Cryptography
CS 555

Week 7:

- AES
- One Way Functions
- **Readings:** Katz and Lindell Chapter 6.2.5, 6.3, 7.1-7.4

Spring 2021

Recap

- Block Ciphers, SPNs, Feistel Networks, DES
- Meet in the Middle, 3DES
- Building Stream Ciphers
  - Linear Feedback Shift Registers (+ Attacks)
  - RC4 (+ Attacks)
  - Trivium

CS 555: Week 7: Topic 1
Block Ciphers (Continued)

Advanced Encryption Standard (AES)

- (1997) US National Institute of Standards and Technology (NIST) announces competition for new block cipher to replace DES

- Fifteen algorithms were submitted from all over the world
  - Analyzed by NIST
- Contestants given a chance to break competitors schemes

- October, 2000 NIST announces a winner Rijndael
  - Vincent Rijmen and Joan Daemen
  - No serious vulnerabilities found in four other finalists
  - Rijndael was selected for efficiency, hardware performance, flexibility etc…

Advanced Encryption Standard

- **Block Size:** 128 bits (viewed as 4x4 byte array)
- **Key Size:** 128, 192 or 256

- Essentially a Substitution Permutation Network
  - **AddRoundKey:** Generate 128-bit sub-key from master key XOR with current state
  - **SubBytes:** Each byte of state array (16 bytes) is replaced by another byte according a a single S-box (lookup table)
  - **ShiftRows –** shift ith row by i bytes
  - **MixColumns –** permute the bits in each column

Substitution Permutation Networks

- S-box a public “substitution function” (e.g. $S \in \mathbf{Perm_8}$).

- S is not part of a secret key, but can be used with one

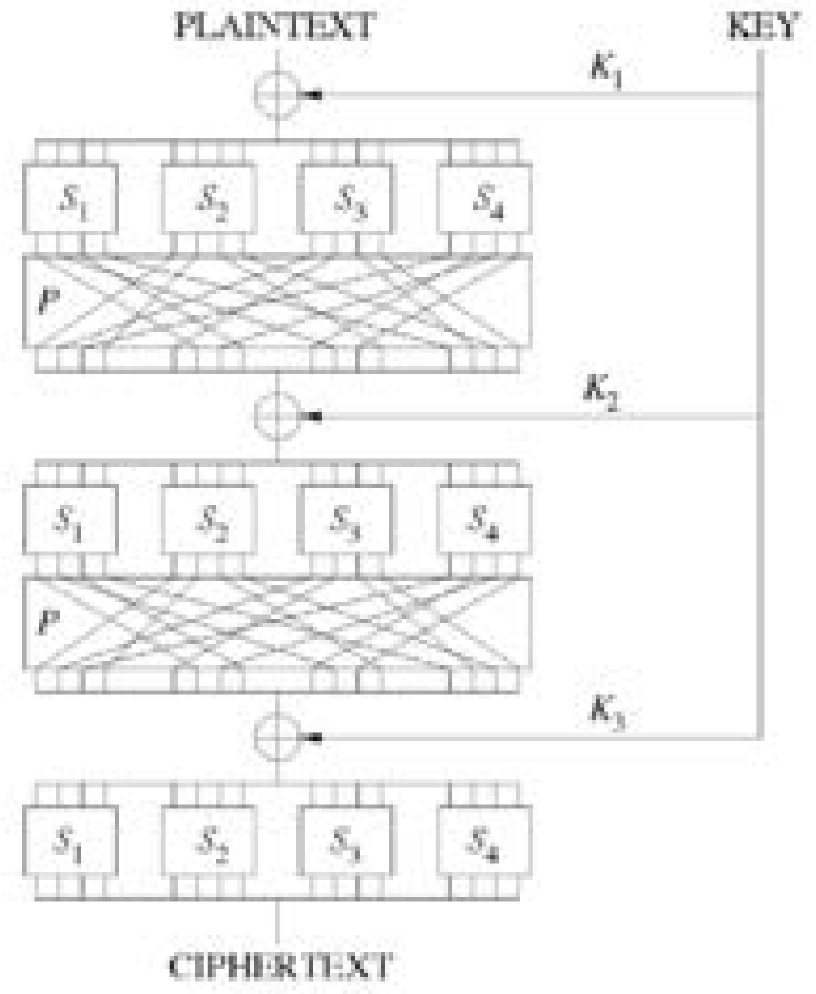
$$f(x) = S(x \oplus k)$$

Note: there are only n! possible bit mixing permutations of [n] as opposed to 2^n! Permutations of $\{0,1\}^n$

Input to round: x, k (k is subkey for current round)

1. **Key Mixing**: Set $x := x \oplus k$
2. **Substitution**: $x := S_1(x_1) \parallel S_2(x_2) \parallel \cdots \parallel S_8(x_8)$
3. **Bit Mixing Permutation**: permute the bits of x to obtain the round output

Substitution Permutation Networks



- **Proposition 6.3:** Let F be a keyed function defined by a Substitution Permutation Network. Then for any keys/number of rounds F_k is a permutation.

- Why? Composing permutations f,g results in another permutation h(x)=g(f(x)).

Advanced Encryption Standard

- Block Size: 128 bits
- Key Size: 128, 192 or 256

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  - **ShiftRows**
  - **MixColumns**

Key Mixing

Substitution

Bit Mixing Permutation

AddRoundKey:

Round Key (16 Bytes)

00001111	**…**	**…**	**…**
10100011	…	…	…
11001100	…	…	…
01111111	…	…	…

$\bigoplus$

State

11110000	**…**	**…**	**…**
01100010	…	…	…
00110000	…	…	…
11111111	…	…	…

$=$

11111111	**…**	**…**	**…**
11000001	…	…	…
11111100	…	…	…
10000000	…	…	…

AddRoundKey:

Round Key (16 Bytes)

	…	…	…
10100011	…	…	…
	…	…	…
	…	…	…

State

11111111	**…**	**…**	**…**
11000001	…	…	…
11111100	…	…	…
10000000	…	…	…

SubBytes (Apply S-box)

S(11111111)	**S(…)**	**S(…)**	**S(…)**
S(11000001)	S(…)	S(…)	S(…)
S(11111100)	S(…)	S(…)	S(…)
S(10000000)	S(…)	S(…)	S(…)

AddRoundKey:

Round Key (16 Bytes)

10100011	…		
		…	
			…

State

S(11111111)	S(…)	S(…)	S(…)
S(11000001)	S(…)	S(…)	S(…)
S(11111100)	S(…)	S(…)	S(…)
S(10000000)	S(…)	S(…)	S(…)

Shift Rows

S(11111111)	S(…)	S(…)	S(…)
S(…)	**S(11000001)**	S(…)	S(…)
S(…)	S(…)	**S(11111100)**	S(…)
S(…)	S(…)	S(…)	**S(10000000)**

AddRoundKey:

Round Key (16 Bytes)

10100011	…		
		…	
			…

State

S(11111111)			
	S(11000001)	S(…)	
S(…)		S(11111100)	
		S(…)	S(10000000)

Mix Columns

Invertible (linear) transformation.

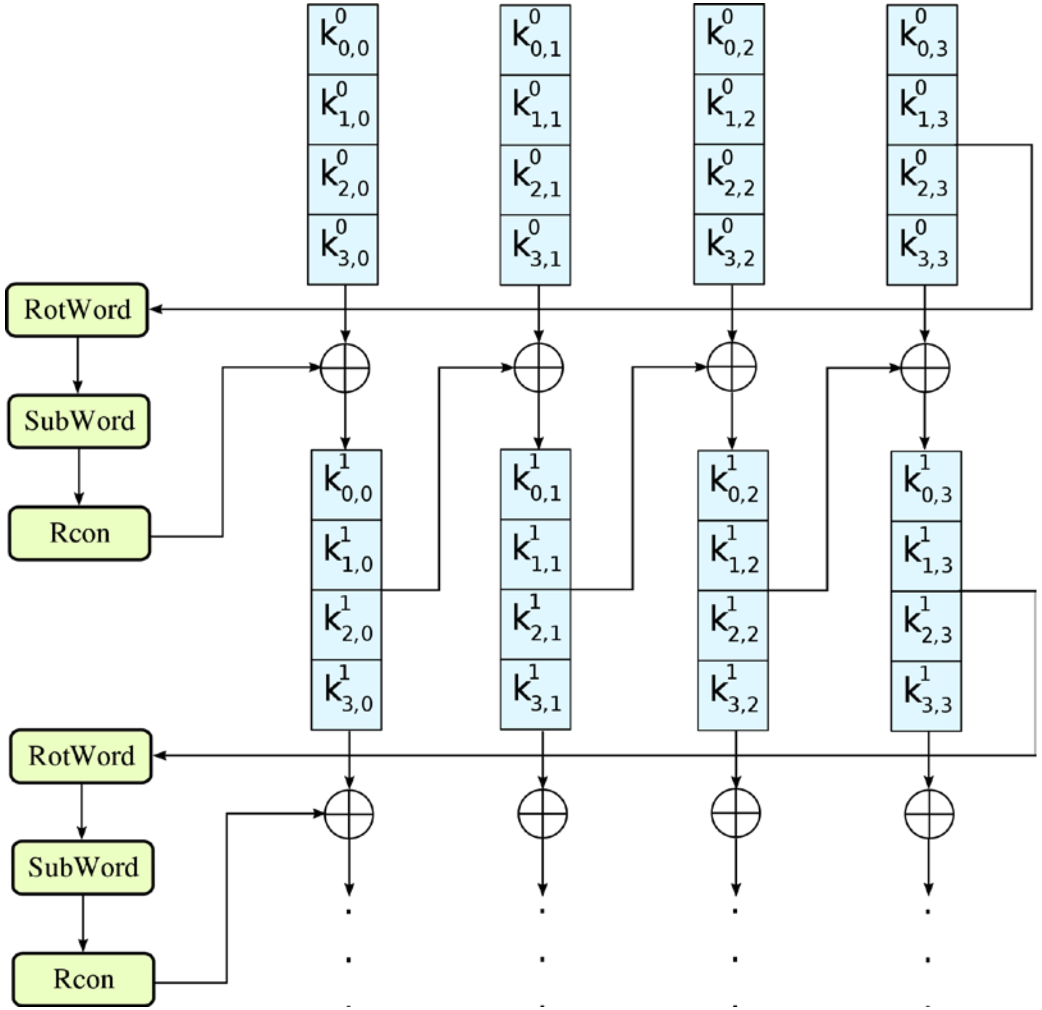
Key property: if inputs differ in b>0 bytes then output differs in 5-b bytes (minimum)

AES

- We just described one round of the SPN

- AES uses
  - 10 rounds (with 128 bit key)
  - 12 rounds (with 192 bit key)
  - 14 rounds (with 256 bit key)

AES-128: Key Schedule



AES Attacks?

- Side channel attacks affect a few specific implementations
  - But, this is not a weakness of AES itself
  - Timing attack on OpenSSL's implementation AES encryption (2005, Bernstein)

- (2009) Related-Key Attack on 11 round version of AES
  - Related Key Attack: Attacker convinces Alice to use two related (but unknown) keys
  - recovers 256-bit key in time 2^{70}
  - But AES is 14 round (with 256 bit key) so the attack doesn't apply in practice
- (2009) Related Key Attack on 192-bit and 256 bit version of AES
  - recovers 256-bit key in time $2^{99.5}$.
- (2011) Key Recovery attack on AES-128 in time $2^{126.2}$.
  - Improved to $2^{126.0}$ for AES-128, $2^{189.9}$ for AES-192 and $2^{254.3}$ for AES-256
- First public cipher approved by NSA for Top Secret information
  - SECRET level (AES-128,AES-192 & AES-256), TOP SECRET level (~~AES-128~~,AES-192 & AES-256)

NIST Recommendations

Ok, as CRHF and in Digital Signatures

Ok, to use for HMAC, Key Derivation and as PRG

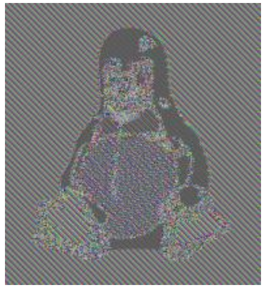
80 bits-security is no longer acceptable

Date	Minimum of Strength	Symmetric Algorithms	Factoring Modulus	Discrete Logarithm Key	Discrete Logarithm Group	Elliptic Curve	Hash (A)	Hash (B)
(Legacy)	80	2TDEA*	1024	160	1024	160	SHA-1**	
2016 - 2030	112	3TDEA	2048	224	2048	224	SHA-224 SHA-512/224 SHA3-224	
2016 - 2030 & beyond	128	AES-128	3072	256	3072	256	SHA-256 SHA-512/256 SHA3-256	SHA-1
2016 - 2030 & beyond	192	AES-192	7680	384	7680	384	SHA-384 SHA3-384	SHA-224 SHA-512/224
2016 - 2030 & beyond	256	AES-256	15360	512	15360	512	SHA-512 SHA3-512	SHA-256 SHA-512/256 SHA-384 SHA-512 SHA3-512

Recommendations from Other Groups (Including NIST): www.keylength.com

AES-GCM

- Note: just because AES is a good block cipher does not mean that all modes of operation that use AES are secure.
  - ECB Penguin



- AES-GCM: authenticated encryption with associated data
  - Increasing deployment: TLS 1.2, TLS 1.3, QUIC
  - Hardware support for AES + AES-GCM in many modern processors

Differential Cryptanalysis

Definition: We say that the differential $(\triangle_x, \triangle_y)$ occurs with probability p in the keyed block cipher F if

$$\Pr\left[F_K(x_1) \oplus F_K(x_1 \oplus \triangle_x) = \triangle_y\right] \geq p$$

Can Lead to Efficient (Round) Key Recovery Attacks

Exploiting Weakness Requires: well over $\frac{1}{p}$ chosen plaintext-ciphertext pairs

Differentials in S-box can lead to (weaker) differentials in SPN.

Linear Cryptanalysis

$$y = F_K(x)$$

Definition: Fixed set of input bits $i_1, \ldots, i_{in}$ and output bits $i_1', \ldots, i_{out}'$ are said to have ε-linear bias if the following holds

$$\left|Pr\left[x_{i_1} \oplus x_{i_2} \ldots \oplus x_{i_{in}} \oplus y_{i_1'} \oplus y_{i_2'} \ldots \oplus y_{i_{out}'}\right]\right| = \varepsilon$$

(randomness taken over the selection of input x and secret key K)

Linear Cryptanalysis

Definition: Fixed set of input bits $i_1, \ldots, i_{in}$ and output bits $i_1{}', \ldots, i_{out}{}'$ are said to have ε-linear bias if the following holds

$$\left|\Pr\left[x_{i_1} \oplus x_{i_2} \ldots \oplus x_{i_{in}} \oplus y_{i_1{}'} \oplus y_{i_2{}'} \ldots \oplus y_{i_{out}{}'}\right] - \frac{1}{2}\right| = \varepsilon$$

(randomness taken over the selection of input x and secret key K, $y = F_K(x)$)

Matsui: DES can be broken with just 2^{43} *known* plaintext/ciphertext pairs.

- Lots of examples needed!
- But the examples do not need to be chosen plaintext/ciphertext pairs…
- One encrypted file can provide a large amounts of known plaintext

Recap

- 2DES, Meet in the Middle Attack
- 3DES
- Stream Ciphers
  - Breaking Linear Feedback Shift Registers
  - Trivium
- AES

CS 555: Week 8: Topic 1: One Way Functions

What are the minimal assumptions necessary for symmetric key-cryptography?

One-Way Functions (OWFs)

$$\mathrm{f}(x) = y$$

Definition: A function $\mathrm{f}: \{0,1\}^* \to \{0,1\}^*$ is one way if it is

1. **(Easy to compute)** There is a polynomial time algorithm (in |x|) for computing f(x).
2. **(Hard to Invert)** Select $\mathrm{x} \leftarrow \{0,1\}^n$ uniformly at random and give the attacker input 1^n, f(x). The probability that a PPT attacker outputs x' such that $\mathrm{f}(x') = f(x)$ is negligible in n.

One-Way Functions (OWFs)

$$\mathrm{f}(x) = y$$

Key Takeaway: One-Way Functions is a *necessary* and *sufficient* assumption for most of symmetric key cryptography.

- From OWFs we can construct PRGs, PRFs, Authenticated Encryption
- From eavesdropping secure encryption (weakest) notion we can construct OWFs

One-Way Functions (OWFs)

$$f(x) = y$$

Remarks:

- **A function that is not one-way is not necessarily always easy to invert (even often)**
- **Any such function can be inverted in time 2^n (brute force)**
- **Length-preserving OWF: $|f(x)| = |x|$**
- **One way permutation: Length-preserving + one-to-one**

One-Way Functions (OWFs)

$$\mathrm{f}(x) = y$$

Remarks:

1. **f(x) does not necessarily hide all information about x.**
2. **If f(x) is one way then so is $\mathbf{f}'(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \parallel \boldsymbol{LSB}(\boldsymbol{x})$.**

One-Way Functions (OWFs)

$$\mathrm{f}(x) = y$$

Remarks:

1. **Actually we usually consider a family of one-way functions**

$$\boldsymbol{f_I}: \{\mathbf{0}, \mathbf{1}\}^{\boldsymbol{I}} \rightarrow \{\mathbf{0}, \mathbf{1}\}^{\boldsymbol{I}}$$

Candidate One-Way Functions

$$f_{ss}(x_1, \ldots, x_n, J) = \left(x_1, \ldots, x_n, \sum_{i \in J} x_i \mod 2^n \right)$$

(Subset Sum Problem is NP-Complete)

Note: $J \subset [n]$ **and** $\mathbf{0} \leq \boldsymbol{x_i} \leq \mathbf{2^n - 1}$

Candidate One-Way Functions

$$f_{ss}(x_1, \ldots, x_n, J) = \left(x_1, \ldots, x_n, \sum_{i \in J} x_i \mod 2^n \right)$$

(Subset Sum Problem is NP-Complete)

Question: Does $\mathrm{P} \neq NP$ imply this is a OWF?

Answer: No! $\mathrm{P} \neq NP$ only implies that any polynomial-time algorithm fails to solve "some instance" of subset sum. By contrast, we require that PPT attacker fails to solve "almost all instances" of subset sum.

Candidate One-Way Functions (OWFs)

$$f_{p,g}(x) = [g^x \bmod p]$$

(Discrete Logarithm Problem)

Note: The existence of OWFs implies $\mathrm{P} \neq NP$ so we cannot be *absolutely certain* that they do exist.

How to Build a PRG with One-Way Functions?

Hard Core Predicates

- Recall that a one-way function f may potentially reveal lots of information about input

- **Example**: $f(x_1,x_2)=(x_1,g(x_2))$, where g is a one-way function.
- **Claim**: f is one-way (even though $f(x_1,x_2)$ reveals half of the input bits!)

Hard Core Predicates

Definition: A predicate $\mathrm{hc}: \{0,1\}^* \to \{0,1\}$ is called a hard-core predicate of a function f if

1. (Easy to Compute) hc can be computed in polynomial time
2. (Hard to Guess) For all PPT attacker A there is a negligible function negl such that we have

$$\mathbf{Pr}_{x \leftarrow \{0,1\}^n}[A(1^n, f(x)) = \mathrm{hc}(x)] \le \frac{1}{2} + negl(n)$$

Attempt 1: Hard-Core Predicate

Consider the predicate

$$\mathrm{hc}(\mathrm{x}) = \bigoplus_{i=1}^{n} x_i$$

Hope: hc is hard core predicate for any OWF.

Counter-example:

$$\mathrm{f(x)} = (\mathrm{g(x)}, \bigoplus_{i=1}^{n} x_i)$$

Trivial Hard-Core Predicate

Consider the function

$$f(x_1,\ldots,x_n) = x_1,\ldots,x_{n-1}$$

f has a trivial hard core predicate

$$\mathrm{hc}(\mathrm{x}) = x_n$$

Not useful for crypto applications (e.g., f is not a OWF)

Attempt 3: Hard-Core Predicate

Consider the predicate

$$\mathrm{hc}(\mathrm{x}, \mathrm{r}) = \bigoplus_{i=1}^{n} x_i r_i$$

(the bits r_1,…, r_n will be selected uniformly at random)

Goldreich-Levin Theorem: (Assume OWFs exist) For any OWF f, hc is a hard-core predicate of g(x,r)=(f(x),r).

Question: Why is g a OWF?

Attempt 3: Hard-Core Predicate

Consider the predicate

$$\mathrm{hc}(\mathrm{x}, \mathrm{r}) = \bigoplus_{i=1}^{n} x_i r_i$$

(the bits $r_1,\ldots, r_n$ will be selected uniformly at random)

Goldreich-Levin Theorem: (Assume OWFs exist) For any OWF f, hc is a hard-core predicate of g(x,r)=(f(x),r).

Intuition: If $\mathbf{Pr}_{x \leftarrow \{0,1\}^n}[A(1^n, g(x,r)) = \mathrm{hc}(x,r)] \geq \frac{1}{2} + \frac{1}{p(n)}$ is non-negligible then we can recover x by repeatedly running $A(1^n$, (f(x),r')) for inputs r' of our choosing.

Using Hard-Core Predicates

Theorem: Given a one-way-permutation f and a hard-core predicate hc we can construct a PRG G with expansion factor $\ell(n) = n + 1$.

Construction:

$$G(s) = f(s) \parallel \mathrm{hc}(s)$$

Intuition: f(s) is actually uniformly distributed

- s is random
- f(s) is a permutation
- Last bit is hard to predict given f(s) (since hc is hard-core for f)

Arbitrary Expansion

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = n + 1$. Then for any polynomial p(.) there is a PRG with expansion factor p(n).

Construction:

- $G^1(x) = G(x) := y \,||\, b.$ (n+1 bits)
- $G^2(x) = G^1(y)||b$ (n+2 bits)
- $G^{i+1}(x) = G^i(y) \,||\, b$ where $G^i(x) = y||b$

n+i+1 bits

First n bits of output

Last i bits of output

And Beyond…

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = n + 1$. Then for any polynomial p(.) there is a PRG with expansion factor p(n).

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = 2n$. Then there is a secure PRF.

Theorem: Suppose that there is a secure PRF then there is a strong pseudorandom permutation.

And Beyond…

Corollary: If one-way functions exist then PRGs, PRFs and strong PRPs all exist.

Corollary: If one-way functions exist then there exist CCA-secure encryption schemes and secure MACs.

Announcements

- Homework 3 due tonight 11:59PM on Gradescope
- Quiz 3 released today
  - Due Saturday, March 6 at 11:30PM on Brightspace
- Midterm on March 11th in class
  - If you are not able to take the exam in class (e.g., quarantine) let me know and we can arrange an alternative
  - Allowed to prepare a 1 page cheat sheet
  - Practice Exam released this weekend

Recap

- One Way Functions/One Way Permutations
- Hard Core Predicate
- PRG with from OWP + Hard Core Predicate (n+1)
- PRG with arbitrary expansion from PRG with expansion (n+1)
  - $G^1(x) = G(x)$ (n+1 bits)
  - $G^{i+1}(x) = G^i(y) \,||\, z$ where $G^i(x) = y \,||\, z$

n+i+1 bits

First n bits of output

Last i bits of output

- **PRGs → PRFs (and PRPs/MACs/authenticated encryption)**

PRFs from PRGs

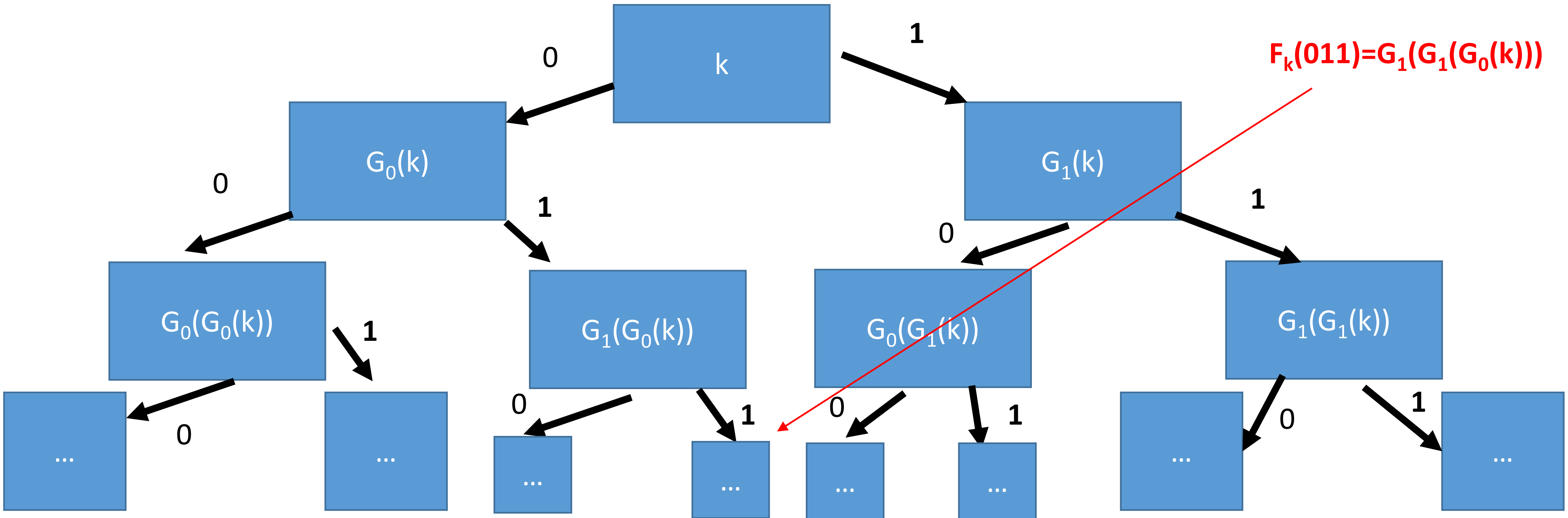
Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = 2n$. Then there is a secure PRF.

Let $G(x) = G_0(x)||G_1(x)$ (first/last n bits of output)

$$\boldsymbol{F_K(x_1, \ldots, x_n) = G_{x_n}\left(\ldots\left(G_{x_2}\left(G_{x_1}(K)\right)\right)\ldots\right)}$$

PRFs from PRGs

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = 2n$. Then there is a secure PRF.



PRFs from PRGs

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = 2n$. Then there is a secure PRF.

Proof:

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right| < \boldsymbol{negl(n)}$$

PRFs from PRGs

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\Pr\left[A\left(r_1 \parallel \cdots \parallel r_{t(n)}\right)\right] - \Pr\left[A\left(G(s_1) \parallel \cdots \parallel G\left(s_{t(n)}\right)\right)\right]\right| < negl(n)$$

Proof by Triangle Inequality: Fix j

$$Adv_j = \left|\Pr\left[A\left(r_1 \parallel \cdots \parallel r_{j+1} \parallel G\left(s_{j+2}\right) \ldots \parallel G\left(s_{t(n)}\right)\right)\right]\right.$$

PRFs from PRGs

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|Pr\left[A\left(r_1 \parallel \cdots \parallel r_{t(n)}\right)\right] - Pr\left[A\left(G(s_1) \parallel \cdots \parallel G\left(s_{t(n)}\right)\right)\right]\right| < negl(n)$$

Proof

$$\left|Pr\left[A\left(r_1 \parallel \cdots \parallel r_{t(n)}\right)\right] - Pr\left[A\left(G(s_1) \parallel \cdots \parallel G\left(s_{t(n)}\right)\right)\right]\right|$$

$$\leq \sum_{j<t(n)} Adv_j$$

$$\leq t(n) \times negl(n) = negl(n)$$

PRFs from PRGs

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right| < \boldsymbol{negl}(\boldsymbol{n})$$

Proof

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right|$$

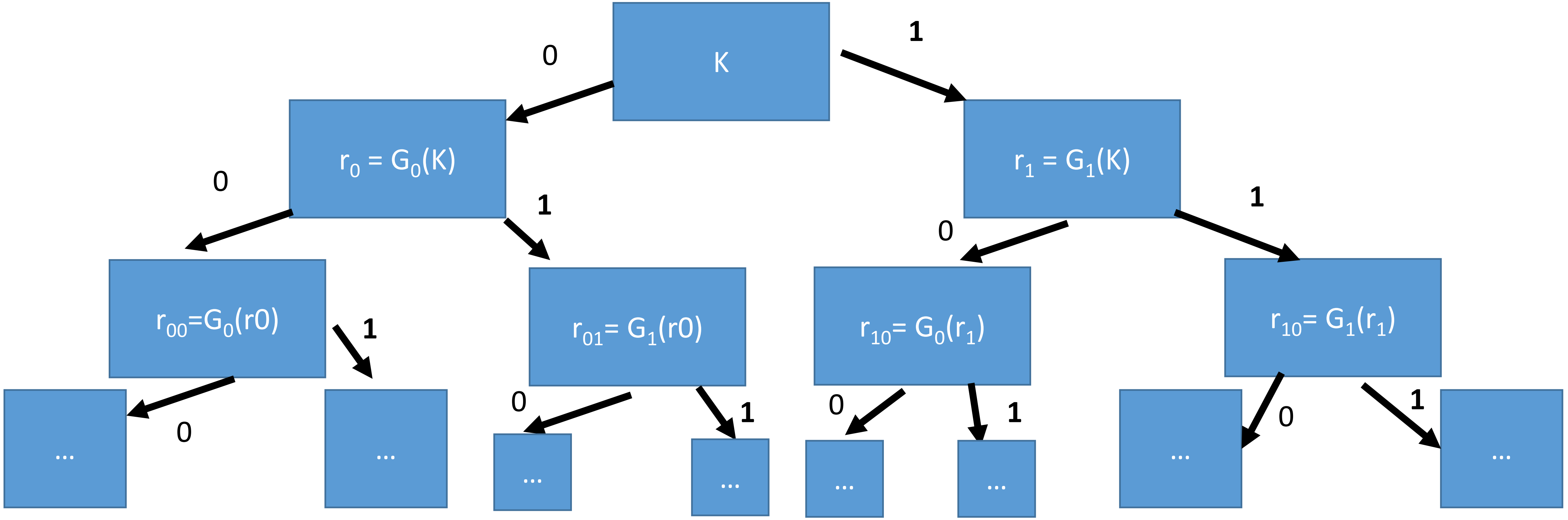
$$\leq \sum_{j<t(n)} \boldsymbol{Adv_j}$$

$$\leq \boldsymbol{t}(\boldsymbol{n}) \times \boldsymbol{negl}(\boldsymbol{n}) = \boldsymbol{negl}(\boldsymbol{n})$$

(QED, Claim 1)

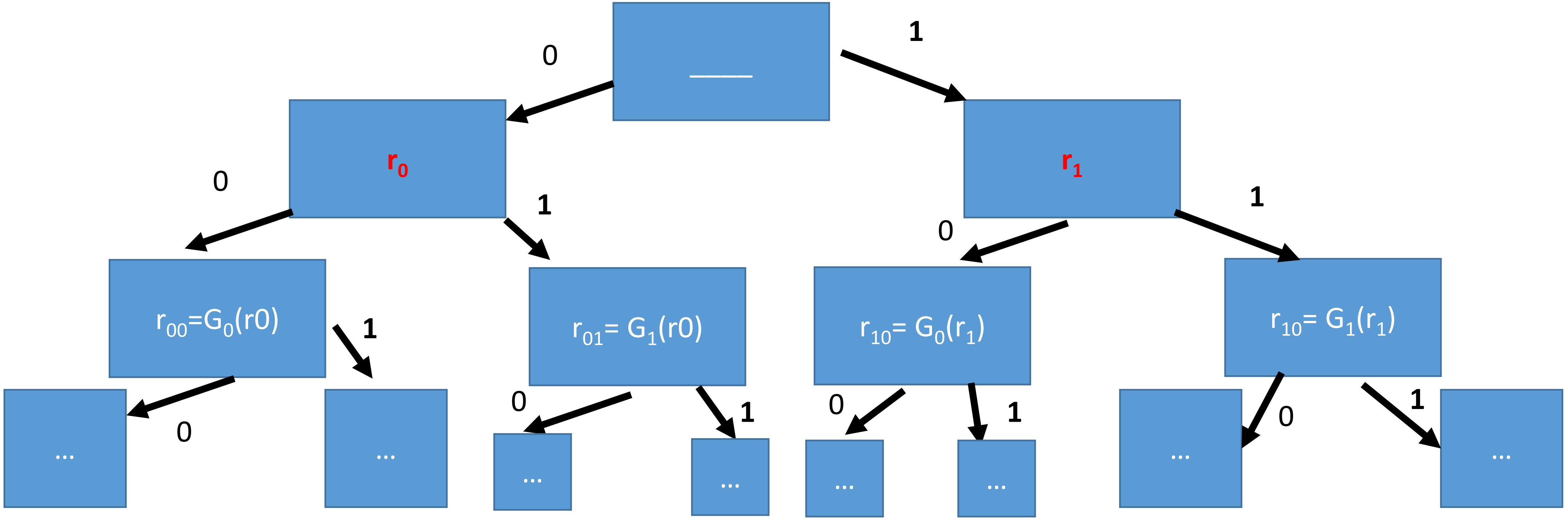
Hybrid H_1 and H_2

- Original Construction: Hybrid H_1



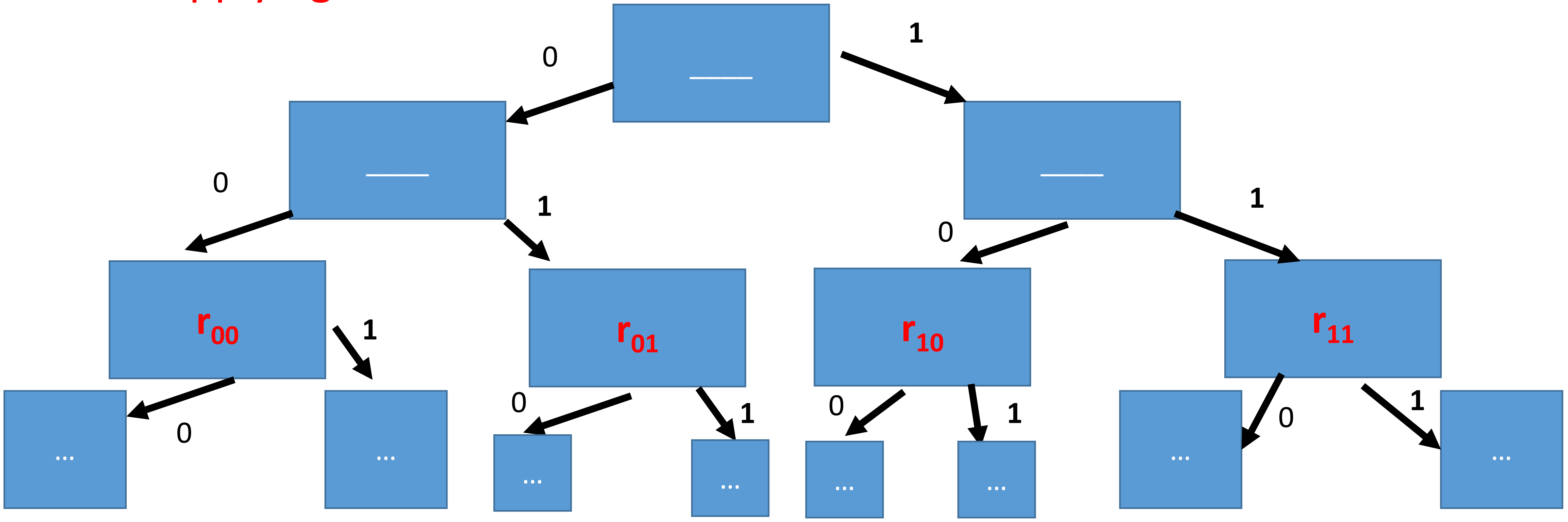
Hybrid H_1 and H_2

- Modified Construction H_2: Pick r_0 and r_1 randomly instead of $r_i = G_i(K)$



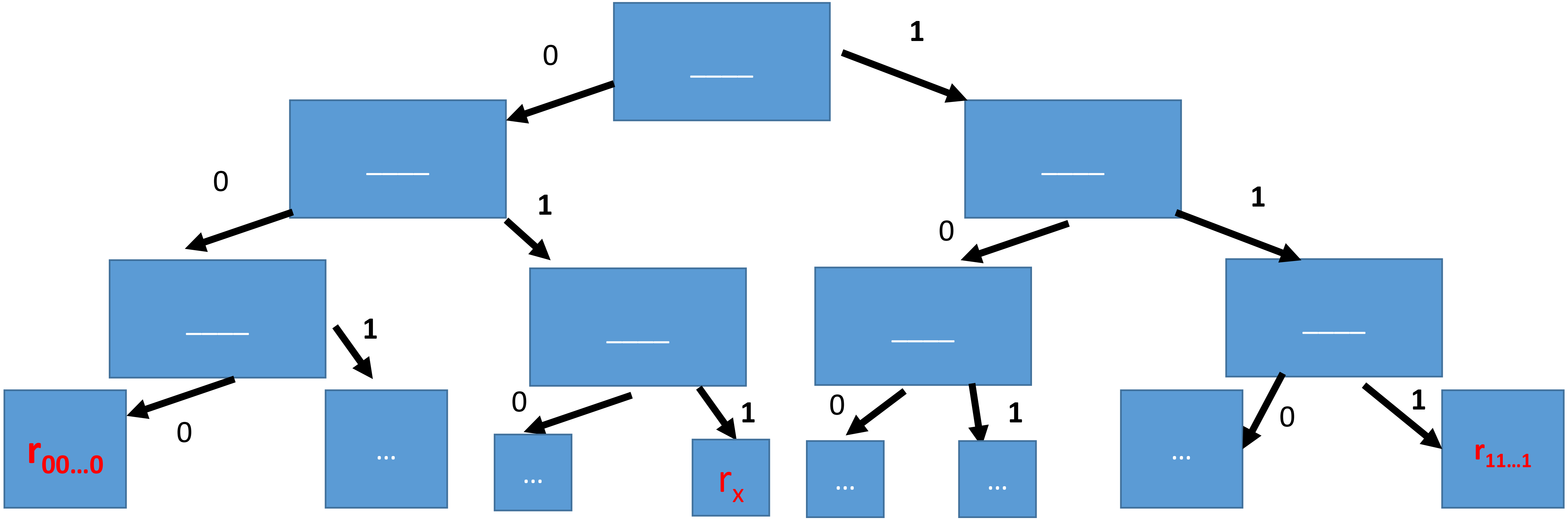
Hybrid H_3

- Modified Construction H_3: Pick r_{00} , r_{01} , r_{10} and r_{11} randomly instead of applying PRG



Hybrid H_n

- Truly Random Function: All output values r_x are picked randomly



Hybrid H_1 vs H_2

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right| < \boldsymbol{negl}(\boldsymbol{n})$$

Claim 2: ***Attacker who makes t(n) queries to F_k (or f) cannot distinguish H_2 from the real game (except with negligible probability).***

Proof Intuition: Follows by Claim 1

Hybrid H_i vs H_i

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right| < \boldsymbol{negl}(\boldsymbol{n})$$

Claim 3: Attacker who makes t(n) queries to F_k (or f) cannot distinguish H_i from H_{i-1} the real game (except with negligible probability).

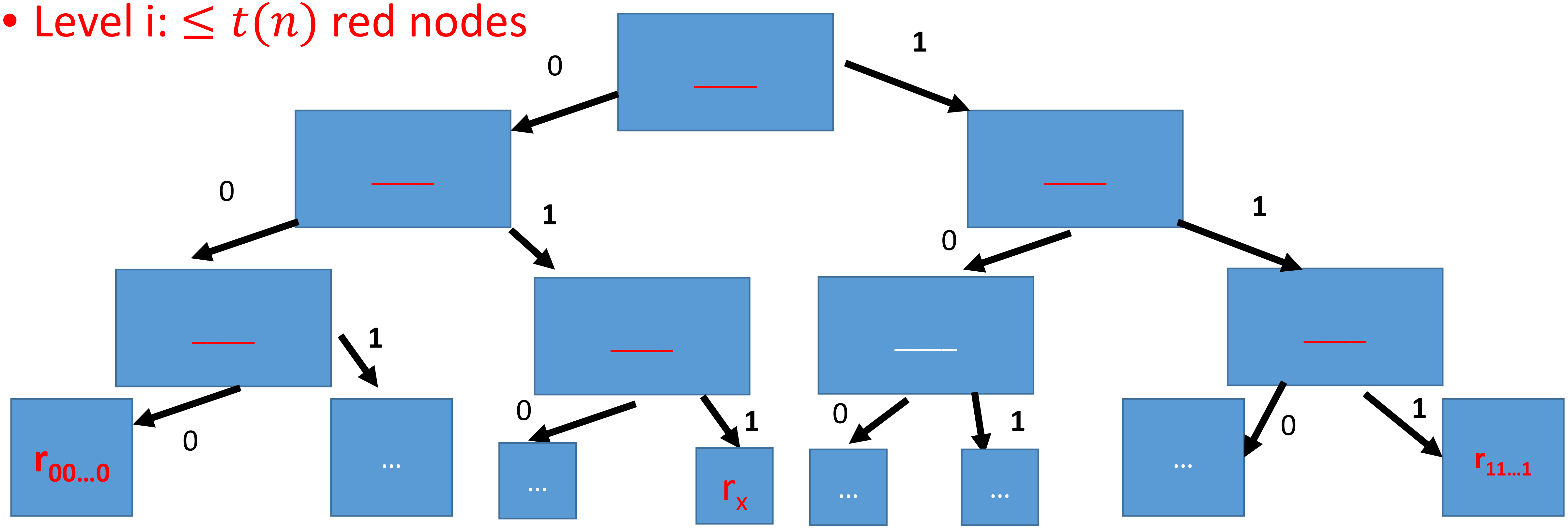
Challenge: Cannot replace 2^i pseudorandom values with random strings at level i

2^i negl(n) is not necessarily negligible if $i = \frac{n}{2}$

Key Idea: Only need to replace t(n) values (note: $t(n)$negl(n) is negligible).

Hybrid H_i

- Red Leaf Nodes: Queried $F_k(x)$ (at most t(n) red leaf nodes)
- Red Internal Nodes: On path from red leaf node to root
- Level i: $\leq t(n)$ red nodes



Hybrid H_i vs H_i

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right| < \boldsymbol{negl}(\boldsymbol{n})$$

Claim 3: Attacker who makes t(n) queries to F_k (or f) cannot distinguish H_i from H_{i-1} the real game (except with negligible probability).

Triangle Inequality: Attacker who makes t(n) queries to F_k (or f) *cannot* distinguish H_1 (real construction) from H_n (truly random function) except with negligible probability.

From OWFs (Recap)

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = n + 1$. Then for any polynomial p(.) there is a PRG with expansion factor p(n).

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = 2n$. Then there is a secure PRF.

Theorem: Suppose that there is a secure PRF then there is a strong pseudorandom permutation.

From OWFs (Recap)

Corollary: If one-way functions exist then PRGs, PRFs and strong PRPs all exist.

Corollary: If one-way functions exist then there exist CCA-secure encryption schemes and secure MACs.

Are OWFs Necessary for Private Key Crypto

- Previous results show that OWFs are <u>sufficient</u>.

- Can we build Private Key Crypto from weaker assumptions?

- **Short Answer:** No, OWFs are also <u>*necessary*</u> for most private-key crypto primitives

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Question: why can we assume that we have an PRG with expansion 2n?

Answer: Last class we showed that a PRG with expansion factor $\ell(n) = n + 1$. Implies the existence of a PRG with expansion p(n) for any polynomial.

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Claim: G is also a OWF!

(Easy to Compute?) ✓

(Hard to Invert?)

Intuition: If we can invert G(x) then we can distinguish G(x) from a random string.

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Claim 1: Any PPT A, given G(s), cannot find s except with negligible probability.

Reduction: Assume (for contradiction) that A can invert G(s) with non-negligible probability p(n).

Distinguisher D(y): Simulate A(y)

Output 1 if and only if A(y) outputs x s.t. G(x)=y.

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Claim 1: Any PPT A, given G(s), cannot find s except with negligible probability.

Intuition for Reduction: If we can find x s.t. G(x)=y then y is not random.

Fact: Select a random 2n bit string y. Then (whp) there does not exist x such that G(x)=y.

Why not?

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Claim 1: Any PPT A, given G(s), cannot find s except with negligible probability.

Intuition: If we can invert G(x) then we can distinguish G(x) from a random string.

Fact: Select a random 2n bit string y. Then (whp) there does not exist x such that G(x)=y.

- Why not? Simple counting argument, 2^{2n} possible y's and 2^n x's.
- Probability there exists such an x is at most 2^{-n} (for a random y)

What other assumptions imply OWFs?

- PRGs → OWFs
- (Easy Extension) PRFs → PRGs → OWFs

- Does secure crypto scheme imply OWFs?
  - CCA-secure? (Strongest)
  - CPA-Secure? (Weaker)
  - EAV-secure? (Weakest)
    - As long as the plaintext is longer than the secret key
  - Perfect Secrecy? X (Guarantee is information theoretic)

EAV-Secure Crypto → OWFs

Proposition 7.29: If there exists a EAV-secure private-key encryption scheme that encrypts messages twice as long as its key, then a one-way function exists.

Recap: EAV-secure.

- Attacker picks two plaintexts m_0,m_1 and is given c=$Enc_K(m_b)$ for random bit b.
- Attacker attempts to guess b.
- No ability to request additional encryptions (chosen-plaintext attacks)
- In fact, no ability to observe any additional encryptions

EAV-Secure Crypto → OWFs

Proposition 7.29: If there exists a EAV-secure private-key encryption scheme that encrypts messages twice as long as its key, then a one-way function exists.

Reduction: $\boldsymbol{f(m,k,r) = Enc_k(m;r)\|m}$.

Input: 4n bits

(For simplicity assume that **Enc**$_k$ accepts n bits of randomness)

Claim: f is a OWF

EAV-Secure Crypto → OWFs

Proposition 7.29: If there exists a EAV-secure private-key encryption scheme that encrypts messages twice as long as its key, then a one-way function exists.

Reduction: $\boldsymbol{f(m,k,r) = Enc_k(m;r) \| m}$.

Claim: f is a OWF

Reduction: If attacker A can invert f, then attacker A' can break EAV-security as follows. Given c=$Enc_k(m_b;r)$ run A(c$\|m_0$). If A outputs (m',k',r') such that $\mathrm{f(m',k',r') = c}\|m_0$ then output 0; otherwise 1;

MACs→ OWFs

In particular, given a MAC that satisfies MAC security (Definition 4.2) against an attacker who sees an arbitrary (polynomial) number of message/tag pairs.

Conclusions: OWFs are necessary and sufficient for all (non-trivial) private key cryptography.

→OWFs are a minimal assumption for private-key crypto.

Public Key Crypto/Hashing?

- OWFs are known to be necessary
- Not known (or believed) to be sufficient.

Computational Indistinguishability

- Consider two distributions X_ℓ and Y_ℓ (e.g., over strings of length ℓ).
- Let D be a distinguisher that attempts to guess whether a string s came from distribution X_ℓ or Y_ℓ.

The advantage of a distinguisher D is

$$Adv_{D,\ell} = \left|Pr_{s\leftarrow X_\ell}[D(s) = 1] - Pr_{s\leftarrow Y_\ell}[D(s) = 1]\right|$$

Definition: We say that an ensemble of distributions $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are <u>computationally indistinguishable</u> if for all PPT distinguishers D, there is a negligible function negl(n), such that we have

$$Adv_{D,n} \leq negl(n)$$