:** It is easy to generate uniformly random elements of $\mathbb{G}_r$
  - **Remark 3:** Any element (besides 1) is a generator of $\mathbb{G}_r$

Can we find a cyclic group where DDH holds?

- **Theorem:** Let p=rq+1 be a random n-bit prime where q is a large λ-bit prime then the set of rth residues modulo p is a cyclic subgroup of order q. Then $\mathbb{G}_r = \left\{[h^r mod\ p] \middle| h \in \mathbb{Z}_p^*\right\}$ is a cyclic subgroup of $\mathbb{Z}_p^*$ of order q.
  - **Closure**: $h^r g^r = (hg)^r$
  - **Inverse** of h^r is $(h^{-1})^r \in \mathbb{G}_r$
  - **Size** $(h^r)^x = h^{[rx\ mod\ rq]} = (h^r)^x = h^{r[x\ mod\ q]} = (h^r)^{[x\ mod\ q]} mod\ p$

Remark: Two known attacks on Discrete Log Problem for $\mathbb{G}_r$(Section 9.2).

- First runs in time $O(\sqrt{q}) = O(2^{\lambda/2})$
- Second runs in time $2^{O\left(\sqrt[3]{n}(\log n)^{2/3}\right)}$

Can we find a cyclic group where DDH holds?

Remark: Two known attacks (Section 9.2).

- First runs in time $O(\sqrt{q}) = O(2^{\lambda/2})$
- Second runs in time $2^{O(\sqrt[3]{n}(\log n)^{2/3})}$, where n is bit length of p

Goal: Set λ and n to balance attacks

$$\lambda = O(\sqrt[3]{n}(\log n)^{2/3})$$

How to sample p=rq+1?

- First sample a random λ-bit prime q and
- Repeatedly check if rq+1 is prime for a random n- λ bit value r

Can we find a cyclic group where DDH holds?

Elliptic Curves Example: Let p be a prime (p > 3) and let A, B be constants. Consider the equation

$$y^2 = x^3 + Ax + B \ mod \ p$$

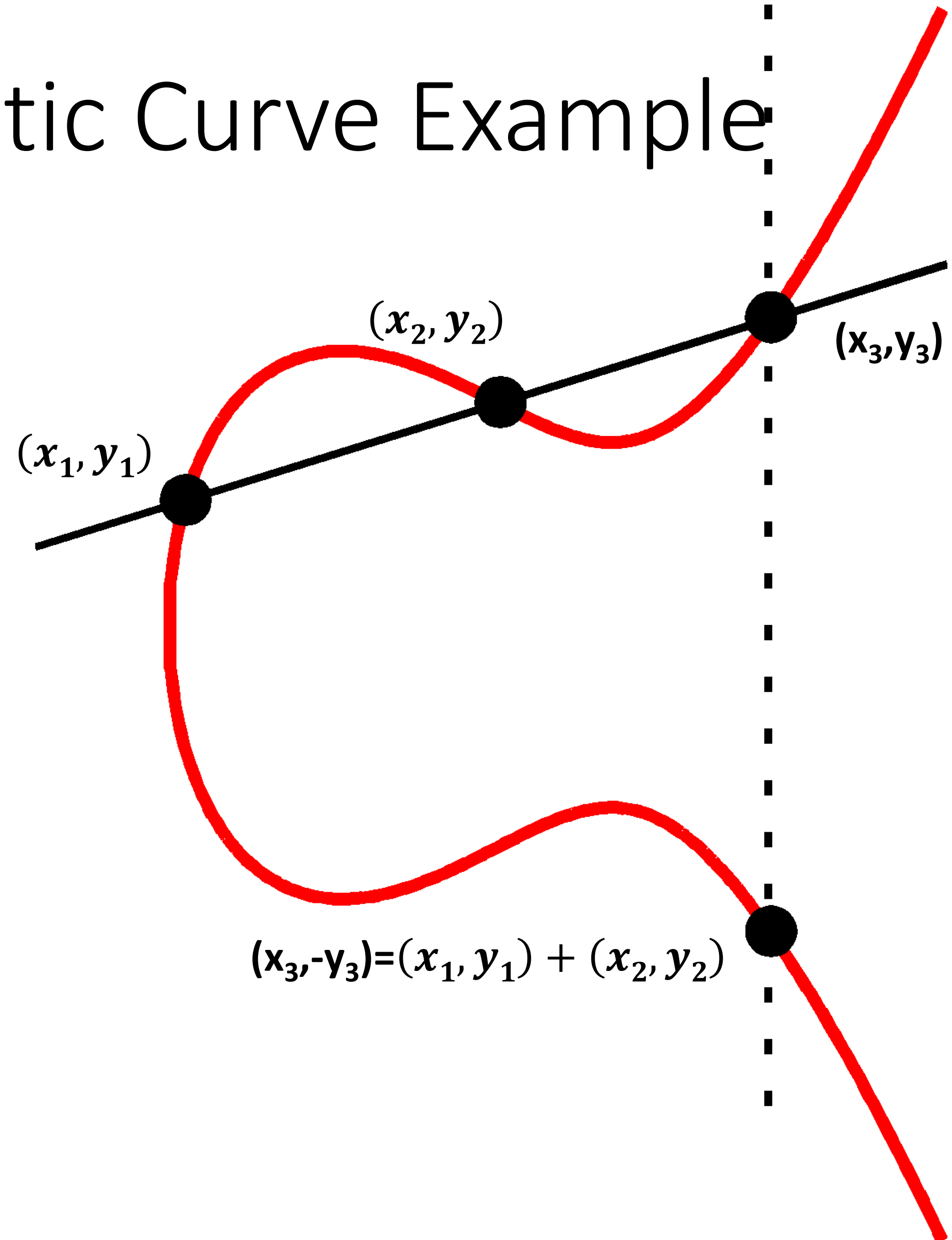
And let

$$E\left(\mathbb{Z}_p\right) = \left\{(x, y) \in \mathbb{Z}_p^2 \middle| y^2 = x^3 + Ax + B \ mod \ p \right\} \cup \{\mathcal{O}\}$$

Note: $\mathcal{O}$ is defined to be an additive identity $(x, y) + \mathcal{O} = (x, y)$

What is $(x_1, y_1) + (x_2, y_2)$?

Elliptic Curve Example



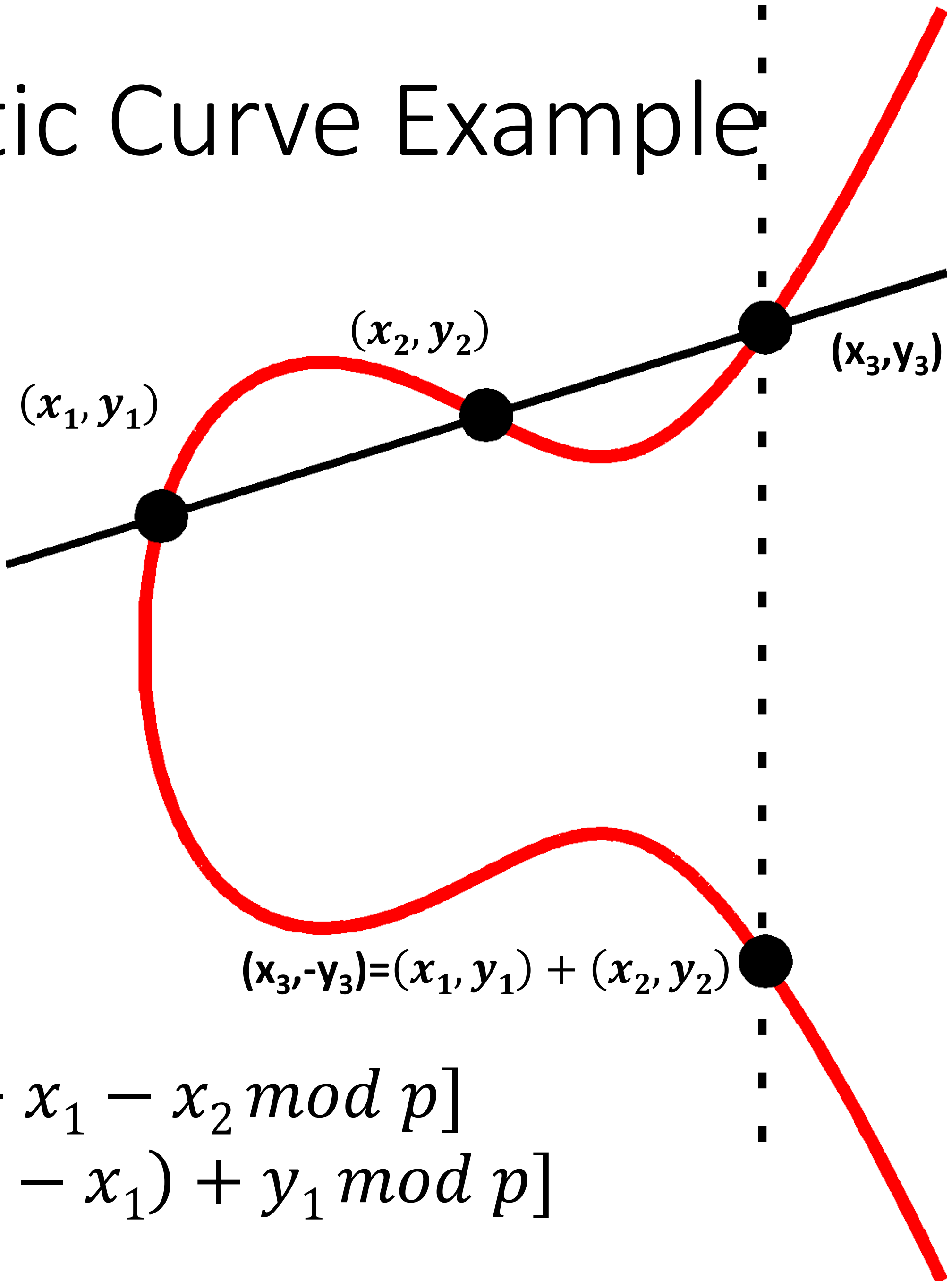
The line passing through $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$ has the equation

$$y = m(x - x_1) + y_1 \, mod \, P$$

Where the slope

$$m = \left[\frac{y_1 - y_2}{x_1 - x_2} \, mod \, p\right]$$

Elliptic Curve Example



Formally, let

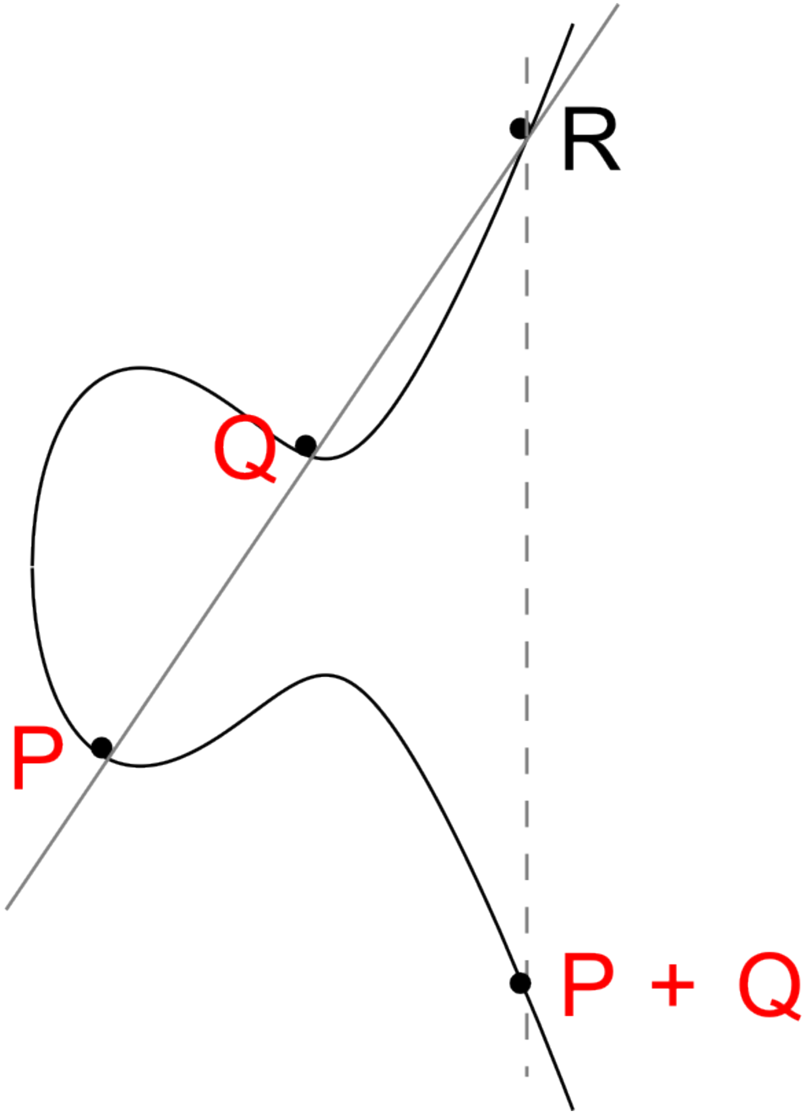
$$m = \left[\frac{y_1 - y_2}{x_1 - x_2} \bmod p\right]$$

Be the slope. Then the line passing through $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$ has the equation

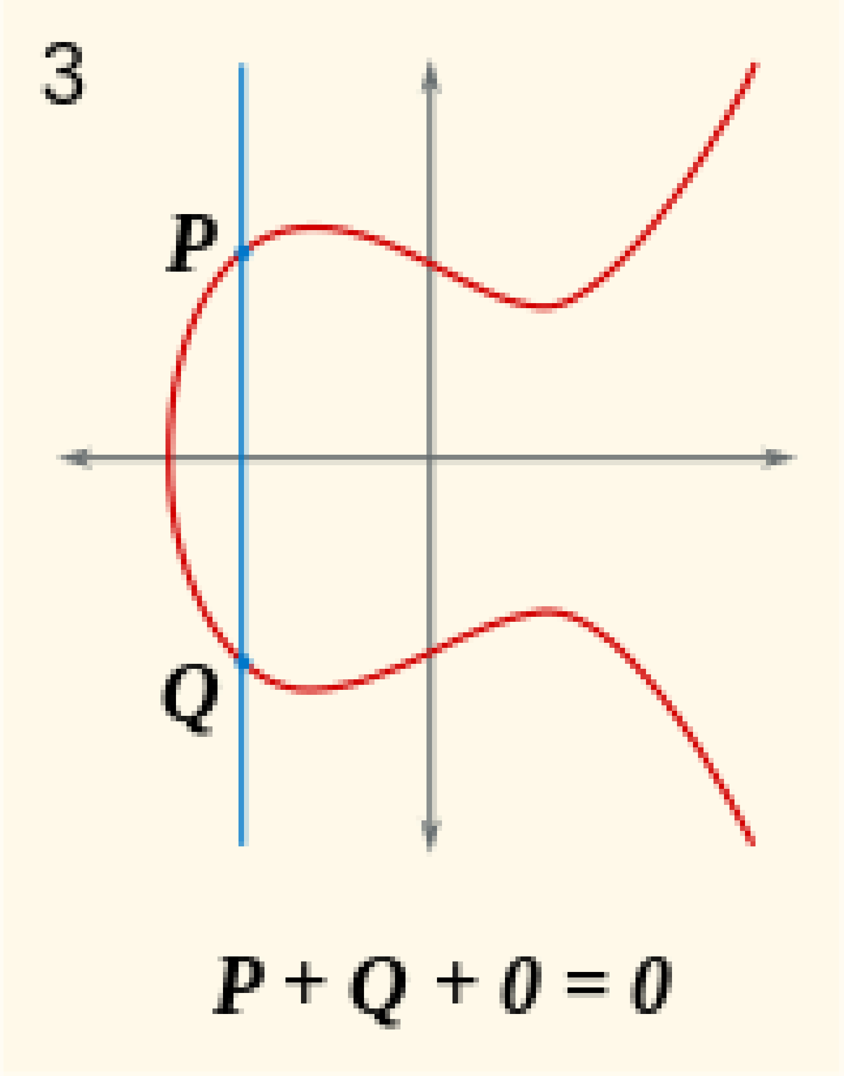
$$y = m(x - x_1) + y_1 \bmod P$$

$$(m(x - x_1) + y_1)^2 \\ = x^3 + Ax + B \bmod p$$

$$x_3 = [m^2 - x_1 - x_2 \bmod p]$$
$$y_3 = [m(x_3 - x_1) + y_1 \bmod p]$$


R
Q
P
P + Q

Elliptic Curve Example



No third point R on the elliptic curve.

P+Q = 0

(Inverse)

Can we find a cyclic group where DDH holds?

Elliptic Curves Example: Let p be a prime (p > 3) and let A, B be constants. Consider the equation

$$y^2 = x^3 + Ax + B \ mod \ p$$

And let

$$E\left(\mathbb{Z}_p\right) = \left\{(x, y) \in \mathbb{Z}_p^2 \middle| y^2 = x^3 + Ax + B \ mod \ p \right\} \cup \{\mathcal{O}\}$$

Fact: $E\left(\mathbb{Z}_p\right)$ defines an abelian group

- For *appropriate curves* the DDH assumption is believed to hold
- If you make up your own curve there is a good chance it is broken…
- NIST has a list of recommendations