Cryptography
CS 555

Week 5:

- Cryptographic Hash Functions
- HMACs
- Generic Attacks
- Random Oracle Model
- Applications of Hashing

Readings: Katz and Lindell Chapter 5, Appendix A.4

Spring 2021

Recap

- Authenticated Encryption + CCA-Security
  - ~~Encrypt and Authenticate [SSL]~~
  - Authenticate then Encrypt [TLS] (Caution Required)
  - Encrypt **then** Authenticate!

$$Enc_K(m) = \left\langle \mathrm{c}, \mathrm{Mac}'_{K_M}(\mathrm{c}) \right\rangle \text{ where } \mathrm{c} = \mathrm{Enc}'_{K_E}(m)$$

- Secure Communication
  - Attacks: Reflection/Replay/Reordering + Defenses
  - AES-GCM
- Cryptographic Hash Functions
  - Definitional Challenges

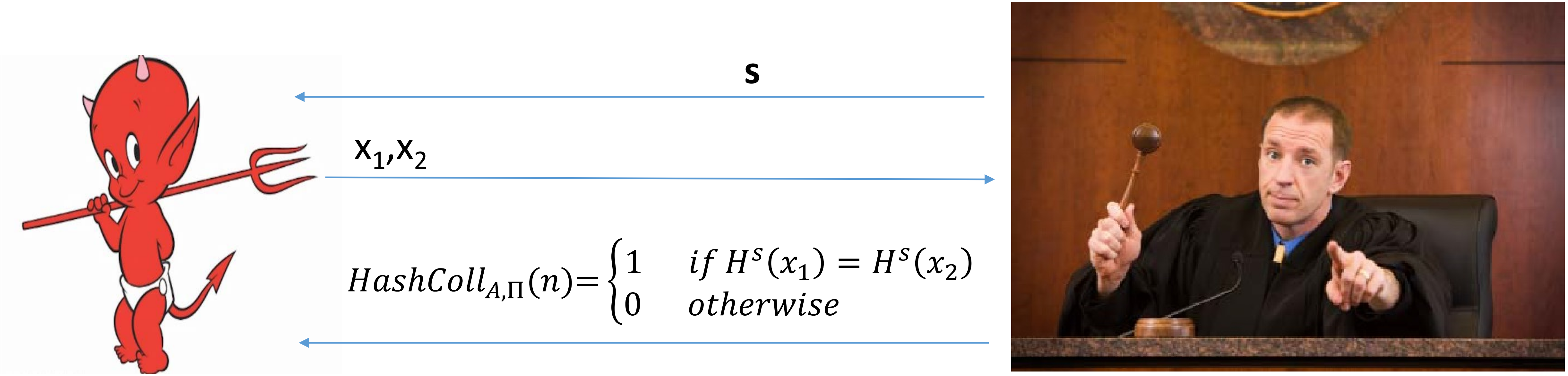
Week 5: Topic 1:
Cryptographic Hash Functions

Keyed Hash Function Syntax

- Two Algorithms
  - $\mathrm{Gen}(1^n; R)$ (Key-generation algorithm)
    - **Input:** Random Bits R
    - **Output:** ~~Secret~~ key s
  - $H^s(m)$ (Hashing Algorithm)
    - **Input:** key s and message $\mathrm{m} \in \{0,1\}^*$ (unbounded length)
    - **Output:** hash value $H^s(m) \in \{0,1\}^{\ell(n)}$

- Fixed length hash function
  - $m \in \{0,1\}^{\ell'(n)}$ with $\ell'(n) > \ell(n)$
  - Example: $m \in \{0,1\}^{2n}$ and $H^s(m) \in \{0,1\}^n$

Collision Experiment ($HashColl_{A,\Pi}(n)$)



s

x_1, x_2

$$HashColl_{A,\Pi}(n) = \begin{cases} 1 & if\ H^s(x_1) = H^s(x_2) \\ 0 & otherwise \end{cases}$$

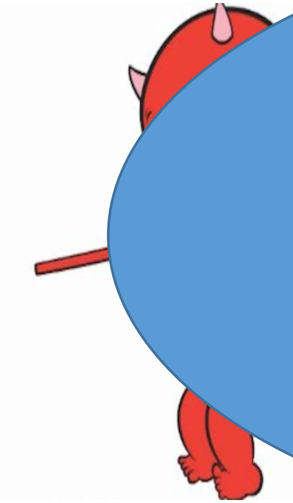
$s = \mathrm{Gen}(1^n; R)$

Definition: (Gen,H) is a collision resistant hash function if

$\forall PPT\ A\ \exists \mu$ (negligible) s.t

$$\Pr\left[HashColl_{A,\Pi}(n) = 1\right] \le \mu(n)$$

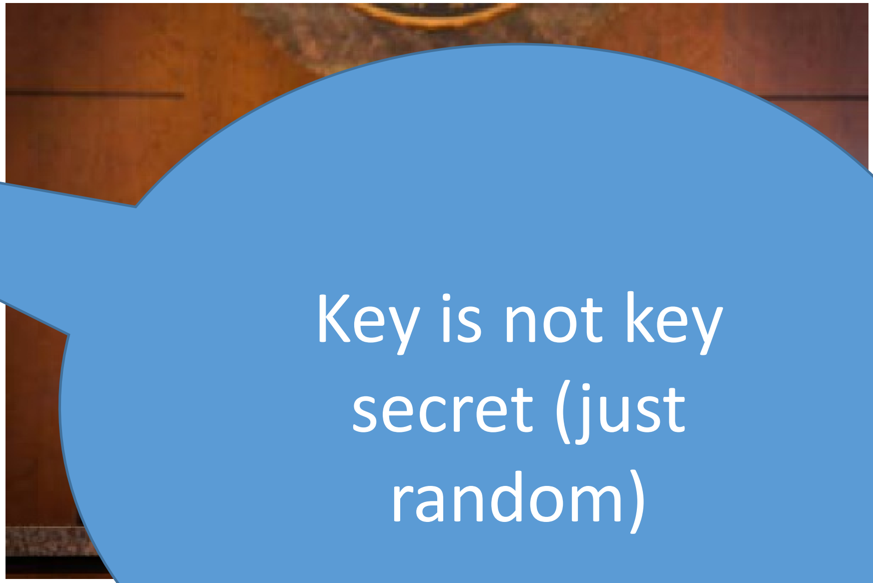
Collision Experiment ($HashColl_{A,\Pi}(n)$)



For simplicity we will sometimes just say that H (or H^s) is a collision resistant hash function

$= H^s(x_2)$

ise



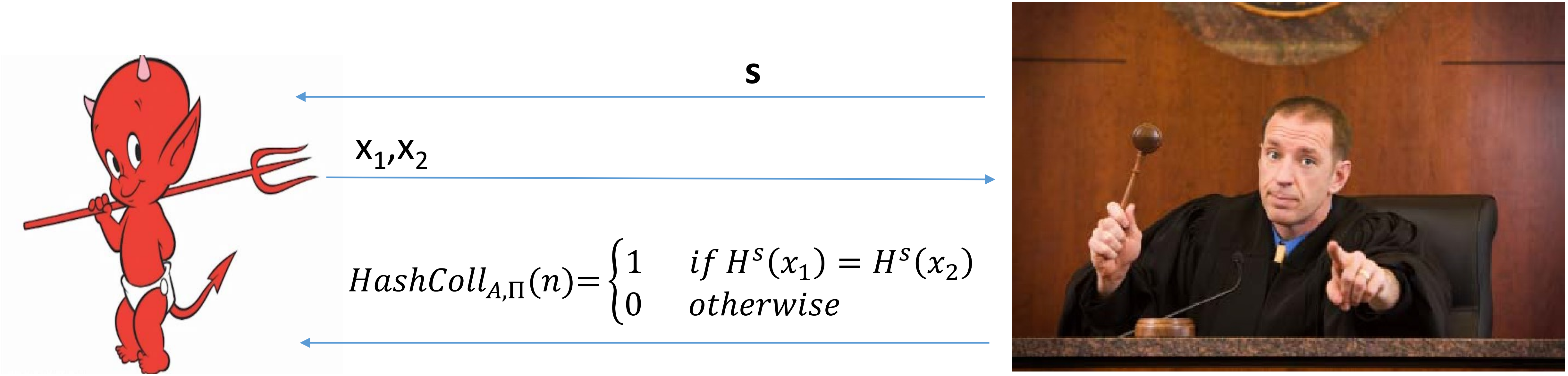
Key is not key secret (just random)

Definition: (Gen,H) is a collision resistant hash function if

$$\forall PPT\ A\ \exists \mu \text{ (negligible) s.t}$$

$$\Pr\left[HashColl_{A,\Pi}(n)=1\right] \leq \mu(n)$$

Concrete Security ($HashColl_{A,\Pi}(n)$)



s

x1,x2

$$HashColl_{A,\Pi}(n)=\begin{cases}1 & if\ H^s(x_1)=H^s(x_2)\\ 0 & otherwise\end{cases}$$

$\mathrm{s} = \mathrm{Gen}(1^n; R)$

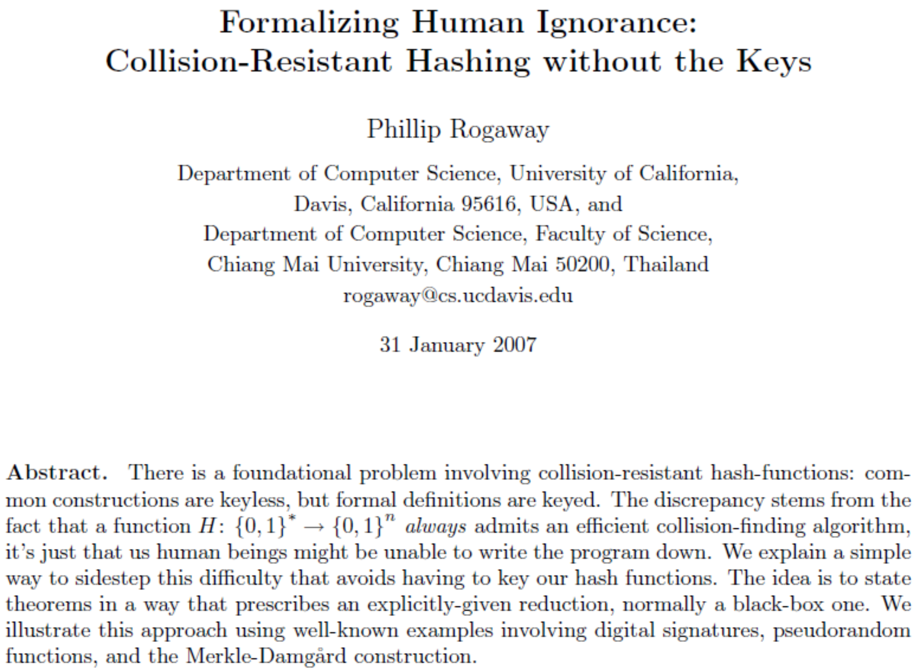
Definition: (Gen,H) is a (t,ε) $-$collision resistant hash function if $\forall\ attackers\ A\ running\ in\ \ time\ at\ most\ t(n)$

$$\Pr\left[HashColl_{A,\Pi}(n)=1\right] \le \varepsilon(n)$$

Theory vs Practice

- Most cryptographic hash functions used in practice are un-keyed
  - Examples: MD5, SHA1, SHA2, SHA3, Blake2B
- Tricky to formally define collision resistance for keyless hash function
  - There is a PPT algorithm to find collisions
  - We just usually can't find this algorithm ☺
  - Guarantee for protocol using H

    If we know an explicit efficient algorithm A breaking our protocol then there is an efficient blackbox reduction transforming A into an efficient collision finding algorithm.



Formalizing Human Ignorance:
Collision-Resistant Hashing without the Keys

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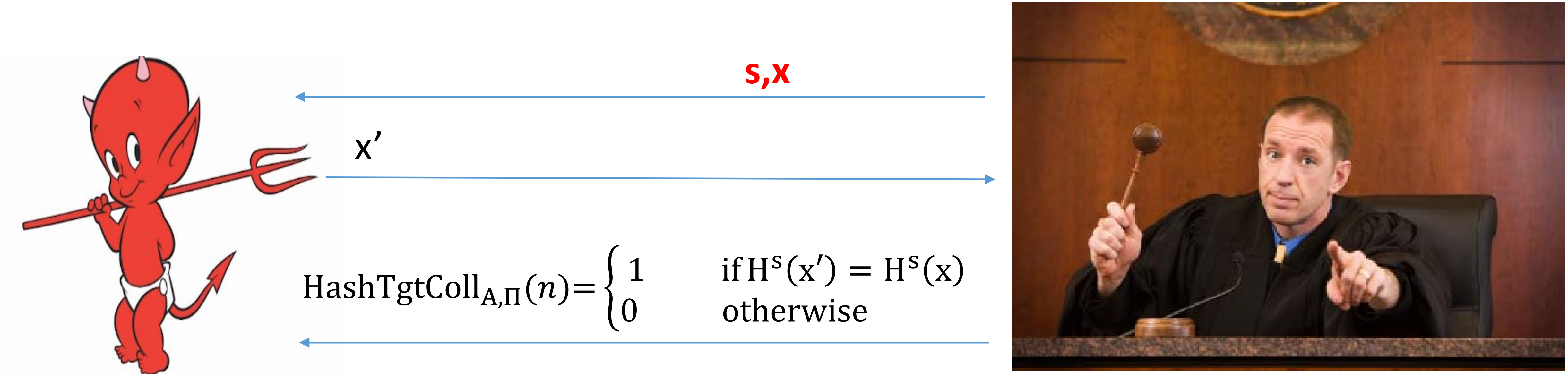
31 January 2007

Abstract. There is a foundational problem involving collision-resistant hash-functions: common constructions are keyless, but formal definitions are keyed. The discrepancy stems from the fact that a function $H\colon \{0,1\}^* \to \{0,1\}^n$ *always* admits an efficient collision-finding algorithm, it's just that us human beings might be unable to write the program down. We explain a simple way to sidestep this difficulty that avoids having to key our hash functions. The idea is to state theorems in a way that prescribes an explicitly-given reduction, normally a black-box one. We illustrate this approach using well-known examples involving digital signatures, pseudorandom functions, and the Merkle-Damgård construction.

Weaker Requirements for Cryptographic Hash

- Target-Collision Resistance



$$\mathrm{HashTgtColl}_{A,\Pi}(n) = \begin{cases} 1 & \text{if } H^s(x') = H^s(x) \\ 0 & \text{otherwise} \end{cases}$$

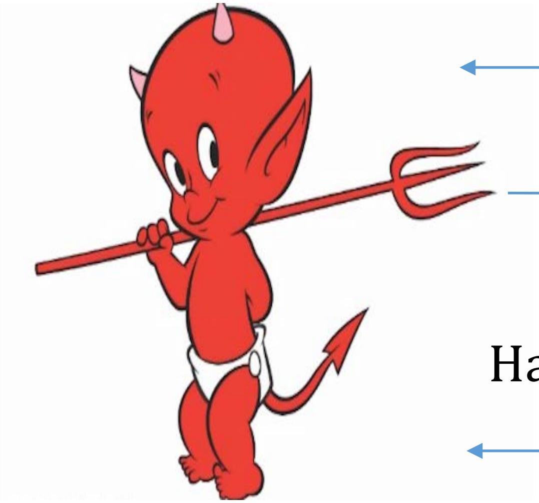
$$s = \mathrm{Gen}(1^n; R)$$

$$x \in \{0,1\}^n$$

Question: Why is collision resistance stronger?

Weaker Requirements for Cryptographic Hash

- Preimage Resistance (One-Wayness)



s, y

x

$$\text{HashPreImgRes}_{A,\Pi}(n) = \begin{cases} 1 & \text{if } H^{s}(x) = y \\ 0 & \text{otherwise} \end{cases}$$

$s = \mathrm{Gen}(1^n; R)$

$y \in \{0,1\}^{\ell(n)}$

Question: Why is collision resistance stronger?

Merkle-Damgård Transform

- Most cryptographic hash functions accept fixed length inputs

- What if we want to hash arbitrary length strings?

Construction: Suppose (Gen,h) fixed length hash function from 2n bits to n bits, define H^s as follows

$$H^s(x_1, \dots, x_d) = h^s\big(h^s\big(h^s\big(h^s\big(\dots h^s(0^n \parallel x_1)\big) \parallel x_{d-1}\big) \parallel x_d\big) \parallel |x|\big)$$

Merkle-Damgård Transform

Construction: (Gen,h) fixed length hash function from 2n bits to n bits

```
$H^s(x)$ =
1. Break x into n bit segments $x_1$,..,$x_d$ (pad last block by 0's)
2. $z_0 = 0^n$ (initialization)
3. For i = 1 to d
    1. $z_i = h^s(z_{i-1} \parallel x_{\mathrm{i}})$
4. Output $z_{d+1} = h^s(z_d \parallel L)$ where $L$ encodes $|x|$ as an n-bit string
```

Merkle-Damgård Transform

Theorem: If (Gen,h) is collision resistant then so is (Gen,H)

Proof: Show that any collision in H^s yields a collision in h^s. Thus a PPT attacker A_H for (Gen,H) can be transformed into PPT attacker A_h for (Gen,h).

Suppose that A_H finds a collision i.e., distinct x and x' such that

$$H^s(x) = H^s(x')$$

(note x and x' may have different lengths)

Merkle-Damgård Transform

Theorem: If (Gen,h) is collision resistant then so is (Gen,H)

Proof: Suppose that $H^s(x) = H^s(x')$. We will extract a collision for h^s.

Case 1: L=|x|=|x'|=L' (proof for case two is similar)

$$H^s(x) = z_{d+1} = h^s(z_d \parallel L) = H^s(x') = z'_{d+1} = h^s(z'_d \parallel L')$$

$$z_d \parallel L =?\, z'_d \parallel L'$$

No → Found collision

$h^s(z_d \parallel L) = h^s(z'_d \parallel L')$

Yes?

$$z_d = h^s(z_{d-1} \parallel x_d) = h^s(z'_{d-1} \parallel x'_d) = z'_d$$

Merkle-Damgård Transform

Theorem: If (Gen,h) is collision resistant then so is (Gen,H)

Proof: Suppose that

$$H^s(x) = H^s(x')$$

Case 1: L=|x|=|x'|=L' (proof for case two is similar)

$$z_d = h^s(z_{d-1} \parallel x_d) = h^s(z'_{d-1} \parallel x'_d) = z'_d$$

$$z_{d-1} \parallel x_d =? \, z'_{d-1} \parallel x'_d$$

No → Found collision

$h^s(z_{d-1} \parallel x_d) = h^s(z'_{d-1} \parallel x'_d)$

Yes?

$$z_{d-1} = h^s(z_{d-2} \parallel x_{d-1}) = h^s(z'_{d-2} \parallel x'_{d-1}) = z'_{d-1}$$

Merkle-Damgård Transform

Theorem: If (Gen,h) is collision resistant then so is (Gen,H)

Proof: Suppose that

$$H^s(x) = H^s(x')$$

Case 1: |x|=|x'| (proof for case two is similar)

If for some i we have $z_{i-1} \parallel x_i \neq z'_{i-1} \parallel x'_i$ then we will find a collision

But x and x' are different so we must have $x_i \neq x'_i$ for some $i \leq d$!

Merkle-Damgård Transform

Theorem (Concrete Version): If (Gen,h) is (t, ε)-collision resistant then (Gen,H) is is (t', ε)-collision resistant for where $t' = O(t)$

Analysis: Run attacker A_H to get pair x and x’ (time t), then compute z_i (resp. $z_i{}'$) values to extract collision.

$$H^s(x) = z_{d+1} = h^s(z_d \parallel L) = H^s(x') = z'_{d+1} = h^s(z'_d \parallel L')$$

$$z_d \parallel L =?\, z'_d \parallel L'$$

No → Found collision

$$h^s(z_d \parallel L) = h^s(z'_d \parallel L')$$

Yes?

$$z_d = h^s(z_{d-1} \parallel x_d) = h^s(z'_{d-1} \parallel x'_d) = z'_d$$

Week 5: Topic 2: HMACs and Generic Attacks

MACs for Arbitrary Length Messages

$\text{Mac}_K(m)$=

- Select random n/4 bit string r
- Let $t_i = \mathrm{Mac}'_K(r \parallel \ell \parallel i \parallel m_i)$ for i=1,…,d
  - (Note: encode i and ℓ as n/4 bit strings)
- **Output** $\langle r, t_1, \dots, t_d \rangle$

Theorem 4.8: If Π’ is a secure MAC for messages of fixed length n, above construction $\Pi = (\mathrm{Mac}, \mathrm{Vrfy})$ is secure MAC for arbitrary length messages.

MACs for Arbitrary Lengt

Disadvantage 1: Long output

Disadvantages: Lose Strong-MAC Guarantee (Multiple valid MACs of same message)

ri

‖ i ‖

- (Note: e and ℓ as n/4 bi

- **Output** $\langle r, t_1, \ldots, t_d \rangle$

Randomized Construction (no canonical verification). Disadvantage?

Theorem 4.8: If Π' i above constructio messages.

Hash and MAC Construction

Start with $\Pi = (\mathrm{Mac}, \mathrm{Vrfy})$, a secure MAC for messages of fixed length, and (Gen$_H$,H) a collision resistant hash function and define Π'

$$Mac'_{\langle K_M, S \rangle}(m) = Mac_{K_M}\left(H^s(m)\right)$$

$$Vrfy'_{\langle K_M, S \rangle}(m, t) = Vrfy_{K_M}(H^s(m), t)$$

Theorem 5.6: Π' is a secure MAC for arbitrary length message assuming that Π is a secure MAC and (Gen$_H$,H) is collision resistant.

Note: If $\mathrm{Vrfy}_{K_M}(m, t)$ is canonical then $\mathrm{Vrfy}'_{\langle K_M, S \rangle}(m, t)$ is canonical.

Hash and MAC Construction

Start with (Mac,Vrfy) a MAC for messages of fixed length and (Gen$_H$,H) a collision resistant hash function

$$Mac'_{\langle K_M, S\rangle}(m) = Mac_{K_M}\left(H^s(m)\right)$$

Theorem 5.6: Above construction is a secure MAC.

Proof Intuition: If attacker successfully forges a valid MAC tag t' for unseen message m' then either

- **Case 1:** $H^s(m') = H^s(m_i)$ for some previously requested message m$_i$
- **Case 2:** $H^s(m') \neq H^s(m_i)$ for every previously requested message m$_i$

Hash and MAC Construction

Theorem 5.6: Above construction is a secure MAC.

Proof Intuition: If attacker successfully forges a valid MAC tag t' for unseen message m' then either

- **Case 1:** $H^s(m') = H^s(m_i)$ for some previously requested message m_i
  - **Attacker can find hash collisions!**
- **Case 2:** $H^s(m') \neq H^s(m_i)$ for every previously requested message m_i
  - **Attacker forged a valid new tag on the "new message" $\boldsymbol{H^s(m')}$**
  - **Violates security of the original fixed length MAC**

Hash and MAC Construction

Start with (Mac,Vrfy) a MAC for messages of fixed length and (Gen$_H$,H) a collision resistant hash function

$$Mac'_{\langle K_M,S\rangle}(m) = Mac_{K_M}\left(H^s(m)\right)$$

Theorem 5.6 (Concrete Version): If Mac is $(t, q_{MAC}, \varepsilon_{MAC}) -$ secure and (Gen$_H$,H) is (t, ε_{Hash}) −collision resistant then $Mac'_{\langle K_M,S\rangle}$ is $(O(t), q_{MAC}, \varepsilon_{MAC} + \varepsilon_{Hash}) - secure$

Proof Intuition: When A succeeds we either get a hash collision (case 1) or a Mac_{K_M} forgery (case 2)

if $\Pr[\text{case } 2] > \varepsilon_{MAC}$ **we could violate** $(t, q_{MAC}, \varepsilon_{MAC}) - \text{secure for } Mac_{K_M}$

Simulate $Mac'_{\langle K_M,S\rangle}$ attacker A

when attacker makes a query $Mac'_{\langle K_M,S\rangle}(m)$ we

1. compute $H^s(m)$ and
2. forward $H^s(m)$ to Mac_{K_M}oracle to get back $Mac_{K_M}\left(H^s(m)\right)$

A's tag yields a Mac_{K_M} forgery for new message with probability at least $\Pr[\text{case } 2] > \varepsilon_{MAC}$

Similar argument If $\Pr[\text{case } 1] > \varepsilon_{HASH}$ **we could violate** (t, ε_{Hash}) −collision resistance for $H^s(.)$

Therefore, A succeeds with probability at most $\varepsilon_{MAC} + \varepsilon_{Hash}$

Recap

- Definition of Collision Resistant Hash Functions (Gen,H)
  - Definitional challenges
  - Gen(1^n) outputs a public seed.

- Merkle-Damgård construction to hash arbitrary length strings
  - Proof of correctness

- Hash and MAC construction
  - Proof of correctness

MAC from Collision Resistant Hash

- Failed Attempt:

$$Mac_{\langle k,S\rangle}(m) = H^s(k \parallel m)$$

Broken if H^suses Merkle-Damgård Transform. Let m_3 encode length of $\mathrm{k} \parallel m_1 \parallel m_2$ and L_3 encode the length of$\mathrm{k} \parallel m_1 \parallel m_2 \parallel m_3$

$$Mac_{\langle k,S\rangle}(m_1 \parallel m_2 \parallel m_3) = h^s(h^s(h^s(h^s(h^s(0^n \parallel k) \parallel m_1) \parallel m_2) \parallel m_3) \parallel L_3)$$
$$= h^s\left(Mac_{\langle k,S\rangle}(m_1 \parallel m_2) \parallel L_3\right)$$

Why does this mean $Mac_{\langle k,S\rangle}$ is broken?

MAC from Collision Resistant Hash

- Failed Attempt: $Mac_{\langle k,S \rangle}(m) = H^s(k \parallel m)$

Broken if H^suses Merkle-Damgård Transform. Let m_3 encode length of $\mathrm{k} \parallel m_1 \parallel m_2$

$$Mac_{\langle k,S \rangle}(m_1 \parallel m_2 \parallel m_3) = h^s(h^s(h^s(h^s(h^s(0^n \parallel k) \parallel m_1) \parallel m_2) \parallel m_3) \parallel L_3)$$
$$= h^s\left(Mac_{\langle k,S \rangle}(m_1 \parallel m_2) \parallel L_3\right)$$

Why does this mean $\boldsymbol{Mac_{\langle k,S \rangle}}$ is broken?

1. **Attacker asks for $\boldsymbol{\tau} = Mac_{\langle k,S \rangle}(m_1 \parallel m_2)$**
2. **Attacker computes $\boldsymbol{\tau}' = h^s(\boldsymbol{\tau} \parallel L_3)$ which is a forgery for the message $m_1 \parallel m_2 \parallel m_3$**

HMAC

$$Mac_{\langle k,S\rangle}(m) = H^s\Big((k\oplus\text{opad}) \parallel H^s\big((k\oplus\text{ipad}) \parallel m\big)\Big)$$

ipad?

HMAC

$$Mac_{\langle k,S\rangle}(m) = H^s\left((k\oplus\text{opad}) \parallel H^s\left((k\oplus\text{ipad}) \parallel m\right)\right)$$

$$\text{ipad} = \text{inner pad}$$
$$\text{opad} = \text{outer pad}$$

Both ipad and opad are fixed constants.

Why use key twice?

Allows us to prove security from *weak collision resistance* of H^s

HMAC Security

$$Mac_{\langle k,S\rangle}(m) = H^s\left((k\oplus\mathrm{opad}) \parallel H^s\big((k\oplus\mathrm{ipad}) \parallel m\big)\right)$$

Theorem (Informal): Assuming that H^s is weakly collision resistant and that (certain other plausible assumptions hold) this is a secure MAC.

Weak Collision Resistance: Give attacker oracle access to $f(m) = \ H^s(k \parallel m)$ (secret key k remains hidden).

Attacker Goal: Find distinct m, m' such that $f(m) = f(m')$

HMAC in Practice

- MD5 can no longer be viewed as collision resistant

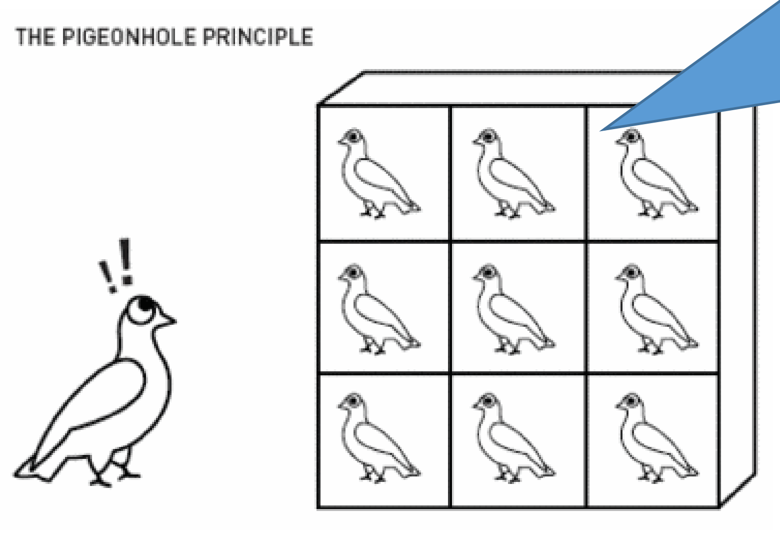
- However, HMAC-MD5 remained unbroken after MD5 was broken
  - Gave developers time to replace HMAC-MD5
  - Nevertheless, don’t use HMAC-MD5!
- HMAC-SHA1 still seems to be okay (temporarily), despite collision
- HMAC is efficient and unbroken
  - CBC-MAC was not widely deployed because it is “too slow”
  - Instead practitioners often used heuristic constructions (which were breakable)

Finding Collisions

- Ideal Hashing Algorithm
  - Random function H from $\{0,1\}^*$ to $\{0,1\}^{\ell}$
  - Suppose attacker has oracle access to H(.)

- **Attack 1:** Evaluate H(.) on $2^{\ell}+1$ distinct inputs.

Can we do better?



Birthday Attack for Finding Collisions

- Ideal Hashing Algorithm
  - Random function H from {0,1}* to $\{0,1\}^{\ell}$
  - Suppose attacker has oracle access to H(.)

- **Attack 2:** Evaluate H(.) on $q = 2^{(\ell/2)+1} + 1$ distinct inputs $x_1,\ldots,x_q$.

$$\Pr[\text{No Collision}] = \Pr[\forall i < j. H(\mathrm{x}_i) \neq H(\mathrm{x}_j)]$$

$$= \Pr[\boldsymbol{D}_2] \prod_{i=3}^{q} Pr[\boldsymbol{D}_i | \boldsymbol{D}_{i-1}, \ldots, \boldsymbol{D}_2]$$

$$\boldsymbol{D}_i = \boldsymbol{event\ that}\ H(x_i) \neq H(x_{i-1}), \ldots, H(x_1)$$

Birthday Attack for Finding Collisions

- Ideal Hashing Algorithm
  - Random function H from {0,1}* to $\{0,1\}^{\ell}$
  - Suppose attacker has oracle access to H(.)

- **Attack 2:** Evaluate H(.) on $q = 2^{(\ell/2)+1} + 1$ distinct inputs $x_1,\ldots,x_q$.

$$\Pr[\forall i < j.\, H(x_i) \neq H(x_j)] =$$

$$1 \times \overbrace{\left(1-\frac{1}{2^{\ell}}\right)}^{\Pr[\overbrace{H(x_2)\neq H(x_1)}^{D_2}]} \times \overbrace{\left(1-\frac{2}{2^{\ell}}\right)}^{\Pr[D_3 \mid D_2]} \times \cdots \times \overbrace{\left(1-\frac{2^{(\ell/2)+1}}{2^{\ell}}\right)}^{\Pr\left[D_q \middle| D_{q-1},\ldots,D_2\right]}$$

Birthday Attack for Finding Collisions

- Ideal Hashing Algorithm
  - Random function H from {0,1}* to $\{0,1\}^{\ell}$
  - Suppose attacker has oracle access to H(.)

- **Attack 2:** Evaluate H(.) on $q = 2^{(\ell/2)+1} + 1$ distinct inputs $x_1,\ldots,x_q$.

$$\Pr[\forall i < j. H(x_i) \neq H(x_j)] = 1\left(1 - \frac{1}{2^{\ell}}\right)\left(1 - \frac{2}{2^{\ell}}\right)\left(1 - \frac{3}{2^{\ell}}\right)\ldots\left(1 - \frac{2^{(\ell/2)+1}}{2^{\ell}}\right)$$

$$\approx \exp\left(\frac{-q(q-1)}{2^{\ell+1}}\right)$$

Birthday Attack for Finding Collisions

- Ideal Hashing Algorithm
  - Random function H from {0,1}* to $\{0,1\}^{\ell}$
  - Suppose attacker has oracle access to H(.)

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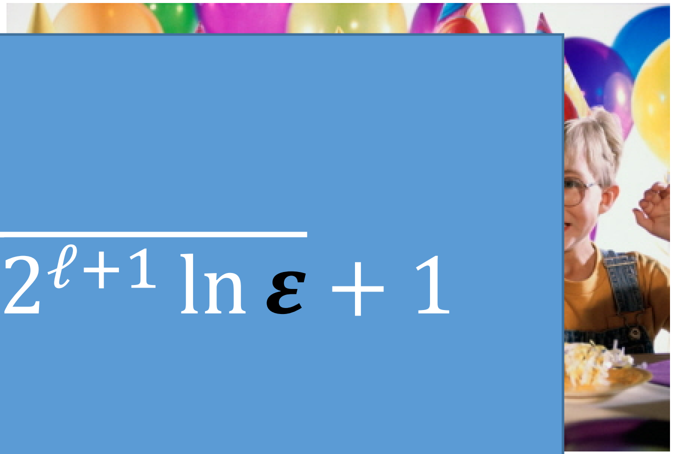
$$\approx \exp\left(\frac{-q(q-1)}{2^{\ell+1}}\right) < \exp\left(\frac{-42^{\ell}}{2^{\ell+1}}\right) = e^{-2} < \frac{1}{2}$$

Birthday Attack for Finding Collisions



- Ideal Hashing Algorit
  - Random function H f
  - Suppose attacker has

$$\exp\left(\frac{-q(q-1)}{2^{\ell+1}}\right) < \boldsymbol{\varepsilon} \text{ for } q > \sqrt{2^{\ell+1}\ln \boldsymbol{\varepsilon}} + 1$$

- **Attack 2:** Evaluate H(

$$\Pr[\forall i < j.\, H(\mathrm{x}_i) \neq H(\mathrm{x}_j)] = \left(1 - \frac{1}{2^{\ell}}\right)\left(1 - \frac{2}{2^{\ell}}\right)\left(1 - \frac{3}{2^{\ell}}\right)\ldots\left(1 - \frac{2^{(\ell/2)+1}}{2^{\ell}}\right)$$

$$\approx \exp\left(\frac{-q(q-1)}{2^{\ell+1}}\right) < \exp\left(\frac{-4 2^{\ell}}{2^{\ell+1}}\right) = e^{-2} < \frac{1}{2}$$

Recap

- Collision Resistant Hash Functions
- Merkle–Damgård Construction
- Applications to MACs
  - Hash and MAC
  - Failed MAC: $Mac_{\langle k,S \rangle}(m) = H^s(k \parallel m)$
  - HMAC
- Birthday Attack: Finds collision in time $q = 2^{(\ell/2)+1} + 1$ (and space q)

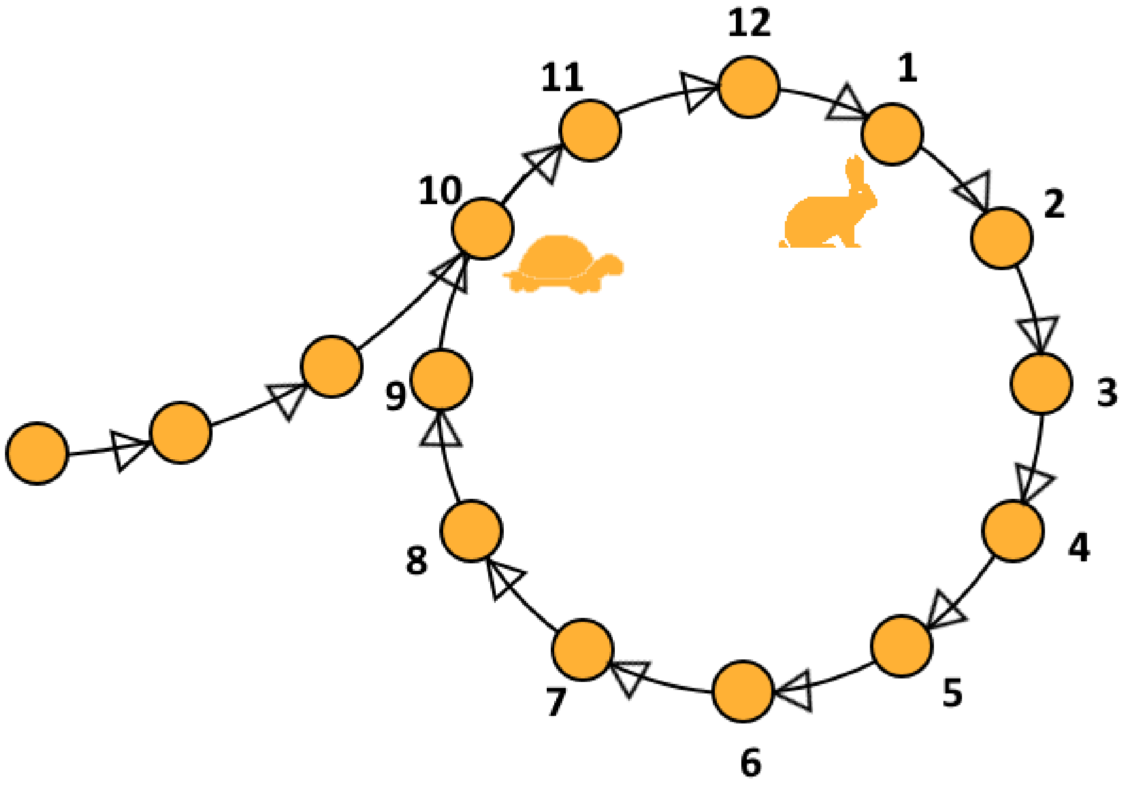
- **Reminder:** Homework 2 Due Tonight
- **Final Exam:** Monday, May 3 at 10:30AM (FRNY B124)

Birthday Attack for Finding Collisions

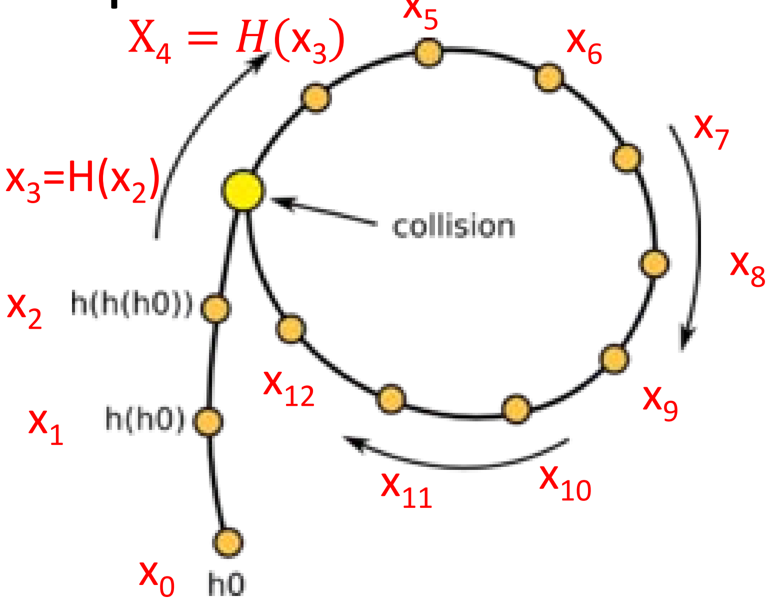
- Ideal Hashing Algorithm
  - Random function H from {0,1}* to $\{0,1\}^{\ell}$
  - Suppose attacker has oracle access to H(.)

- **Attack 2:** Evaluate H(.) on $q = 2^{(\ell/2)+1} + 1$ distinct inputs $x_1,\ldots,x_q$.
- Store values $(x_i, H(x_i))$ in a hash table of size q
  - Requires time/space $O(q) = O\left(\sqrt{2^{\ell}}\right)$
  - Can we do better?

Floyd’s Cycle Finding Algorithm



- A cycle denotes a hash collision
    - $x_3=H(x_2)=H(x_{12})$
- Occurs after $O\left(2^{\ell/2}\right)$ steps by birthday paradox
- First attack phase detects cycle
- Second phase identifies collision



- Analogy: Cycle detection in linked list
- Can traverse “linked list” by computing H

Small Space Birthday Attack

- **Attack 2:** Select random x_0, define $x_i := H(x_{i-1})$
  - Initialize: $x=x_0$ and $x'=x_0$
  - Repeat for i=1,2,…
    - $x:=H(x)$ now $x = x_i$
    - $x':=H(H(x'))$ now $x' = x_{2i}$
    - If $x=x'$ then break
  - Reset $x=x_0$ and set $x'=x$ and remember i
  - Repeat for j=1 to i
    - If $H(x) = H(x')$ then output x,x'
    - Else $x:= H(x)$, $x' = H(x)$

Now $x=x_j$ AND $x' = x_{i+j}$

Claim: for some $k \leq i$ the collision is

$x_k=H(x_{k-1})=H(x_{k+i-1})$

Proof: Let C be length of cycle,

Let k= #steps before cycle

2i-k = i-k mod C ➔ i= mod C

Hare takes 2i-k total steps inside cycle, looping around before ending in same place

Tortoise takes i-k steps inside cycle (equivalent to k backwards steps)

Initially, for phase 2 we have $x'=x_i$ and $x = x_0$ after j=k-1 steps we have $x=x_{k-1}$ and

$x'=x_{i+K-1}=x_{k+C-1}$

Small Space Birthday Attack

- **Attack 2:** Select random x_0, define $x_i = H(x_{i-1})$
  - Initialize: $x=x_0$ and $x'=x_0$
  - Repeat for i=1,2,…
    - $x:=H(x)$ now $x = x_i$
    - $x':=H(H(x'))$ now $x' = x_{2i}$
    - If x=x' then break
  - Reset $x=x_0$ and set $x'=x$
  - Repeat for j=1 to i
    - If H(x) = H(x') then output x,x'
    - Else $x:= H(x)$, $x' = H(x)$ Now $x=x_j$ AND $x' = x_{i+j}$

Finds collision after $O\left(2^{\ell/2}\right)$ steps in expectation

Small Space Birthday Attack

- Can be adapted to find "meaningful collisions" if we have a large message space O$\left(2^{\ell}\right)$

- **Example**: $\mathrm{S} = S_1 \cup S_2$ with $|S_1| = |S_2| = 2^{\ell-1}$
  - S_1 = Set of positive recommendation letters
  - S_2 = Set of negative recommendation letters

- **Goal**: find $z_1 \in S_1, z_2 \in S_2$, such that H(z$_1$) = H(z$_2$)

- Can adapt previous attack by defining an injective mapping $\mathrm{b}\colon \{0,1\}^{\ell} \rightarrow S$

$$\mathrm{x_i} = H(\mathrm{b}(\mathrm{x_{i-1}}))$$

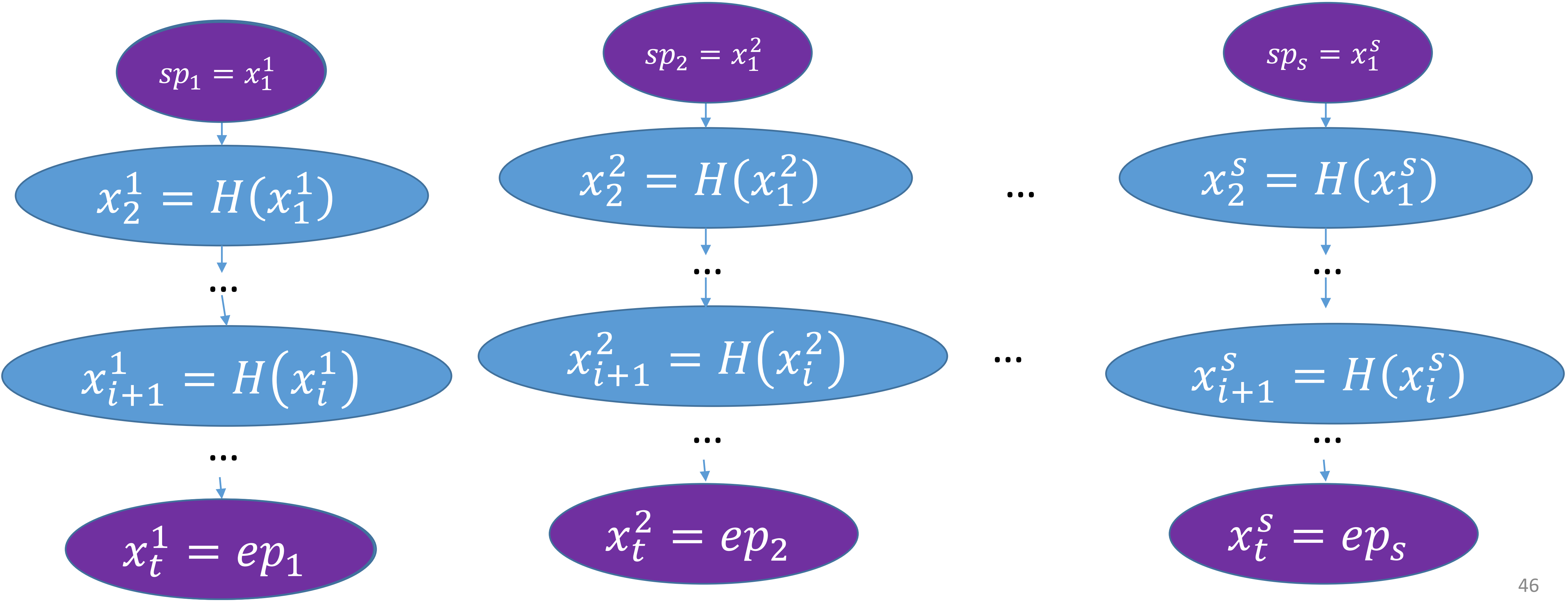
- If $\mathrm{x_i} = \mathrm{x_{i+j}}$ then $H\left(\mathrm{b}(\mathrm{x_{i-1}})\right) = H\left(\mathrm{b}(\mathrm{x_{i+j-1}})\right)$ ➔ Colliding inputs are both in S

Pre-Computation Attacks for Targeted Collision

- **Challenger:** Picks random x and sends y=H(x) to attacker
- **Attacker's Goal:** Find some x' (not necessarily x) s.t. y=H(x')
- **Brute-Force Attack:** Requires $2^{\ell-1}$ queries to H on average.
- **Pre-Computation Attack:** What if we know we will need to do this multiple times?
  - Pre-Processing Cost (one-time cost): $O(2^{\ell})$
  - Post-Processing Cost: $\ll 2^{\ell}$ (is this possible?)
- **Applications:**
  - Targeted Hash Inversion, MAC forgery, Signature Forgery, Key-Recovery, Password Cracking etc…

Pre-Computation Attacks for Targeted Collision

- Precomputation ($t \times s$ steps, $2s$ memory)



$sp_1 = x_1^1$

$x_2^1 = H(x_1^1)$

…

$x_{i+1}^1 = H(x_i^1)$

…

$x_t^1 = ep_1$

$sp_2 = x_1^2$

$x_2^2 = H(x_1^2)$

…

$x_{i+1}^2 = H(x_i^2)$

…

$x_t^2 = ep_2$

…

…

$sp_s = x_1^s$

$x_2^s = H(x_1^s)$

…

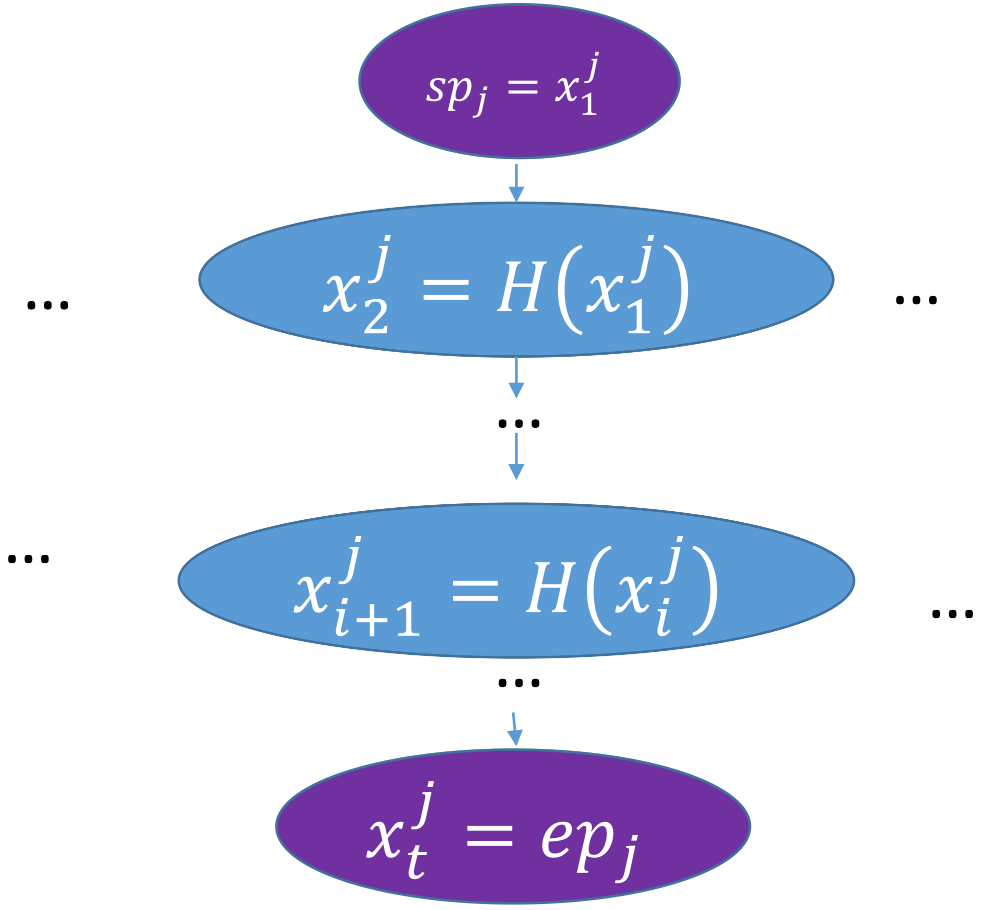
$x_{i+1}^s = H(x_i^s)$

…

$x_t^s = ep_s$

Pre-Computation Attacks for Targeted Collision

- Precomputation ($t \times s$ steps, $2s \times \ell$ memory)



$sp_j = x_1^j$

… $x_2^j = H\left(x_1^j\right)$ …

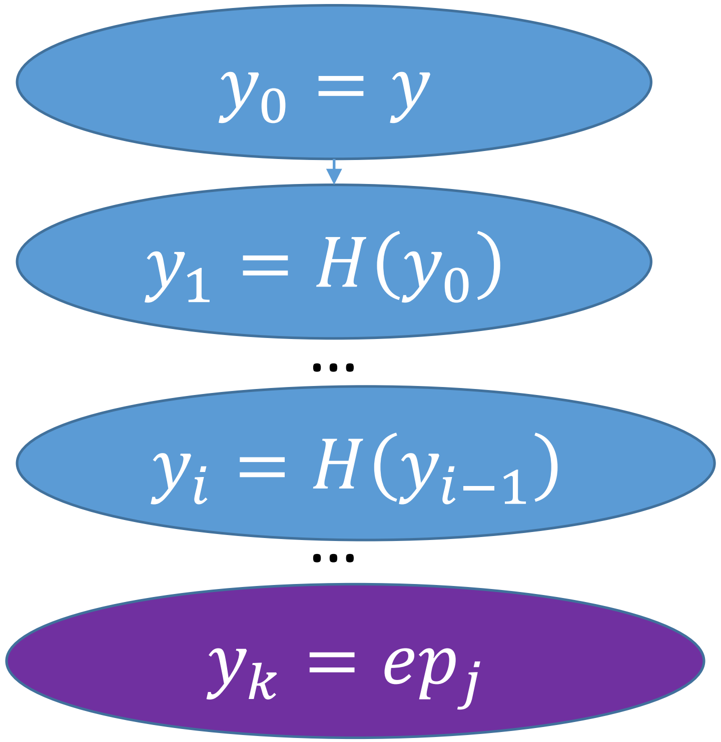
…

… $x_{i+1}^j = H\left(x_i^j\right)$ …

…

$x_t^j = ep_j$

- **Goal:** Find collision for target $y = H(x)$



$y_0 = y$

$y_1 = H(y_0)$

…

$y_i = H(y_{i-1})$

…

$y_k = ep_j$

Pre-Computation Attack sion

- Precomputation ($t \times s$ steps, $2s \times$

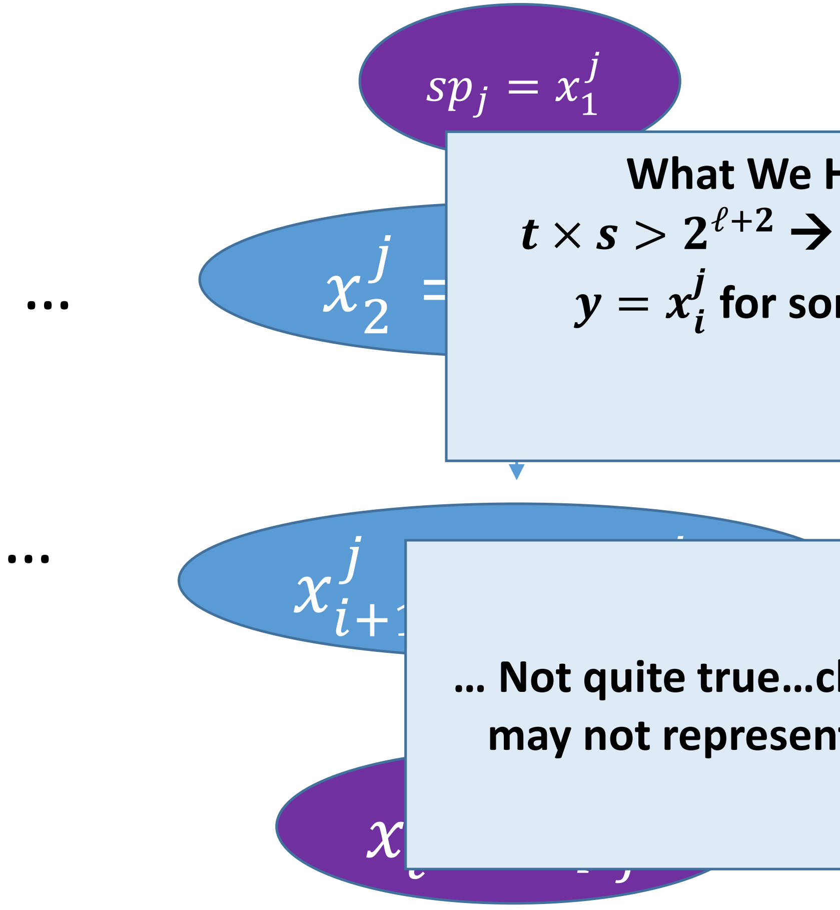
Suppose $y = x_i^j$ for some $i \le t, j \le s$

→

$y = H\left(x_{i-1}^j\right) = H^{i-1}(sp_j)$

(takes t steps to recover x_{i-1}^j from sp_j)

- **Goal:** F collision for target $y = H(x)$

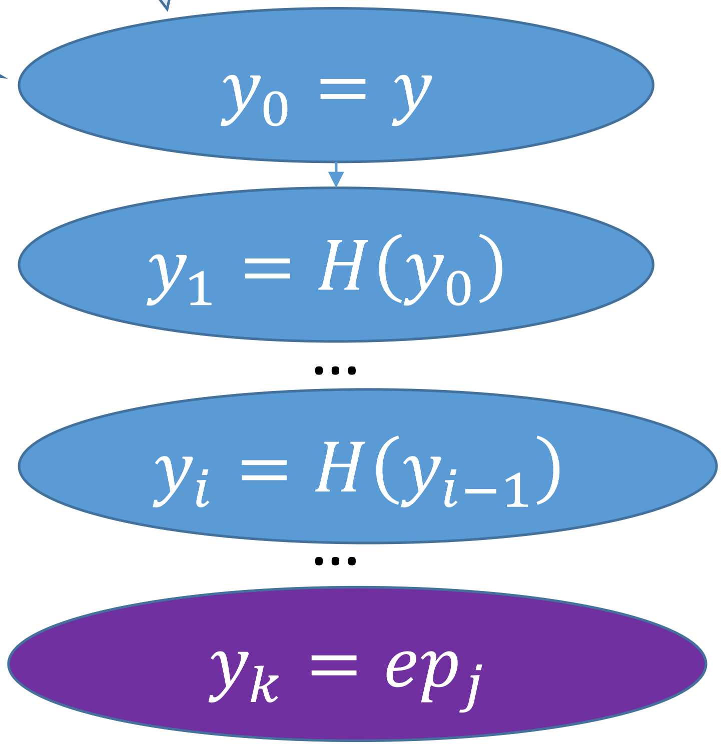


What We Hope is True:

$t \times s > 2^{\ell+2}$ → good chance that

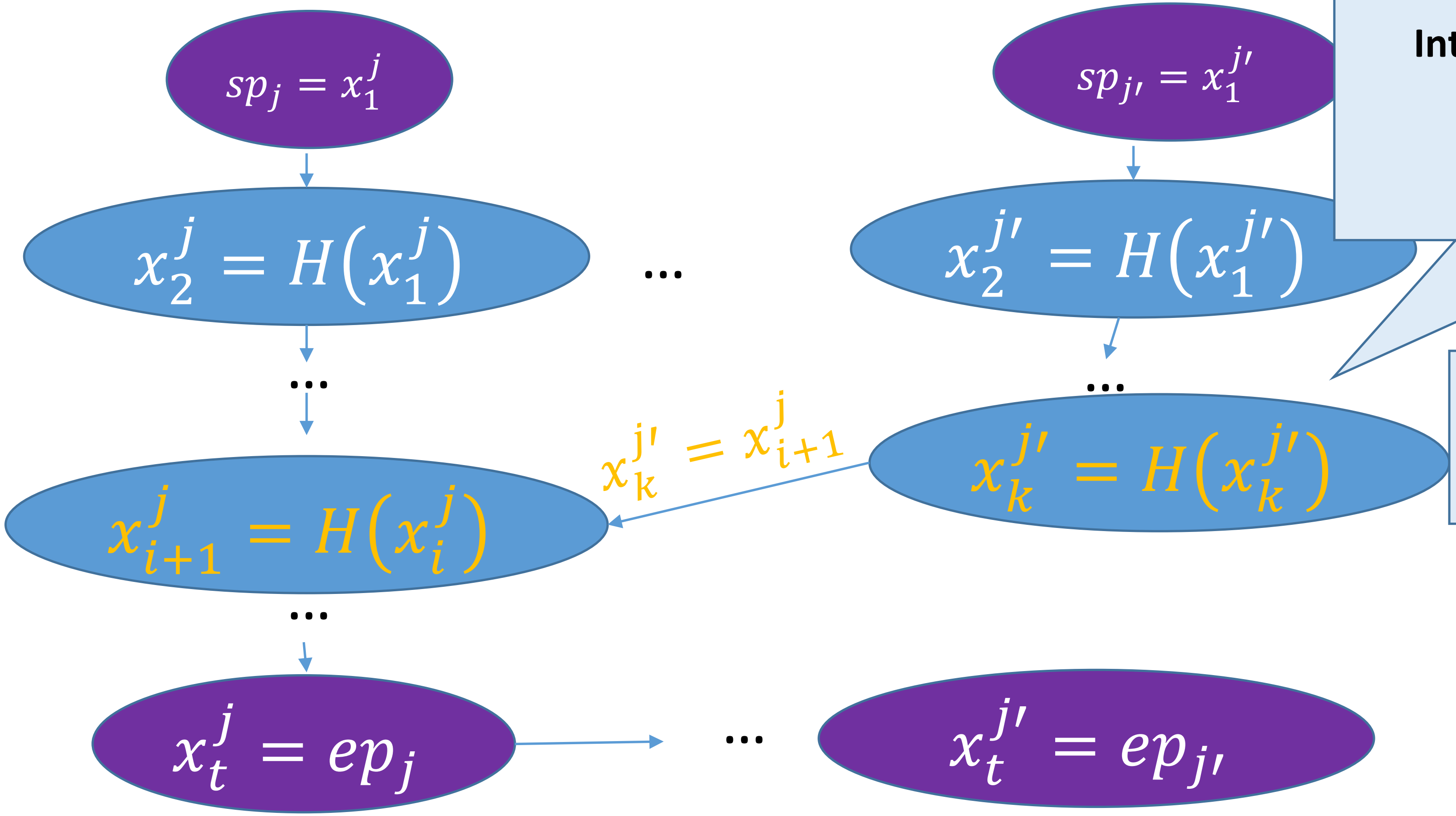
$y = x_i^j$ for some $i \le t, j \le s$

… Not quite true…chains can intersect and may not represent $t \times s$ *distinct* points



Intersecting Chains

- Precomputation ($t \times s$ steps, $2s$ memory)



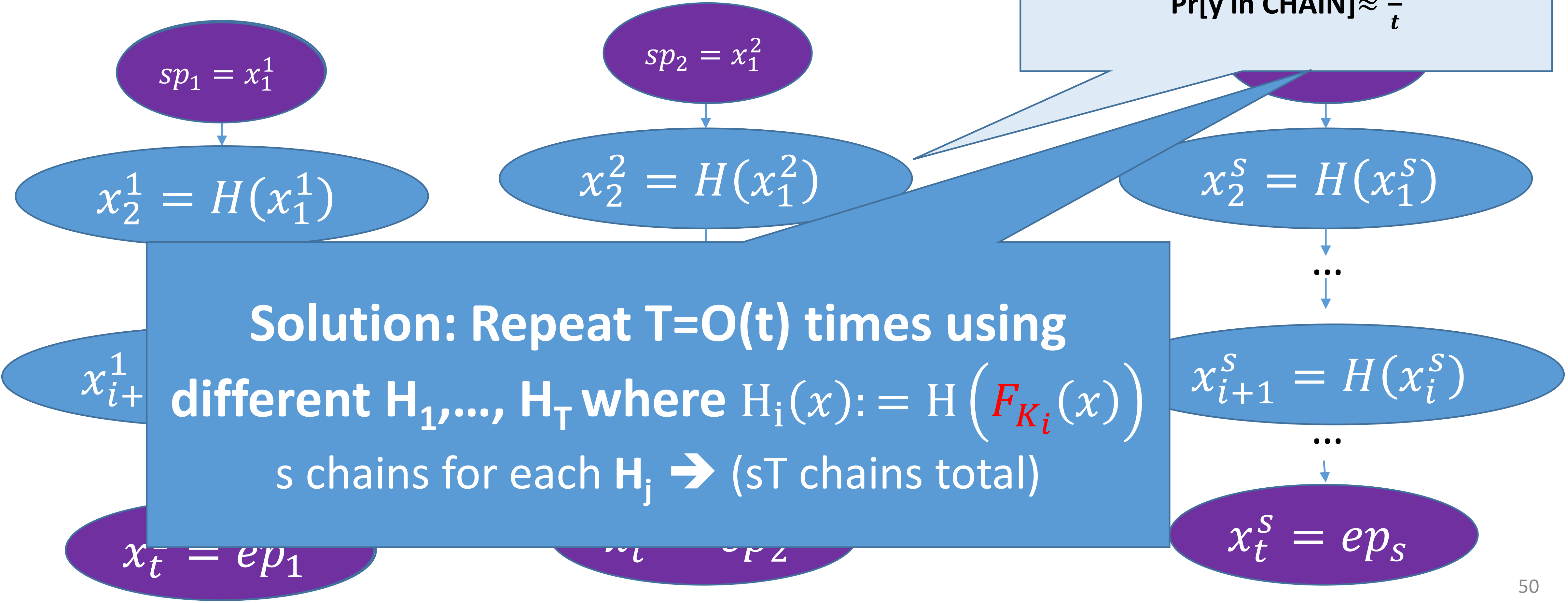
Intersecting chains contain $\ll st$ distinct points.

After initial intersection the chains merge together ☹

Targeted Collision Attacks

- Precomputation ($t \times s$ steps, $2s$ memory)

Fact: If $t^2 \times s < 2^\ell$ then chains contain $\Omega(ts)$ *distinct* points, but then Pr[y in CHAIN]$\approx \frac{1}{t}$

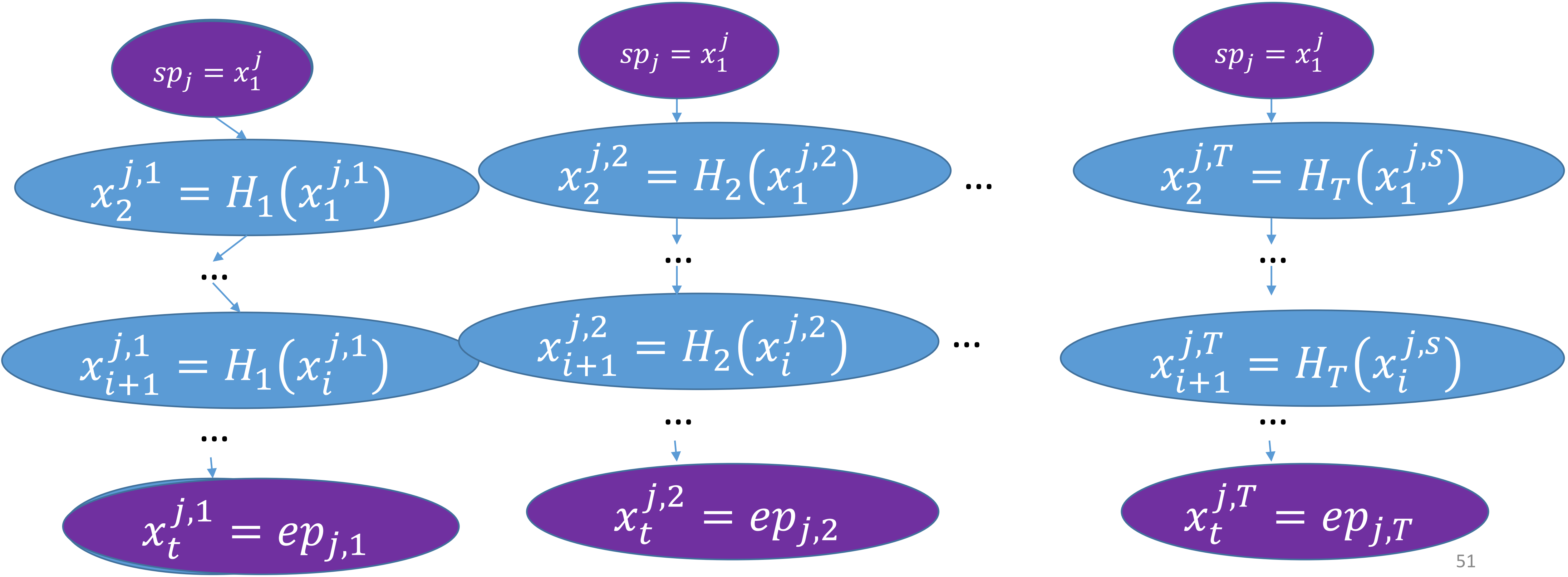


Solution: Repeat T=O(t) times using different H$_1$,…, H$_T$ where $\mathrm{H_i}(x) := \mathrm{H}\left(F_{K_i}(x)\right)$

s chains for each **H$_j$** ➔ (sT chains total)

Targeted Collision Attacks

- Precomputation (stT steps, $2sT$ memory)



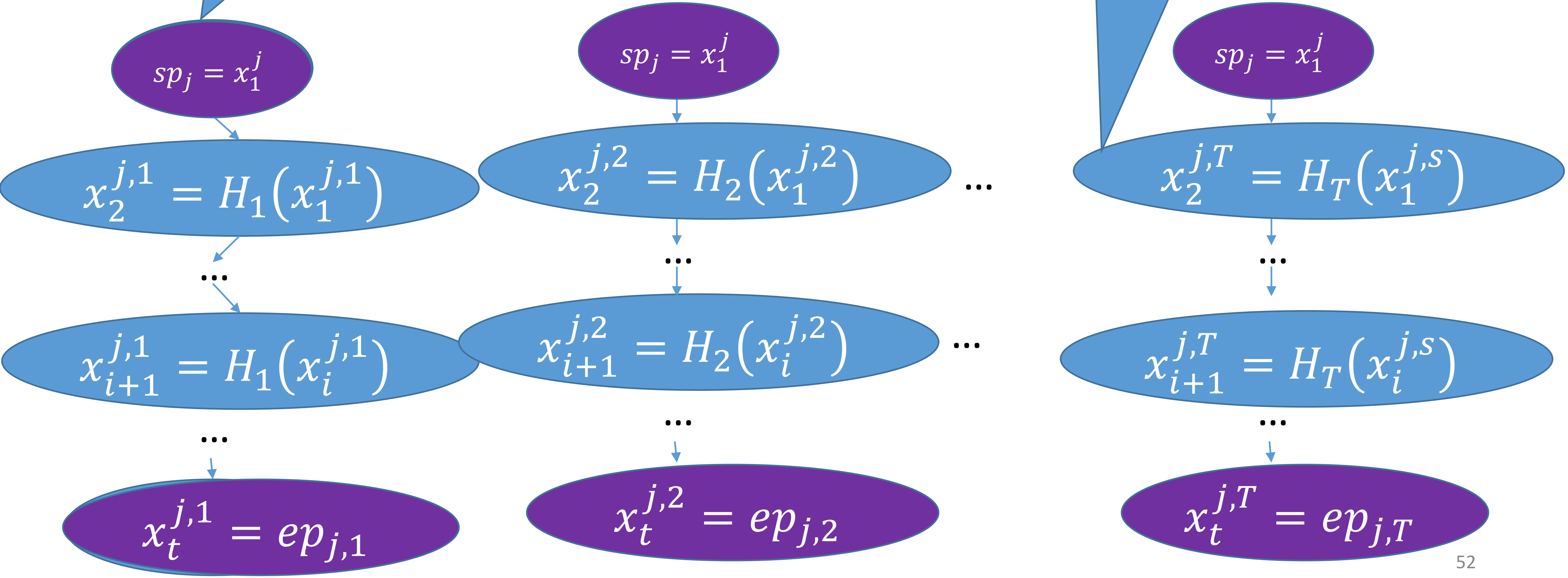
$sp_j = x_1^j$
$x_2^{j,1} = H_1\left(x_1^{j,1}\right)$
…
$x_{i+1}^{j,1} = H_1\left(x_i^{j,1}\right)$
…
$x_t^{j,1} = ep_{j,1}$
$sp_j = x_1^j$
$x_2^{j,2} = H_2\left(x_1^{j,2}\right)$
…
$x_{i+1}^{j,2} = H_2\left(x_i^{j,2}\right)$
…
$x_t^{j,2} = ep_{j,2}$
…
…
$sp_j = x_1^j$
$x_2^{j,T} = H_T\left(x_1^{j,s}\right)$
…
$x_{i+1}^{j,T} = H_T\left(x_i^{j,s}\right)$
…
$x_t^{j,T} = ep_{j,T}$

T Attacks

Repeat for each starting point sp_j with $1 \leq j \leq s$

$H_i(x) = H\left(F_{K_i}(x)\right)$

- Pre omputation (stT steps, $2sT$ memory)



$sp_j = x_1^j$

$x_2^{j,1} = H_1\left(x_1^{j,1}\right)$

…

$x_{i+1}^{j,1} = H_1\left(x_i^{j,1}\right)$

…

$x_t^{j,1} = ep_{j,1}$

$sp_j = x_1^j$

$x_2^{j,2} = H_2\left(x_1^{j,2}\right)$

…

$x_{i+1}^{j,2} = H_2\left(x_i^{j,2}\right)$

…

$x_t^{j,2} = ep_{j,2}$

…

…

$sp_j = x_1^j$

$x_2^{j,T} = H_T\left(x_1^{j,s}\right)$

…

$x_{i+1}^{j,T} = H_T\left(x_i^{j,s}\right)$

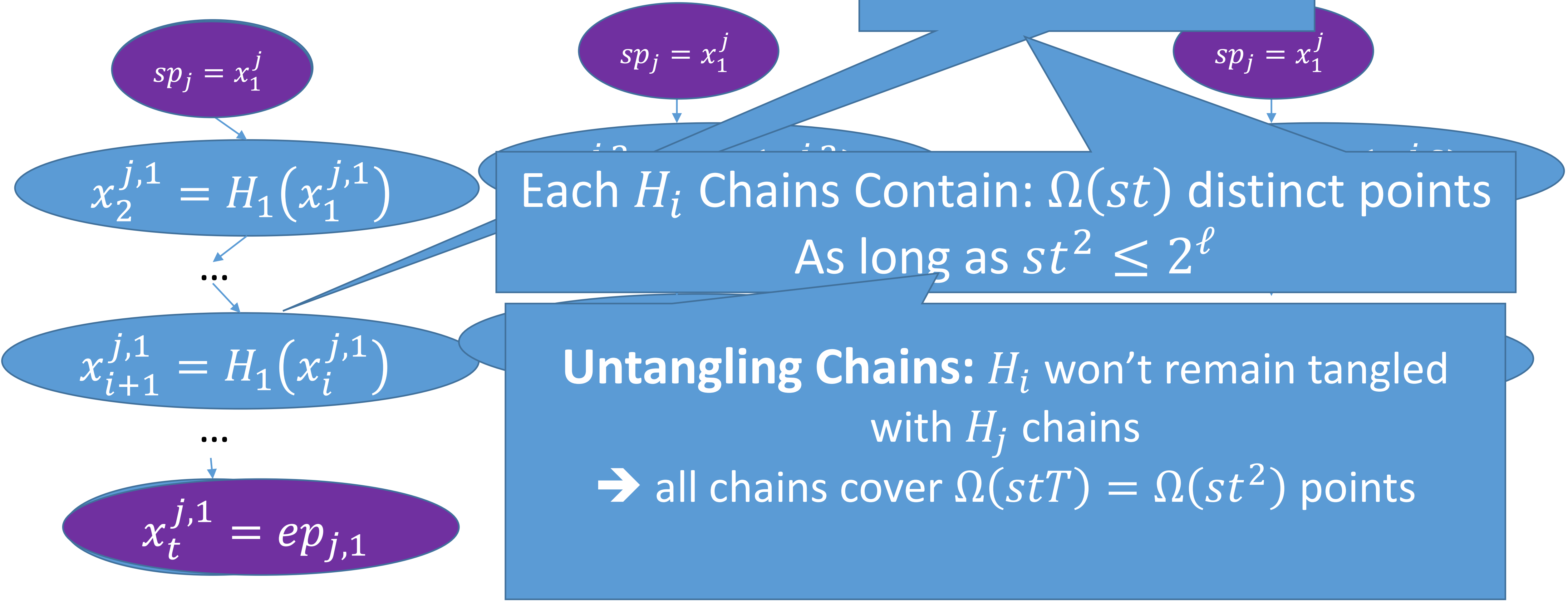
…

$x_t^{j,T} = ep_{j,T}$

Targeted Collision Attacks

- Precomputation (st^2 steps, $2st$ mem

$$H_i(x) = H\left(F_{K_i}(x)\right)$$



$sp_j = x_1^j$
$sp_j = x_1^j$
$sp_j = x_1^j$
$x_2^{j,1} = H_1\left(x_1^{j,1}\right)$
…
$x_{i+1}^{j,1} = H_1\left(x_i^{j,1}\right)$
…
$x_t^{j,1} = ep_{j,1}$

Each H_i Chains Contain: $\Omega(st)$ distinct points

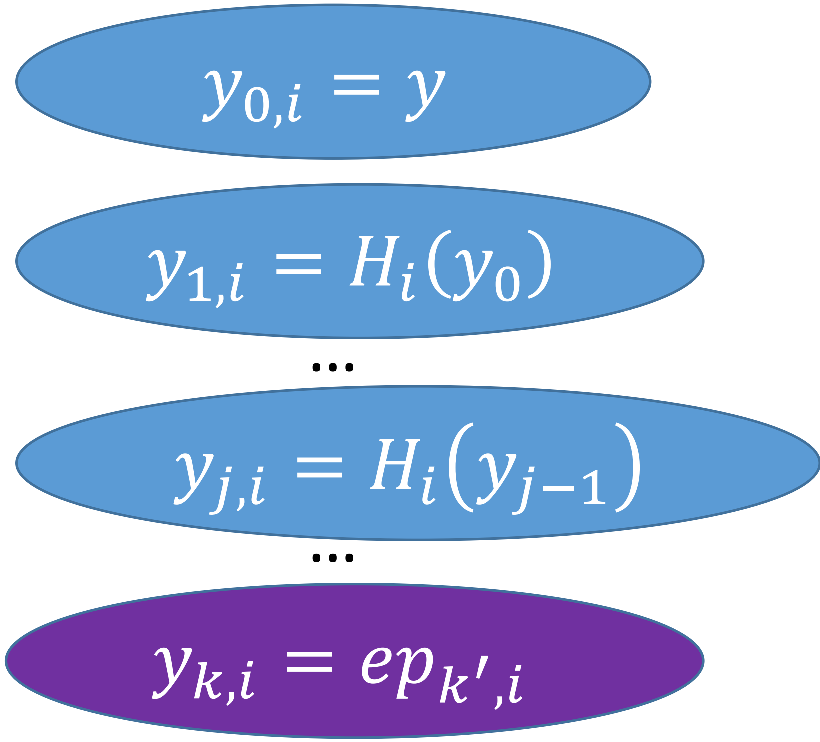
As long as $st^2 \leq 2^\ell$

Untangling Chains: H_i won't remain tangled with H_j chains

➔ all chains cover $\Omega(stT) = \Omega(st^2)$ points

Post-Processing

```
Input: y
For each $i \leq T$  // Compute T chains of length t
    $y' \coloneqq y$         // Start each chain at $y$
    For each $j \leq t$
        $y' := H_i(y')$    // $y' = y_{j,i}$
        For each k’ such that $y' = ep_{k',i}$
            $w' \coloneqq sp_{k'}$   // recompute $H_i$ chain at $sp_{k'}$
            For each $j' \leq t$
                If $y == H_i(w')$ return $F_{K_i}(\boldsymbol{w}')$ else $w' := H_i(w')$
```



$y_{0,i} = y$

$y_{1,i} = H_i(y_0)$

…

$y_{j,i} = H_i(y_{j-1})$

…

$y_{k,i} = ep_{k',i}$

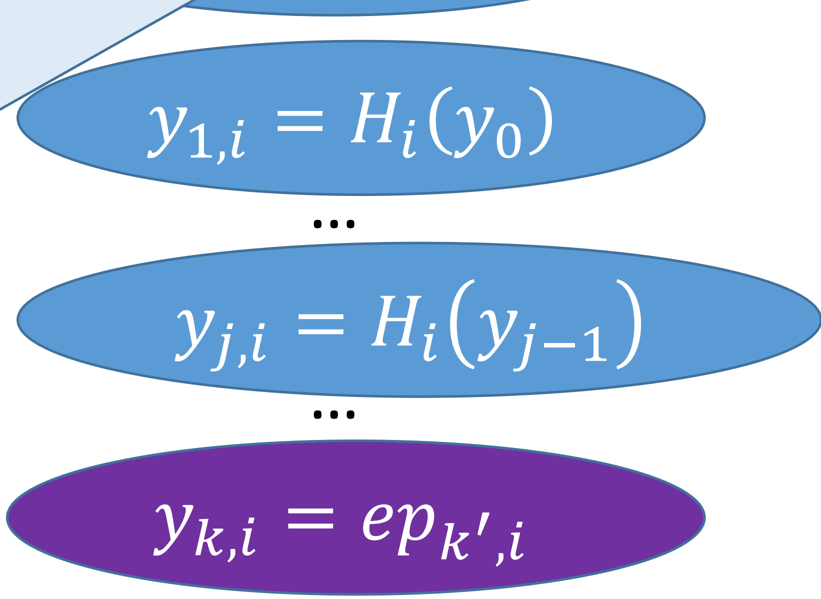
Post-Processing

Observation 1: If y is on any of the chains i.e., $y = x_k^{j,i}$ for some $i \leq T, j \leq t, k \leq s$

→ We will hit the endpoint $y' = ep_{k',i}$

→ We will find a pre-image of y

```
Input: y
For each $i \leq T$   // Compute T chains of length t
    $y' \coloneqq y$        // Start each chain at $y$
    For each $j \leq t$
        $y' := H_i(y')$    // $y' = y_{j,i}$
        For each k' such that $y' = ep_{k',i}$
            $w' \coloneqq sp_{k'}$   // recompute $H_i$ chain at $sp_{k'}$
            For each $j' \leq t$
                If $y == H_i(w')$ return $F_{K_i}(\mathbf{w'})$ else $w' := H_i(w')$
```



$y_{1,i} = H_i(y_0)$

…

$y_{j,i} = H_i(y_{j-1})$

…

$y_{k,i} = ep_{k',i}$

Post-Processing

```
Input: y
For each $i \leq T$   // Compute T chains of length t
    $y' := y$          // Start each chain at $y$
    For each $j \leq t$
        $y' := H_i(y')$     // $y' = y_{j,i}$
        For each k' such that $y' = ep_{k',i}$
            $w' := sp_{k'}$   // recompute $H_i$ chain at $sp_{k'}$
            For each $j' \leq t$
                If $y == H_i(w')$ return $F_{K_i}(w')$ else $w' := H_i(w')$
```

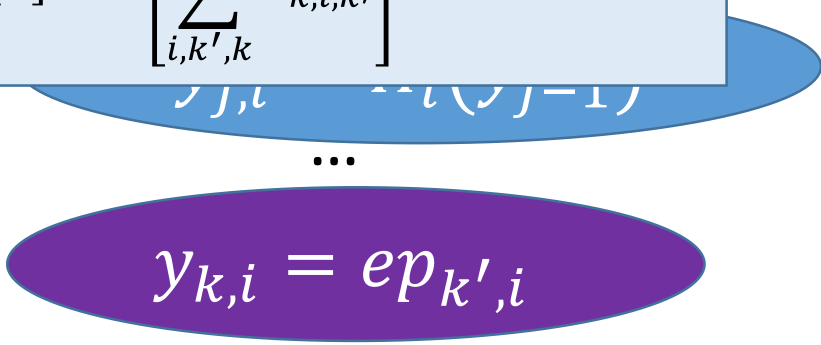
Observation 2: If $y' = ep_{k',i}$ when y is not on the H_i chain starting at $sp_{k'}$ **then we waste t steps checking this chain.**

Let $Z_{k,i,k'}$ be an indicator random variable for the event that $y_{k,i} = ep_{k',i}$ even though y is not on the chain

$$\mathbf{E}[Z_{k,i,k'}] = \mathbf{Pr}[y_{k,i} = ep_{k',i}] \approx 2^{-\ell}$$

Let Z be total number of false positives

$$\mathbf{E}[Z] = \mathbf{E}\left[\sum_{i,k',k} Z_{k,i,k'}\right] \approx stT2^{-\ell}$$


$y_{j,i} = H_i(y_{j-1})$

…

$y_{k,i} = ep_{k',i}$

Post-Processing

```
Input: y
For each $i \leq T$  // Compute T chains of length t
  $y' := y$        // Start each chain at $y$
  For each $j \leq t$
    $y' := H_i(y')$    // $y' = y_{j,i}$
    For each k' such that $y' = ep_{k',i}$
      $w' := sp_{k'}$   // recompute $H_i$ chain at $sp_{k'}$
      For each $j' \leq t$
        If $y == H_i(w')$ return $F_{K_i}(\mathbf{w'})$ else $w' := H_i(w')$
```

Let Z be total number of false positives

$$\mathbf{E}[Z] = \mathbf{E}\left[\sum_{i,k',k} Z_{k,i,k'}\right] \approx stT2^{-\ell}$$

Total Running Time: $O(Tt + Zt)$

If $stT \approx 2^{\ell}$ **and** $T = O(t)$ **then total running time is** $O(t^2)$

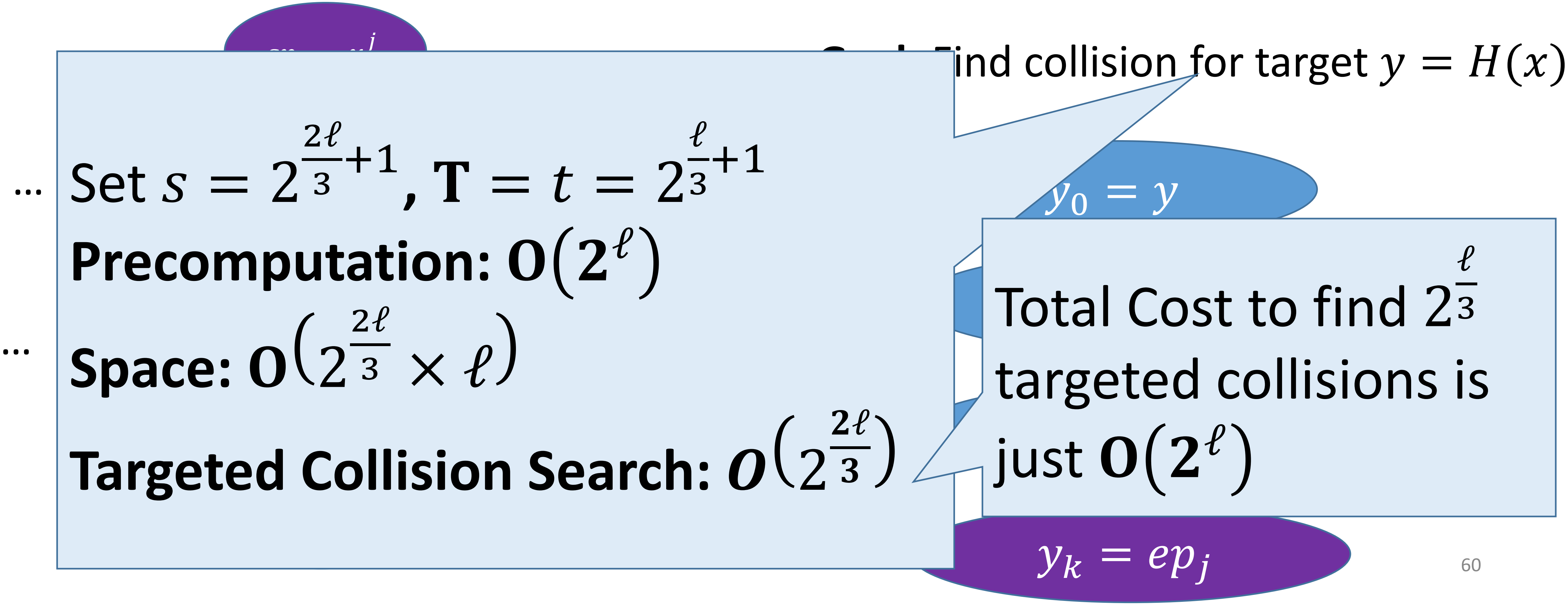
$y_{j,i} = H_i(y_{j-1})$

...

$y_{k,i} = ep_{k',i}$

Targeted Collision Attacks

- Precomputation ($tT \times s$ steps, $2sT \times \ell$ memory)



Set $s = 2^{\frac{2\ell}{3}+1}$, $\mathbf{T} = t = 2^{\frac{\ell}{3}+1}$

Precomputation: $\mathbf{O}\left(\mathbf{2}^{\ell}\right)$

Space: $\mathbf{O}\left(2^{\frac{2\ell}{3}} \times \ell\right)$

Targeted Collision Search: $\boldsymbol{O}\left(2^{\frac{2\ell}{3}}\right)$

Total Cost to find $2^{\frac{\ell}{3}}$ targeted collisions is just $\mathbf{O}\left(\mathbf{2}^{\ell}\right)$

Applications

- Key-Recovery Attacks on Block Cipher $E: \mathcal{K} \times \{0,1\}^n \to \{0,1\}^n$
  - Pre-Computation: $O(|\mathcal{K}|)$
  - **Crack** $2^{\frac{n}{3}}$ **secret keys in total time** $\boldsymbol{O}(|\mathcal{K}|)$ with space $s = \boldsymbol{O}\left(2^{\frac{2n}{3}}\right)$
  - Run prior attack with "hash function" $\mathrm{H}: \{0,1\}^n \to \{0,1\}^n$
    - $\mathrm{H}(K) = E_K(r)$ for some random (fixed) $r \in \{0,1\}^n$
- Password Cracking
  - Attacker is given $H'(x_1)$,…, $H'(x_k)$ for passwords $x_1, \ldots, x_k \in \mathcal{PWDs}$ with $|\mathcal{PWDs}| \ll |\mathcal{K}|$
  - **Goal:** Recover passwords $x_1, \ldots, x_k$
  - Can crack **all** $k = |\mathcal{PWDs}|^{1/3}$ passwords in total time $\boldsymbol{O}(|\mathcal{PWDs}|)$ with space $s = \boldsymbol{O}\left(|\mathcal{PWDs}|^{2/3}\right)$
  - Domain Challenge: $\mathrm{H}': |\mathcal{PWDs}| \to \{0,1\}^n$ with $|\mathcal{PWDs}| \ll 2^n$
    - Define (pseudo)random mapping $\mu: \{0,1\}^n \to \mathcal{PWDs}$
    - Run prior attack with "hash function" $\mathrm{H}: \mathcal{PWDs} \to \mathcal{PWDs}$ as $\mathrm{H}(x) = \mu\left(\mathrm{H}'(x)\right)$

Week 5: Topic 3:
Random Oracle Model +
Hashing Applications

When Collision Resistance Isn’t Enough

- **Example**: Message Commitment
  - Alice sends Bob: $c = H^s(r \parallel m)$ (e.g., predicted winner of NCAA Tournament)
  - Alice can later reveal message (e.g., after the tournament is over)
    - Just send r and m (note: r has fixed length)
    - Why can Alice not change her message?
      - Collision Resistance ➔ Alice can’t find r’ and m’ s.t. $c = H^s(r' \parallel m')$
  - In the meantime Bob shouldn’t learn *anything* about m

- **Problem**: Let (Gen,H’) be collision resistant then so is (Gen,H)

$$H^s(x_1, \dots, x_d) = H'^s(x_1, \dots, x_d) \parallel x_d$$

When Collision Resistance Isn’t Enough

- **Problem**: Let (Gen,H’) be collision resistant then so is (Gen,H)

$$H^s(x_1,\dots,x_d) = H'^s(x_1,\dots,x_d) \parallel x_d$$

Note: An H^s collision trivially yields a H'^s collision

- (Gen,H) definitely does not hide all information about input $(x_1,\dots,x_d)$

- **Conclusion**: Collision resistance is not sufficient for message commitment

The Tension

- **Example**: Message Commitment
  - Alice sends Bob: $\mathrm{H}^s(r \parallel m)$ (e.g., predicted winners of NCAA Final Four)
  - Alice can later reveal message (e.g., after the Final Four is decided)
  - In the meantime Bob shouldn’t learn anything about m

This is still a reasonable approach in practice!

- No attacks when instantiated with any reasonable candidate (e.g., SHA3)
- Cryptographic hash functions seem to provide “something” beyond collision resistance, but how do we model this capability?

Random Oracle Model

- Model hash function H as a truly random function
- Algorithms can only interact with H as an oracle
  - **Query**: x
  - **Response**: H(x)
- If we submit the same query you see the same response
- If x has not been queried, then the value of H(x) is uniform

- **Real World:** H instantiated as cryptographic hash function (e.g., SHA3) of fixed length (no Merkle-Damgård)

Back to Message Commitment

- **Example**: Message Commitment
  - Alice sends Bob: $\mathrm{H}(r \parallel m)$ (e.g., predicted winners of NCAA Final Four)
  - Alice can later reveal message (e.g., after the Final Four is decided)
    - Just send r and m (note: r has fixed length)
    - Why can Alice not change her message?
  - In the meantime Bob shouldn't learn *anything* about m

- **Random Oracle Model**: Above message commitment scheme is secure (Alice cannot change m + Bob learns nothing about m)
- Security Definition + Proof later…

Random Oracle Model: Pros

- It is easier to prove security in Random Oracle Model

- Suppose we are simulating attacker A in a reduction
    - **Extractability**: When A queries H at x we **see this query** and learn x (and can easily find H(x))
    - **Programmability**: We can set the value of H(x) to a value of our choice
        - As long as the value is correctly distribute i.e., close to uniform
- Both **Extractability** and **Programmability** are useful tools for a security reduction!

Random Oracle Model: Pros

- It is easier to prove security in Random Oracle Model

- Provably secure constructions in random oracle model are often much more efficient (compared to provably secure construction is “standard model”

- Sometimes we only know how to design provably secure protocol in random oracle model

Random Oracle Model: Cons

- Lack of formal justification
- Why should security guarantees translate when we instantiate random oracle with a real cryptographic hash function?

- We can construct (contrived) examples of protocols which are
  - Secure in random oracle model…
  - But broken in the real world

Random Oracle Model: Justification

“A proof of security in the random-oracle model is significantly better than no proof at all.”

- **Evidence of sound design** (any weakness involves the hash function used to instantiate the random oracle)
- **Empirical Evidence for Security**

> “there have been no successful real-world attacks on schemes proven secure in the random oracle model”

Hash Function Application: Fingerprinting

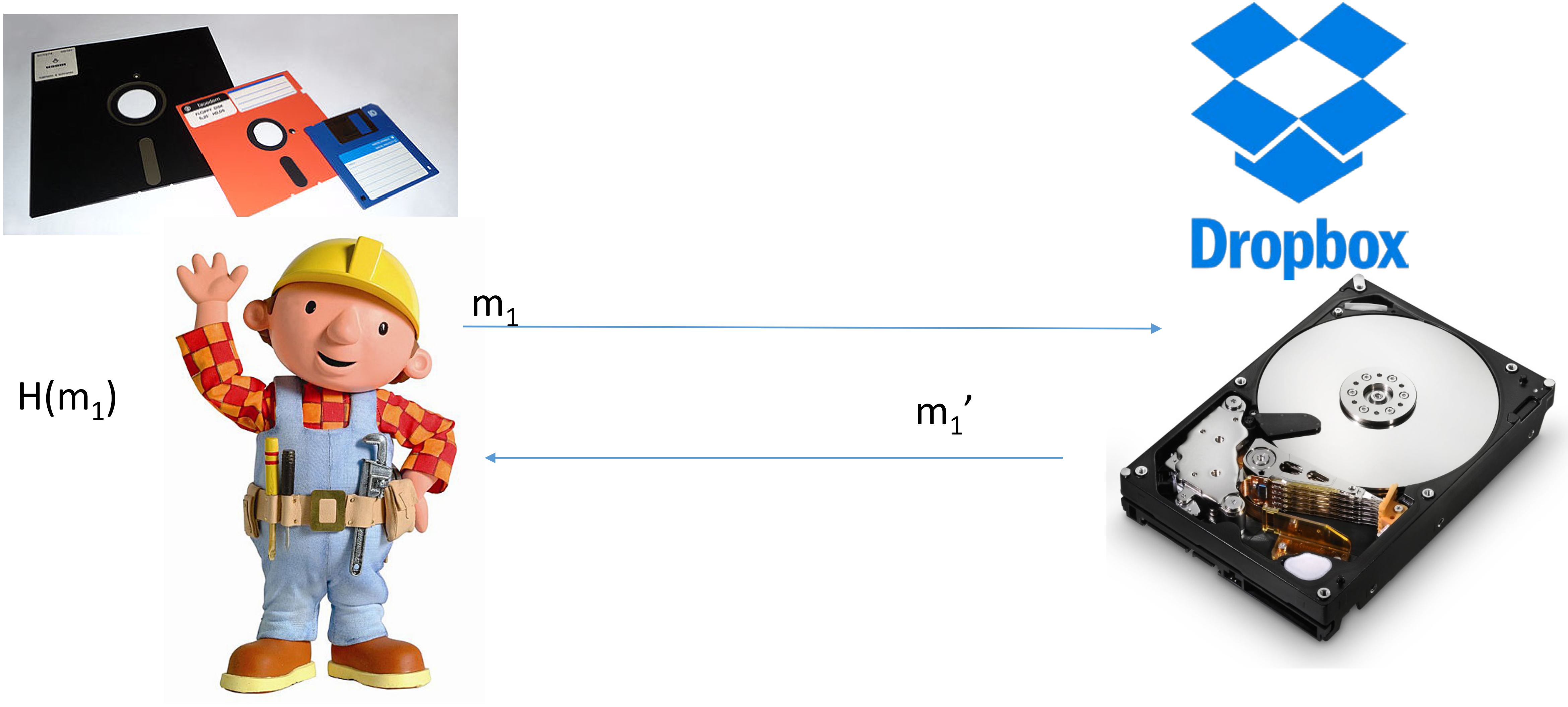
- The hash h(x) of a file x is a unique identifier for the file
  - Collision Resistance → No need to worry about another file y with H(y)=H(y)

- Application 1: Virus Fingerprinting

- Application 2: P2P File Sharing

- Application 3: Data deduplication

Tamper Resistant Storage



m_1

$H(m_1)$

m_1'

Tamper Resistant Storage

File Index	Hash
1	H(m_1)
2	H(m_2)
3	H(m_3)

Disadvantage: Too many hashes to store

Dropbox

m_1, m_2, m_3

Send file 1

m_1'

Tamper Resistant Storage

Disadvantage: Need all files to compute hash m_1,m_2,m_3

$H(m_1,m_2,m_3)$

m_1,m_2,m_3

Send file 1

m_1'

Dropbox

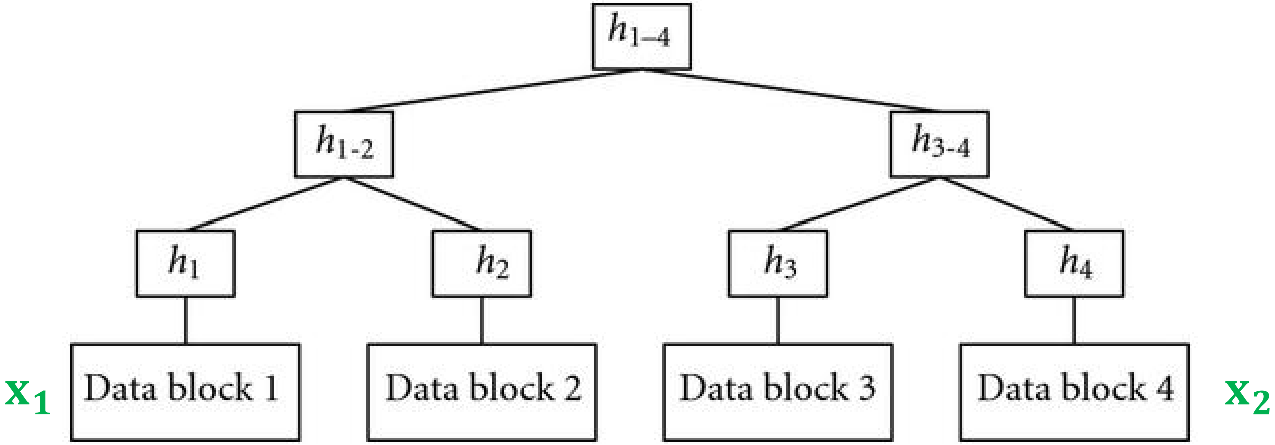
Merkle Trees

$$\mathbf{MT^{s}}(x) := \mathrm{h}^{s}(x)$$

$$\mathbf{MT^{s}}\left(\mathbf{x}_1, \ldots, \mathbf{x}_{2^i}\right) :=$$

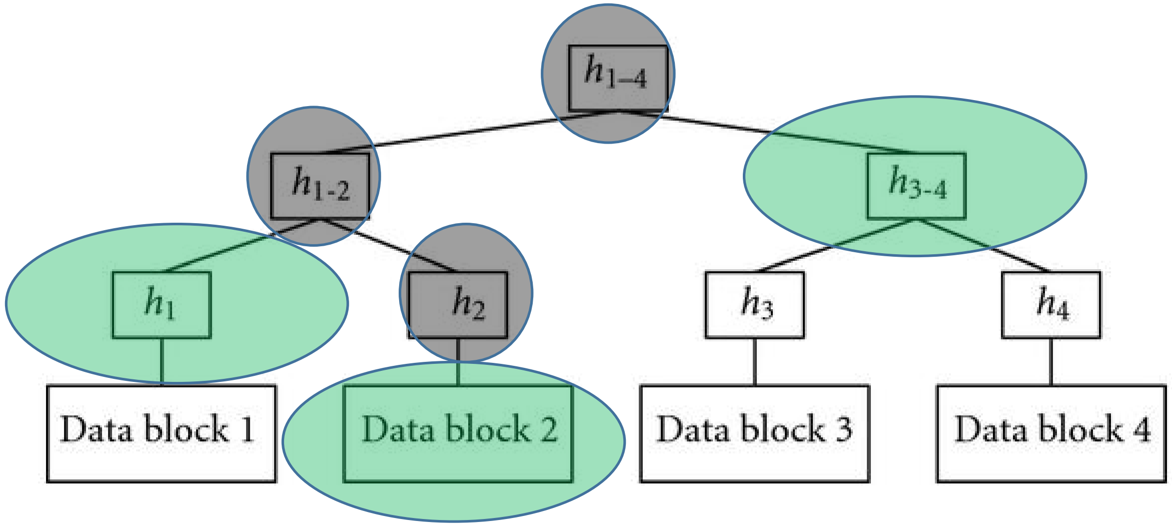
$$\mathrm{h}^{s}\left(\mathbf{MT^{s}}\left(\mathbf{x}_1, \ldots, \mathbf{x}_{2^{i-1}}\right), \mathbf{MT^{s}}\left(\mathbf{x}_{2^{i-1}+1}, \ldots, \mathbf{x}_{2^i}\right)\right)$$

Theorem: Let (Gen, h^s) be a collision resistant hash function then $\mathbf{MT^{s}}$ is collision resistant.



Merkle Trees

- **Proof of Correctness for data block 2**



- **Verify that root matches**
- **Proof consists of just log(n) hashes**
  - **Verifier only needs to permanently store only one hash value**

Tamper Resistant Storage

Root: H_{1-4}

m_1,m_2,m_3,m_4

Send file 2

m_2',h_1,h_{3-4}

Dropbox

Commitment Schemes

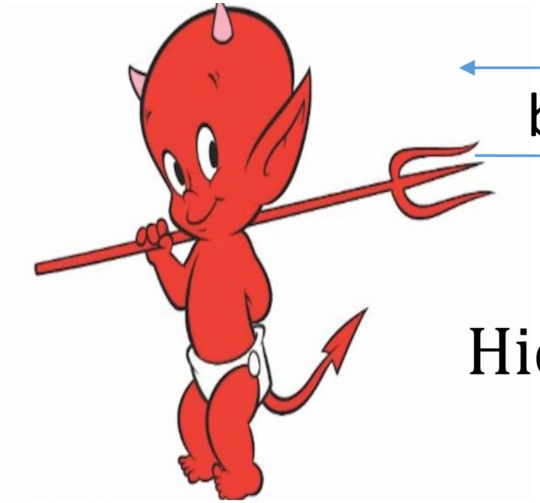
- Alice wants to commit a message m to Bob
  - And possibly reveal it later at a time of her choosing
- Properties
  - Hiding: commitment reveals nothing about m to Bob
  - Binding: it is infeasible for Alice to alter message

Commitment Hiding $\left(\text{Hiding}_{A,Com}(n)\right)$

m$_0$,m$_1$

commit(r,m$_b$)

b’



$$\text{Hiding}_{A,Com}(n) = \begin{cases} 1 & \text{if } b = b' \\ 0 & otherwise \end{cases}$$

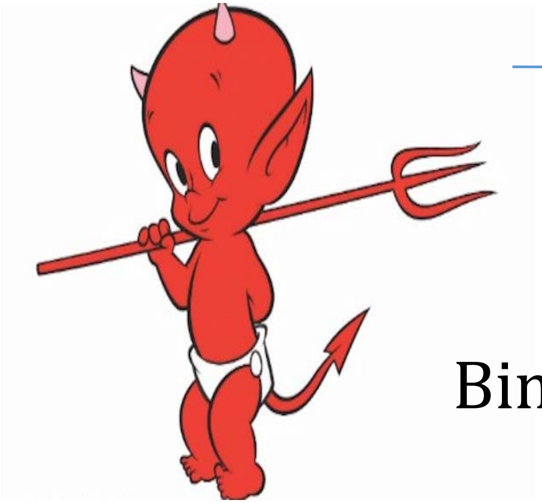
r = Gen(.)

Bit b

$$\forall PPT\ A\ \exists\mu\ (\text{negligible})\ \text{s.t}$$

$$\Pr\left[\text{Hiding}_{A,Com}(n) = 1\right] \le \frac{1}{2} + \mu(n)$$

Commitment Binding ($\text{Binding}_{A,Com}(n)$)



r_0, r_1, m_0, m_1 →

$$\text{Binding}_{A,Com}(n) = \begin{cases} 1 & \text{if } \textbf{commit}(\mathbf{r_0},\mathbf{m_0}) \textbf{= commit}(\mathbf{r_1},\mathbf{m_1}) \\ 0 & otherwise \end{cases}$$

$$\forall PPT\ A\ \exists \mu \text{ (negligible) s.t}$$
$$\Pr\left[\text{Binding}_{A,Com}(n) = 1\right] \le \mu(n)$$

Secure Commitment Scheme

- **Definition:** A secure commitment scheme is **hiding** and **binding**
- **Hiding**

$$\forall PPT\ A\ \exists\mu\ (\text{negligible})\ \text{s.t}$$
$$\Pr\left[\text{Hiding}_{A,Com}(n) = 1\right] \leq \frac{1}{2} + \mu(n)$$

- **Binding**

$$\forall PPT\ A\ \exists\mu\ (\text{negligible})\ \text{s.t}$$
$$\Pr\left[\text{Binding}_{A,Com}(n) = 1\right] \leq \mu(n)$$

Commitment Scheme in Random Oracle Model

- **Commit**(r,m):=H(m|r)
- **Reveal**(c):= (m,r)

Theorem: In the random oracle model this is a secure commitment scheme.

Proof Intuition: Let BAD event that attacker queries $\mathrm{H}(r \parallel m')$ for any message m’ on any of q queries

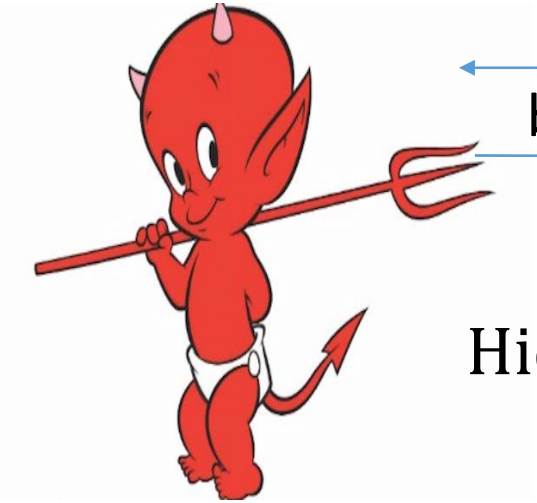
- As long as the event BAD never occurs Bob learns nothing about m (in an information theoretic sense)
- If r is a random n-bit string then $\Pr[\mathrm{BAD}] \leq \frac{\mathrm{q}}{2^{\mathrm{n}}}$

Commitment Hiding $\left(\mathrm{Hiding}_{A,Com}(n)\right)$

m$_0$,m$_1$ →

← **H(r,m$_b$)**

b' →



$$\mathrm{Hiding}_{A,Com}(n) = \begin{cases} 1 & if b = b' \\ 0 & otherwise \end{cases}$$

r = Gen(1^n)

Bit b

$$\forall \cancel{PPT}\ A\ making\ at\ most\ q(n)\ queries$$

$$\Pr\left[\mathrm{Hiding}_{A,Com}(n) = 1\right] \le \frac{1}{2} + \frac{q(n)}{2^n}$$

Other Applications

- Password Hashing

- Key Derivation

Next Week

- Stream Ciphers
- Block Ciphers
- Feistel Networks
- DES, 3DES
- Read Katz and Lindell 6.1-6.2