Cryptography
CS 555

Topic 25: Discrete LOG, DDH + Attacks on Plain RSA

Recap

- Plain RSA
- Public Key (pk): N = pq, e such that $\mathrm{GCD}\big(\mathrm{e}, \phi(N)\big) = 1$
  - $\phi(N) = (p-1)(q-1)$ for distinct primes p and q
- Secret Key (sk): N, d such that ed=1 mod $\phi(N)$
- **Encrypt(pk=(N,e),m)** $= m^{\boldsymbol{e}}\ mod\ N$
- **Decrypt(sk=(N,d),c)** $= c^{\boldsymbol{d}}\ mod\ N$

- Decryption Works because
  $[c^d \text{mod N}] = [m^{ed} \text{mod N}] = \left[m^{[ed\ mod\ \boldsymbol{\phi}(\boldsymbol{N})]} \text{mod N}\right] = [m \text{ mod N}]$

RSA-Assumption

RSA-Experiment: RSA-INV$_{A,n}$

1. **Run KeyGeneration**(1^n) **to obtain (N,e,d)**
2. **Pick uniform** $\mathrm{y} \in \mathbb{Z}_{\mathrm{N}}^{*}$
3. Attacker A is given N, e, y and outputs $\mathrm{x} \in \mathbb{Z}_{\mathrm{N}}^{*}$
4. Attacker wins (RSA-INV$_{A,n}$=1) if $x^e = \mathrm{y} \bmod \mathrm{N}$

$$\forall PPT\ A\ \exists \mu\ (\text{negligible})\ \text{s.t}\ \Pr[\text{RSA–INVA}_{,n} = 1] \leq \mu(n)$$

(Plain) RSA Discussion

- We have not introduced security models like CPA-Security or CCA-security for Public Key Cryptosystems

- However, notice that (Plain) RSA Encryption is stateless and deterministic.

→Plain RSA is not secure against chosen-plaintext attacks

- Plain RSA is also highly vulnerable to chosen-ciphertext attacks
  - Attacker intercepts ciphertext c of secret message m
  - Attacker generates ciphertext c’ for secret message 2m
  - Attacker asks for decryption of c’ to obtain 2m
  - Divide by 2 to recover original message m

(Plain) RSA Discussion

- However, notice that (Plain) RSA Encryption is stateless and deterministic.

→Plain RSA is not secure against chosen-plaintext attacks

- In a public key setting the attacker does have access to an encryption oracle

- Encrypted messages with low entropy are vulnerable to a brute-force attack.
  - If $m < B$ then attacker can recover m after at most B queries to encryption oracle (using public key)

More Weaknesses: Plain RSA with small e

- (Small Messages) If $m^e < N$ then we can decrypt $c = m^e \bmod N$ directly e.g., $m=c^{(1/e)}$

- (Partially Known Messages) If an attacker knows first 1-(1/e) bits of secret message $m = m_1 \| ??$ then he can recover m given
$$\mathbf{Encrypt}(pk, m) = m^e \text{mod N}$$

Theorem[Coppersmith]: If p(x) is a polynomial of degree e then in polynomial time (in log(N), e) we can find all m such that p(m) = 0 mod N and $|m| < N^{(1/e)}$

More Weaknesses: Plain RSA with small e

Theorem[Coppersmith]: If p(x) is a polynomial of degree e then in polynomial time (in log(N), e) we can find all m such that p(m) = 0 mod N and |m|<$N^{(1/e)}$

Example: e = 3, $m = m_1 \| m_2$ and attacker knows $m_1(2k\ bits)$ and $\boldsymbol{c} = (m_1 \| m_2)^e \mathrm{mod}\ \mathrm{N}$, but not $m_2(k\ bits)$

$$p(x) = \left(2^k m_1 + x\right)^3 - c$$

Polynomial has a small root mod N at x= m_2 and coppersmith's method will find it!

Recovering Encrypted Message faster than Brute-Force

Claim: Let m < 2^n be a secret message. For some constant $\alpha = \frac{1}{2} + \varepsilon$. We can recover m in in time $T = 2^{\alpha n}$ with high probability.

For r=1,…,T

  let $\mathrm{x_r} = [cr^{-e} mod\ N]$, where $r^{-e} = (r^{-1})^e mod\ N$

Sort $\mathbf{L} = \{(\boldsymbol{r}, \boldsymbol{xr})\}_{\boldsymbol{r=1}}^{\boldsymbol{T}}$ **(by the x_r values)**

For s=1,…,T

  if $[s^e mod\ N] = xr$ for some r **then**

    return $[rs\ mod\ N]$

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if $[s^e mod\ N] = xr$ for some r **then**

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Analysis: $[rs\ mod\ N] = [r(x_r)^d\ mod\ N]$

$$= [r(cr^{-e})^d\ mod\ N] = [rr^{-ed}(c)^d\ mod\ N]$$

$$= [rr^{-1}m\ mod\ N] = \mathrm{m}$$

Recovering Encrypted Message faster than Brute-Force

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Sort $\mathbf{L} = \{(\boldsymbol{r}, \boldsymbol{xr})\}_{\boldsymbol{r=1}}^{\boldsymbol{T}}$ **(by the $\mathbf{x_r}$ values)**

For s=1,…,T

  if $[s^e mod\ N] = xr$ for some r **then**

    return $[rs\ mod\ N]$

Fact: some constant $\alpha = \frac{1}{2} + \varepsilon$ setting $T = 2^{\alpha n}$ with high probability we will find a pair **s** and $\mathbf{x_r}$ with $[s^e mod\ N] = xr$.

Recovering Encrypted Message faster than Brute-Force

Claim: Let m < 2^n be a secret message. For some constant $\alpha = \frac{1}{2} + \varepsilon$. We can recover m in in time $T = 2^{\alpha n}$ with high probability.

Roughly $\sqrt{B}$ steps to find a secret message m < B

(Recap) Finite Groups

Definition: A (finite) group is a (finite) set $\mathbb{G}$ with a binary operation $\circ$ (over G) for which we have

- (**Closure**:) For all $\mathrm{g}, \mathrm{h} \in \mathbb{G}$ we have $\mathrm{g} \circ \mathrm{h} \in \mathbb{G}$
- **(Identity**:) There is an element $\mathrm{e} \in \mathbb{G}$ such that for all $\mathrm{g} \in \mathbb{G}$ we have
$$\mathrm{g} \circ \mathrm{e} = \mathrm{g} = \mathrm{e} \circ \mathrm{g}$$
- **(Inverses**:) For each element $\mathrm{g} \in \mathbb{G}$ we can find $\mathrm{h} \in \mathbb{G}$ such that $\mathrm{g} \circ \mathrm{h} = \mathrm{e}$. We say that h is the inverse of g.
- (**Associativity:**) For all $\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3 \in \mathbb{G}$ we have
$$(\mathrm{g}_1 \circ \mathrm{g}_2) \circ \mathrm{g}_3 = \mathrm{g}_1 \circ (\mathrm{g}_2 \circ \mathrm{g}_3)$$

We say that the group is **abelian** if

- (**Commutativity:**) For all $\mathrm{g}, \mathrm{h} \in \mathbb{G}$ we have $\mathrm{g} \circ \mathrm{h} = \mathrm{h} \circ \mathrm{g}$

Finite Abelian Groups (Examples)

- **Example 1:** $\mathbb{Z}_N$ when $\circ$ denotes addition modulo N
- Identity: 0, since 0 $\circ$ x =[0+x mod N] = [x mod N].
- Inverse of x? Set x^{-1}=N-x so that [x^{-1}+x mod N] = [N-x+x mod N] = 0.

- **Example 2:** $\mathbb{Z}_N^*$ when $\circ$ denotes multiplication modulo N
- Identity: 1, since 1$\circ$ x =[1(x) mod N] = [x mod N].
- Inverse of x? Run extended GCD to obtain integers a and b such that

$$ax + bN = \gcd(x, N) = 1$$

Observe that: x^{-1} = a. Why?

Cyclic Group

- Let $\mathbb{G}$ be a group with order $\mathrm{m} = |\mathbb{G}|$ with a binary operation $\circ$ (over G) and let g $\in \mathbb{G}$ be given consider the set
$$\langle g \rangle = \{g^0, g^1, g^2, \ldots\}$$

Fact: $\langle g \rangle$ defines a subgroup of $\mathbb{G}$.

- Identity: g^0
- Closure: $g^i \circ g^j = g^{i+j} \in \langle g \rangle$
- g is called a "generator" of the subgroup.

Fact: Let $\mathrm{r} = |\langle g \rangle|$ then $g^i = g^j$ if and only if $i = j \mod r$. Also m is divisible by r.

Finite Abelian Groups (Examples)

Fact: Let p be a prime then $\mathbb{Z}_{p-1}^{*}$ is a cyclic group of order p-1.

- **Note**: A generator g of this group must have gcd(g,p-1)=1

Example (non-generator): p=7, g=2

$<2>=\{1,2,4\}$

Example (generator): p=7, g=5

$<2>=\{1,5,4,6,2,3\}$

Discrete Log Experiment $\mathrm{DLog}_{A,G}(n)$

1. Run G(1n) to obtain a cyclic group $\mathbb{G}$ of order q (with $\|q\| = n$) and a generator g such that $< \mathrm{g} >= \mathbb{G}$.
2. Select $\mathrm{h} \in \mathbb{G}$ uniformly at random.
3. Attacker A is given $\mathbb{G}$, q, g, h and outputs integer x.
4. Attacker wins ($\mathrm{DLog}_{A,G}(n)=1$) if and only if $g^x=h$.

We say that the discrete log problem is hard relative to generator G if

$$\forall PPT\ A\ \exists\mu\ (\mathrm{negligible})\ \mathrm{s.t}\ \Pr[\mathrm{DLog}_{A,n} = 1] \le \mu(n)$$

Diffie-Hellman Problems

Computational Diffie-Hellman Problem (CDH)

- Attacker is given $\mathrm{h}_1 = g^{x_1} \in \mathbb{G}$ and $\mathrm{h}_2 = g^{x_2} \in \mathbb{G}$.
- Attackers goal is to find $g^{x_1 x_2} = (\mathrm{h}_1)^{x_2} = (\mathrm{h}_2)^{x_1}$
- **CDH Assumption**: For all PPT A there is a negligible function negl upper bounding the probability that A succeeds

Decisional Diffie-Hellman Problem (DDH)

- Let $\mathrm{z}_0 = g^{x_1 x_2}$ and let $\mathrm{z}_1 = g^{r}$, where x_1,x_2 and r are random
- Attacker is given g^{x_1}, g^{x_2} and z_b (for a random bit b)
- Attackers goal is to guess b
- **DDH Assumption**: For all PPT A there is a negligible function negl such that A succeeds with probability at most ½ + negl(n).

Secure key-agreement with DDH

1. Alice publishes g^{x_A} and Bob publishes g^{x_B}
2. Alice and Bob can both compute $K_{A,B} = g^{x_B\, x_A}$ but to Eve this key is indistinguishable from a random group element (by DDH)

Remark: Protocol is vulnerable to Man-In-The-Middle Attacks if Bob cannot validate g^{x_A}.

Can we find a cyclic group where DDH holds?

- **Example 1:** $\mathbb{Z}_p^*$ where p is a random n-bit prime.
  - CDH is believed to be hard
  - DDH is *not* hard (Exercise 13.15)

- **Theorem:** Let p=rq+1 be a random n-bit prime where q is a large λ-bit prime then the set of r-th residues modulo p is a cyclic subgroup of order q. Then $\mathbb{G} = \{[hr \; mod \; p] \middle| h \in \mathbb{Z}_p^*\}$ is a cyclic subgroup of $\mathbb{Z}_p^*$ of order q.
  - **Remark 1:** DDH is believed to hold for such a group
  - **Remark 2:** It is easy to generate uniform elements
  - **Remark 3:** Any element (besides 1) is a generator of $\mathbb{G}$

Can we find a cyclic group where DDH holds?

- **Theorem:** Let p=rq+1 be a random n-bit prime where q is a large λ-bit prime then the set of rth residues modulo p is a cyclic subgroup of order q. Then $\mathbb{G} = \{[hr\ mod\ p] \mid h \in \mathbb{Z}_p^*\}$ is a cyclic subgroup of $\mathbb{Z}_p^*$ of order q.
  - **Closure**: $h^r g^r = (hg)^r$
  - **Inverse** of h^r is $(h^{-1})^r \in \mathbb{G}$
  - **Size** $(h^r)^x = h^{[rx\ mod\ rq]} = (h^r)^x = h^{r[x\ mod\ q]} = (h^r)^{[x\ mod\ q]} mod\ p$

Remark: Two known attacks (Section 9.2).

- First runs in time $O(\sqrt{q}) = O(2^{\lambda/2})$
- Second runs in time $2^{O(\sqrt[3]{n}(\log n)^{2/3})}$

Can we find a cyclic group where DDH holds?

Remark: Two known attacks (Section 9.2).
- First runs in time $O(\sqrt{q}) = O(2^{\lambda/2})$
- Second runs in time $2^{O\left(\sqrt[3]{n}(\log n)^{2/3}\right)}$, where n is bit length of p

Goal: Set λ and n to balance attacks

$$\lambda = O\left(\sqrt[3]{n}(\log n)^{2/3}\right)$$

How to sample p=rq+1?
- First sample a random λ-bit prime q and
- Repeatedly check if rq+1 is prime for a random n- λ bit value r

Can we find a cyclic group where DDH holds?

Elliptic Curves Example: Let p be a prime (p > 3) and let A, B be constants. Consider the equation

$$y^2 = x^3 + Ax + B \bmod p$$

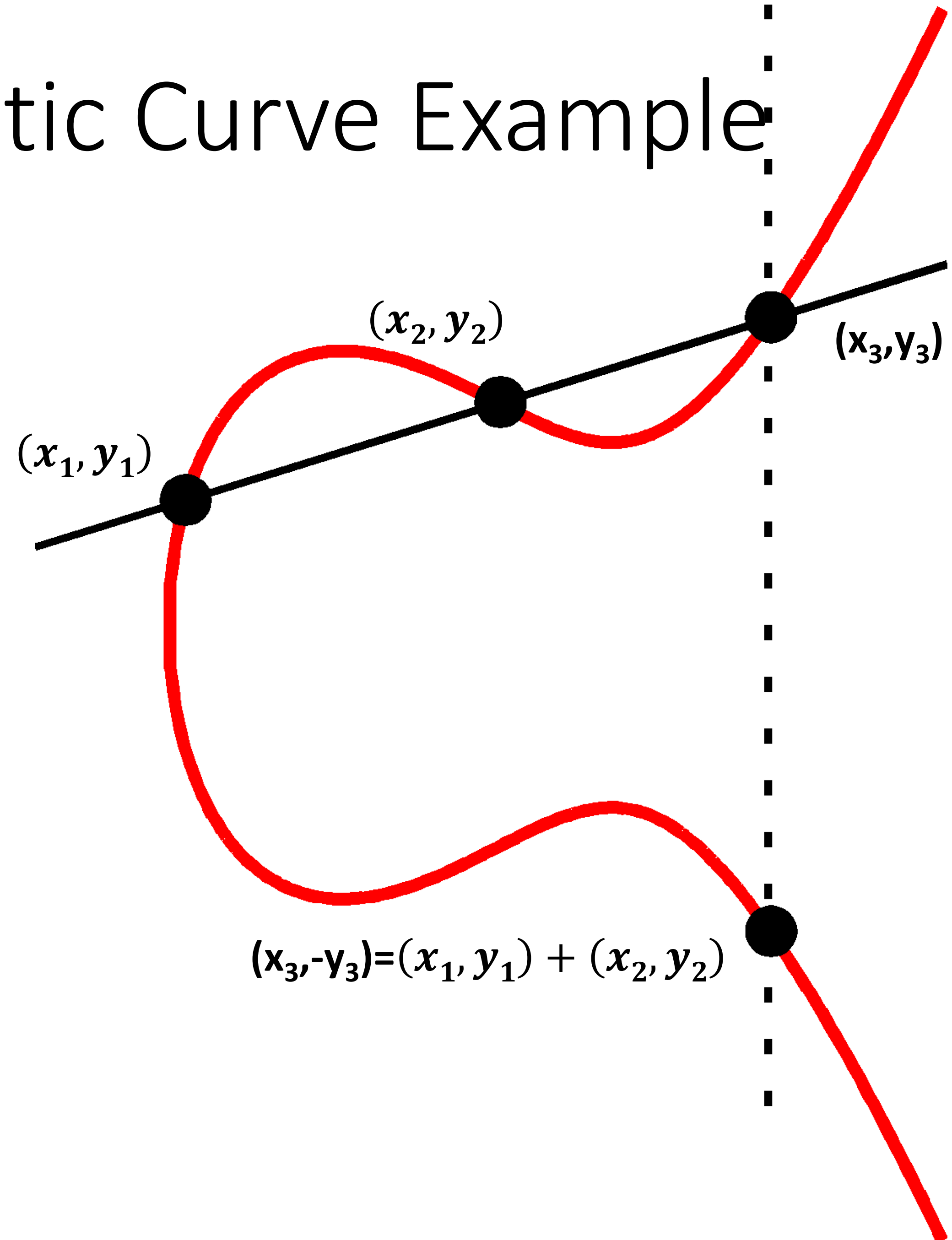
And let

$$E\left(\mathbb{Z}_p\right) = \left\{(x, y) \in \mathbb{Z}_p^2 \middle| y^2 = x^3 + Ax + B \bmod p \right\} \cup \{\mathcal{O}\}$$

Note: $\mathcal{O}$ is defined to be an additive identity $(x, y) + \mathcal{O} = (x, y)$

What is $(x_1, y_1) + (x_2, y_2)$?

Elliptic Curve Example



Formally, let

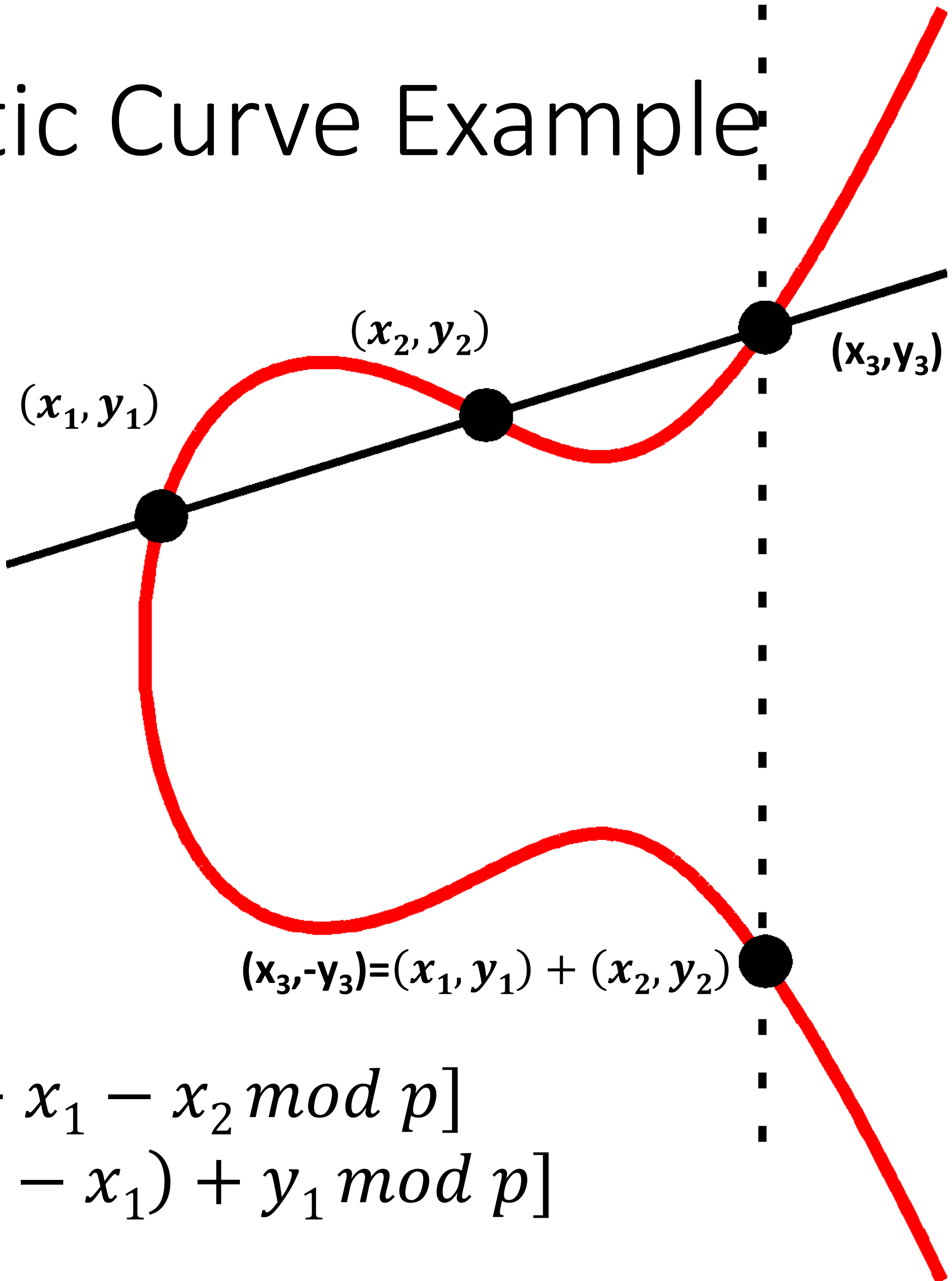
$$m = \left[\frac{y_1 - y_2}{x_1 - x_2} \; mod\, p\right]$$

be the slope.

Then the line passing through $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$ has the equation

$$y = m(x - x_1) + y_1 \; mod\, P$$

Elliptic Curve Example



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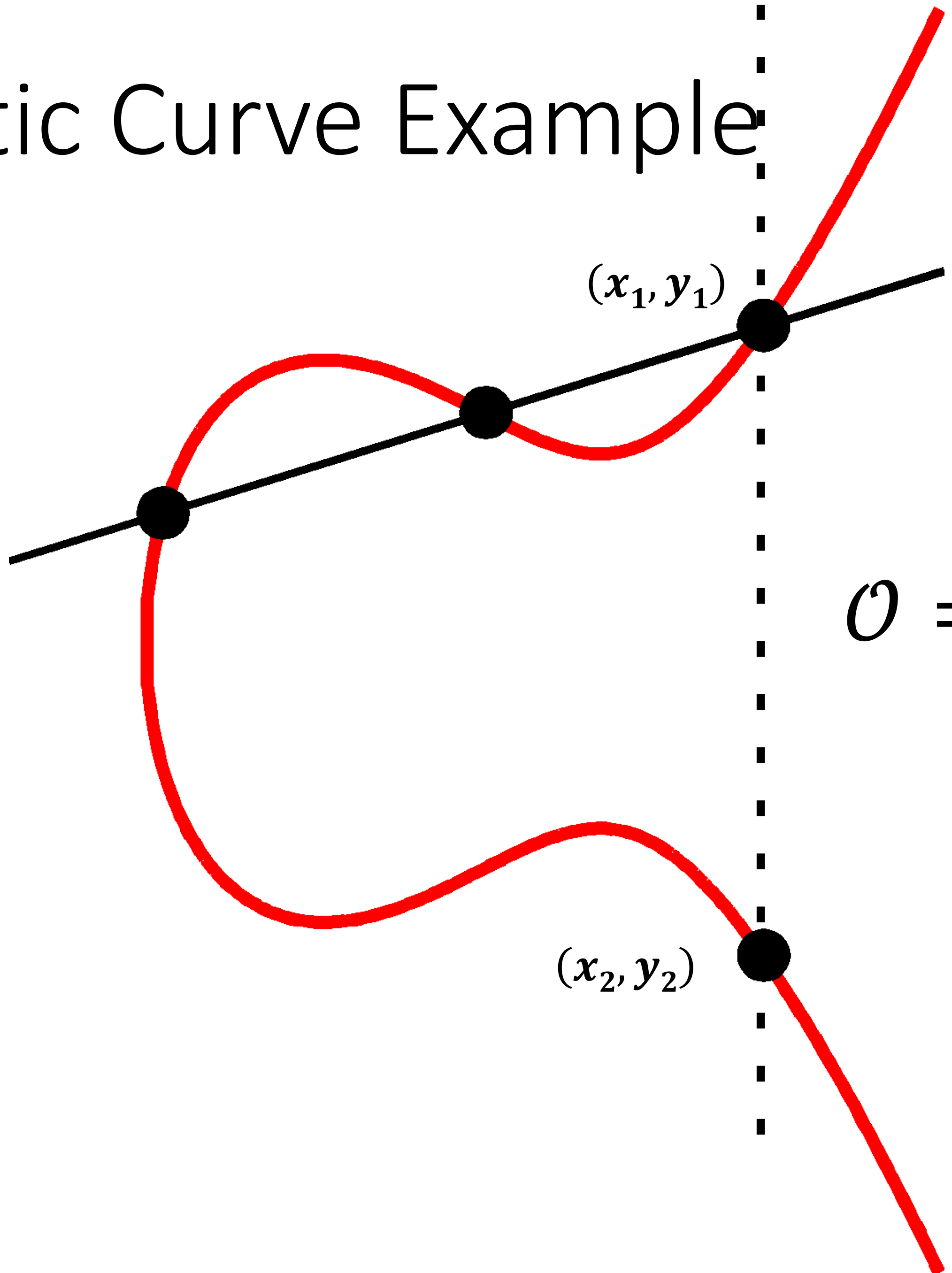
$$y = m(x - x_1) + y_1 \; mod \; P$$

$$(m(x - x_1) + y_1)^2$$
$$= x^3 + Ax + B \; mod \; p$$

$$x_3 = [m^2 - x_1 - x_2 \, mod \; p]$$
$$y_3 = [m(x_3 - x_1) + y_1 \, mod \; p]$$

Elliptic Curve Example



$$\mathcal{O} = (x_1, y_1) + (x_2, y_2)$$

Can we find a cyclic group where DDH holds?

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Fact: $E\left(\mathbb{Z}_p\right)$ defines an abelian group

- For *appropriate curves* the DDH assumption is believed to hold
- If you make up your own curve there is a good chance it is broken…
- NIST has a list of recommendations

Next Week: Spring Break!

- Next class on Monday, March 20th.

- Read Katz and Lindell 8.4

- DDH Applications