Midterm Exam

- **Date:** Tuesday, October 16th
- **Time:** 3PM-4:15PM (in class)
- **Location:** Lawson B134 (right here)
- Closed Book/No Calculator

Note: Our TA (Duc Le) will proctor the midterm

Content: Includes today's lecture (chapters 1-7)

Preparation:

- You may prepare one 3x5 inch index card (double sides).
- Take the practice final
- Review homework solutions, book, lecture notes etc.

Final Exam (Tentative)

- **Date:** Tuesday, December 11th (Subject to Change*)
- **Time:** 8AM (Subject to Change*)
- **Location:** LWSN B151 (Subject to Change*)

* Purdue will not reimburse you for flight re-booking fees

Cryptography
CS 555

Week 8:

- One-Way Functions (Part 2)

Recap

Corollary: If one-way functions exist then PRGs, PRFs and strong PRPs all exist.

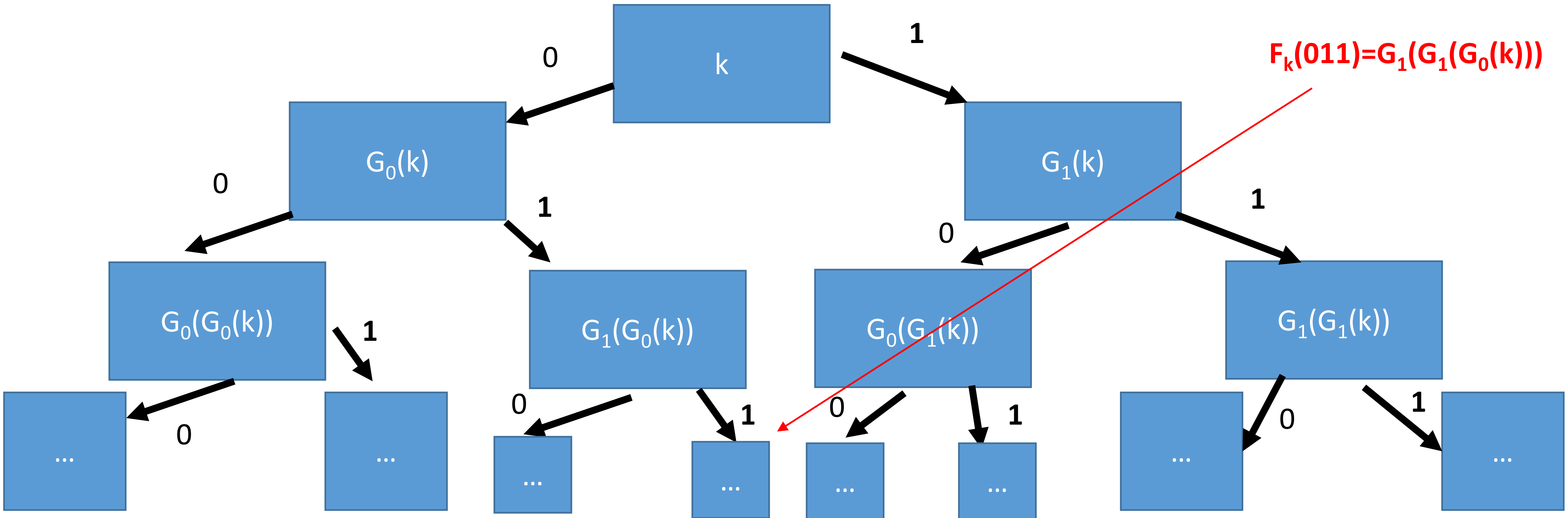
Corollary: If one-way functions exist then there exist CCA-secure encryption schemes and secure MACs.

We saw how to build PRGs from One-Way-Permutations…

PRFs from PRGs

$$G(x) := \overbrace{G_0(x)}^{\text{n-bits}} \,||\, \overbrace{G_1(x)}^{\text{n-bits}}$$

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = 2n$. Then there is a secure PRF.



PRFs from PRGs

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = 2n$. Then there is a secure PRF.

Proof:

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\mathbf{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \mathbf{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right| < \boldsymbol{negl(n)}$$

PRFs from PRGs

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|Pr\left[A\left(r_1 \parallel \cdots \parallel r_{t(n)}\right)\right] - Pr\left[A\left(G(s_1) \parallel \cdots \parallel G\left(s_{t(n)}\right)\right)\right]\right| < negl(n)$$

Proof by Triangle Inequality: Fix j

$$Adv_j$$

$$= \left|Pr\left[A\left(r_1 \parallel \cdots \parallel r_{j+1} \parallel G\left(s_{j+2}\right) \ldots \parallel G\left(s_{t(n)}\right)\right)\right]\right.$$

PRFs from PRGs

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right| < \boldsymbol{negl(n)}$$

Proof

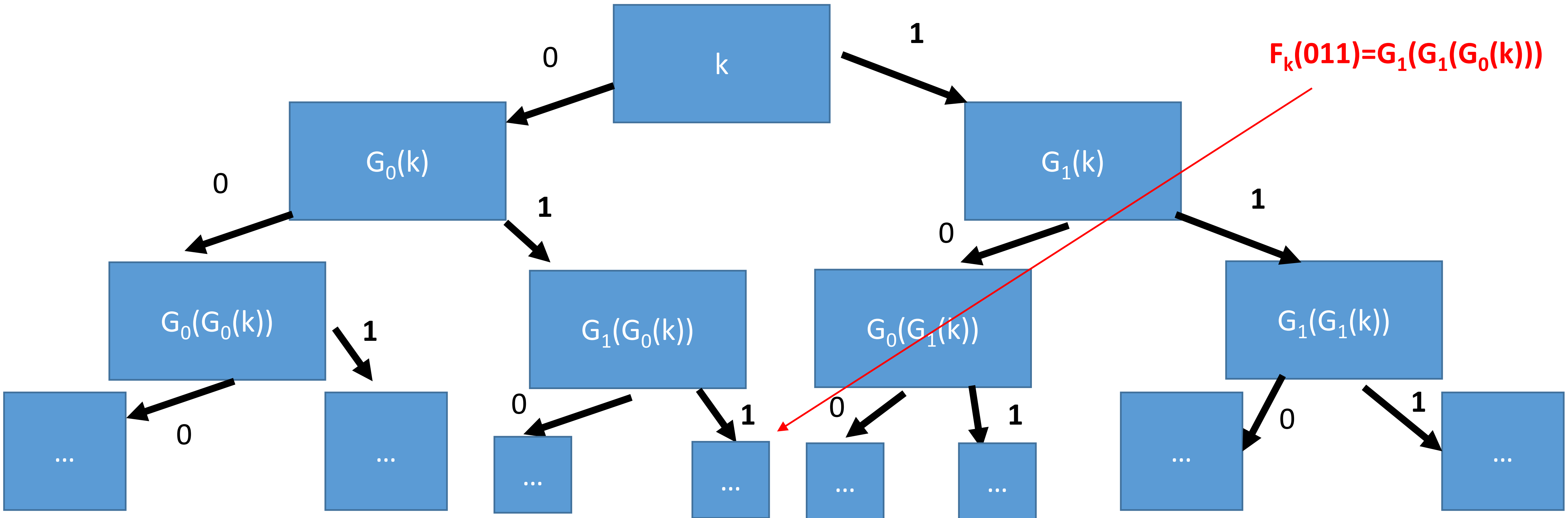
$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s_1}) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right|$$

$$\leq \sum_{\boldsymbol{j<t(n)}} \boldsymbol{Adv_j}$$

$$\leq \boldsymbol{t(n)} \times \boldsymbol{negl(n)} = \boldsymbol{negl(n)}$$

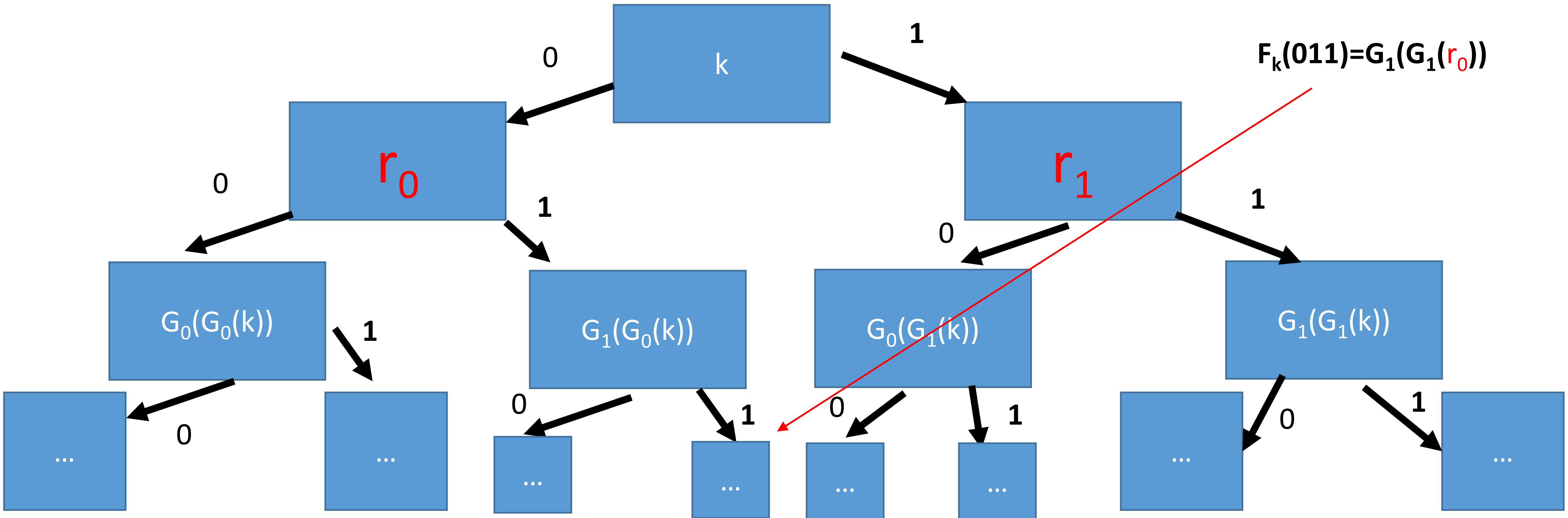
Hybrid H_0 (Real Construction)

$$\mathbf{G(x) := \overbrace{G_0(x)}^{\text{n-bits}} \,||\, \overbrace{G_1(x)}^{\text{n-bits}}}$$

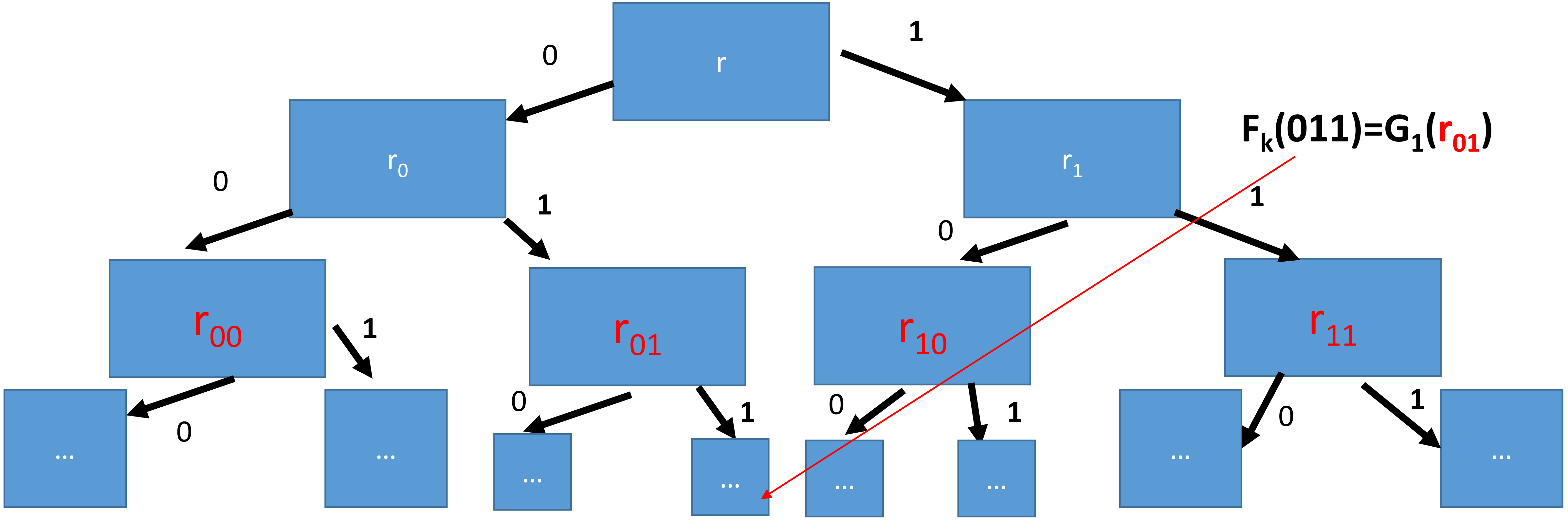


Hybrid H_1 (Real Construction)

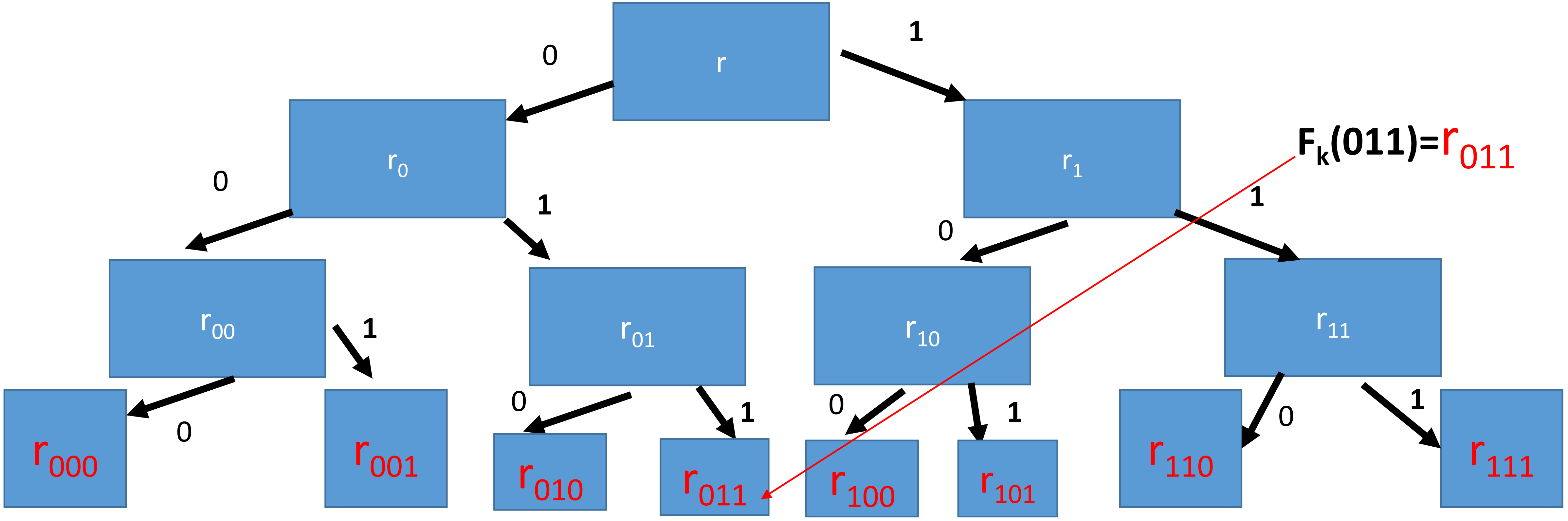
$$G(x) := \underbrace{G_0(x)}_{\text{n-bits}} \,||\, \underbrace{G_1(x)}_{\text{n-bits}}$$



Hybrid H_2



Hybrid H_n (truly random function!)



Hybrid H_1 vs H_2

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r}_1 \parallel \cdots \parallel \boldsymbol{r}_{t(n)}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}(\boldsymbol{s}_1) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s}_{t(n)}\right)\right)\right]\right| < \boldsymbol{negl}(\boldsymbol{n})$$

Claim 2: *Attacker who makes t(n) oracle queries to our function cannot distinguish H_i from H_{i+1} (except with negligible probability).*

Proof: Indistinguishability follows by Claim 1

Let $x_1,\ldots x_t$ denote the t queries. Let $y_1,\ldots,y_t$ denote first i bits of each query.

(H_{i+1} vs H_i: replaced $\boldsymbol{G}\left(\boldsymbol{r}_{y_i}\right)$ with $\boldsymbol{r}_{y_i \parallel 0} \parallel \boldsymbol{r}_{y_i \parallel 1}$)

Triangle Inequality

Claim 1: For any t(n) and any PPT attacker A we have

$$\left|\boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{r_1} \parallel \cdots \parallel \boldsymbol{r_{t(n)}}\right)\right] - \boldsymbol{Pr}\left[\boldsymbol{A}\left(\boldsymbol{G}\left(\boldsymbol{s_1}\right) \parallel \cdots \parallel \boldsymbol{G}\left(\boldsymbol{s_{t(n)}}\right)\right)\right]\right| < \boldsymbol{negl(n)}$$

Claim 2: ***Attacker who makes t(n) queries to F_k (or f) cannot distinguish H_2 from the real game (except with negligible probability).***

→ Triangle Inequality: Attacker cannot distinguish F_k (H_0) from f (H_n).

From OWFs (Recap)

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = n + 1$. Then for any polynomial p(.) there is a PRG with expansion factor p(n).

Theorem: Suppose that there is a PRG G with expansion factor $\ell(n) = 2n$. Then there is a secure PRF.

Theorem: Suppose that there is a secure PRF then there is a strong pseudorandom permutation.

From OWFs (Recap)

Corollary: If one-way functions exist then PRGs, PRFs and strong PRPs all exist.

Corollary: If one-way functions exist then there exist CCA-secure encryption schemes and secure MACs.

Are OWFs Necessary for Private Key Crypto

- Previous results show that OWFs are <u>sufficient</u>.

- Can we build Private Key Crypto from weaker assumptions?

- **Short Answer:** No, OWFs are also <u>*necessary*</u> for most private-key crypto primitives

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Question: why can we assume that we have an PRG with expansion 2n?

Answer: Last class we showed that a PRG with expansion factor $\ell(n) = n + 1$. Implies the existence of a PRG with expansion p(n) for any polynomial.

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Claim: G is also a OWF!

(Easy to Compute?) ✓

(Hard to Invert?)

Intuition: If we can invert G(x) then we can distinguish G(x) from a random string.

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Claim 1: Any PPT A, given G(s), cannot find s except with negligible probability.

Reduction: Assume (for contradiction) that A can invert G(s) with non-negligible probability p(n).

Distinguisher D(y): Simulate A(y)

Output 1 if and only if A(y) outputs x s.t. G(x)=y.

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Claim 1: Any PPT A, given G(s), cannot find s except with negligible probability.

Intuition for Reduction: If we can find x s.t. G(x)=y then y is not random.

Fact: Select a random 2n bit string y. Then (whp) there does not exist x such that G(x)=y.

Why not?

PRGs → OWFs

Proposition 7.28: If PRGs exist then so do OWFs.

Proof: Let G be a secure PRG with expansion factor $\ell(n) = 2n$.

Claim 1: Any PPT A, given G(s), cannot find s except with negligible probability.

Intuition: If we can invert G(x) then we can distinguish G(x) from a random string.

Fact: Select a random 2n bit string y. Then (whp) there does not exist x such that G(x)=y.

- Why not? Simple counting argument, 2^{2n} possible y's and 2^n x's.
- Probability there exists such an x is at most 2^{-n} (for a random y)

What other assumptions imply OWFs?

- PRGs → OWFs
- (Easy Extension) PRFs → PRGs → OWFs

- Does secure crypto scheme imply OWFs?
  - CCA-secure? (Strongest)
  - CPA-Secure? (Weaker)
  - EAV-secure? (Weakest)
    - As long as the plaintext is longer than the secret key
  - Perfect Secrecy? X (Guarantee is information theoretic)

EAV-Secure Crypto → OWFs

Proposition 7.29: If there exists a EAV-secure private-key encryption scheme that encrypts messages twice as long as its key, then a one-way function exists.

Recap: EAV-secure.

- Attacker picks two plaintexts m_0,m_1 and is given $c=Enc_K(m_b)$ for random bit b.
- Attacker attempts to guess b.
- No ability to request additional encryptions (chosen-plaintext attacks)
- In fact, no ability to observe any additional encryptions

EAV-Secure Crypto → OWFs

Proposition 7.29: If there exists a EAV-secure private-key encryption scheme that encrypts messages twice as long as its key, then a one-way function exists.

Reduction: $\boldsymbol{f(m,k,r) = Enc_k(m;r) \| m}$.

Input: 4n bits

(For simplicity assume that $\mathbf{Enc}_k$ accepts n bits of randomness)

Claim: f is a OWF

EAV-Secure Crypto → OWFs

Proposition 7.29: If there exists a EAV-secure private-key encryption scheme that encrypts messages twice as long as its key, then a one-way function exists.

Reduction: $\boldsymbol{f(m,k,r) = Enc_k(m;r) \| m}$.

Claim: f is a OWF

Reduction: If attacker A can invert f, then attacker A' can break EAV-security as follows. Given c=$Enc_k(m_b;r)$ run A(c$\|m_0$). If A outputs (m',k',r') such that $\mathrm{f(m',k',r') = c} \| m_0$ then output 0; otherwise 1;

MACs→ OWFs

In particular, given a MAC that satisfies MAC security (Definition 4.2) against an attacker who sees an arbitrary (polynomial) number of message/tag pairs.

Conclusions: OWFs are necessary and sufficient for all (non-trivial) private key cryptography.

→OWFs are a minimal assumption for private-key crypto.

Public Key Crypto/Hashing?

- OWFs are known to be necessary
- Not known (or believed) to be sufficient.

Computational Indistinguishability

- Consider two distributions X_ℓ and Y_ℓ (e.g., over strings of length ℓ).
- Let D be a distinguisher that attempts to guess whether a string s came from distribution X_ℓ or Y_ℓ.

The advantage of a distinguisher D is

$$Adv_{D,\ell} = \left| Pr_{s \leftarrow X_\ell}[D(s) = 1] - Pr_{s \leftarrow Y_\ell}[D(s) = 1] \right|$$

Definition: We say that an ensemble of distributions $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are computationally indistinguishable if for all PPT distinguishers D, there is a negligible function negl(n), such that we have

$$Adv_{D,n} \leq negl(n)$$

Computational Indistinguishability

The advantage of a distinguisher D is

$$Adv_{D,\ell} = \left| Pr_{s \leftarrow \mathrm{X}_\ell}[D(s) = 1] - Pr_{s \leftarrow \mathrm{Y}_\ell}[D(s) = 1] \right|$$

- Looks similar to definition of PRGs
  - X_n is distribution $G(U_n)$ and
  - Y_n is uniform distribution $U_{\ell(n)}$ over strings of length $\ell(n)$.

Computational Indistinguishability

Definition: We say that an ensemble of distributions $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are <u>computationally indistinguishable</u> if for all PPT distinguishers D, there is a negligible function negl(n), such that we have

$$Adv_{D,n} \leq negl(n)$$

Theorem 7.32: Let t(n) be a polynomial and let $P_n = X_n^{t(n)}$ and $Q_n = Y_n^{t(n)}$ then the ensembles $\{P_n\}_{n\in\mathbb{N}}$ and $\{Q_n\}_{n\in\mathbb{N}}$ are <u>computationally indistinguishable</u>

Computational Indistinguishability

Definition: We say that an ensemble of distributions $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are <u>computationally indistinguishable</u> if for all PPT distinguishers D, there is a negligible function negl(n), such that we have

$$Adv_{D,n} \leq negl(n)$$

Fact: Let $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be <u>computationally indistinguishable</u> and let $\{Z_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ be <u>computationally indistinguishable</u>

Then

$\{X_n\}_{n\in\mathbb{N}}$ and $\{Z_n\}_{n\in\mathbb{N}}$ are <u>computationally indistinguishable</u>

Practice Problems

- Suppose that f is a OWF. Build another OWF f' s.t. f' is not collision resistant.
- Suppose that $h^s(.)$ is collision resistant hash function mapping 2n-bit strings to n-bit strings. Show that $f(s,x)= (s,h^s(x))$ is a one-way function.
- Suppose that $h^s(.)$ is collision resistant hash function mapping 2n-bit strings to n-bit strings. Show that $f(s,x)= h^s(x)$ is not necessarily a OWF.
- $\boldsymbol{f(m,k,r) = Enc_k(m;r)||m}$ is a OWF. What about $\boldsymbol{f(m,k,r) = Enc_k(m;r)}$? Is it necessarily One-Way?