# Course Business

- **Homework 3 Due Now**
- **Homework 4 Released**
- Professor Blocki is travelling, but will be back next week

# Cryptography
# CS 555

**Week 11:**

- Discrete Log/DDH
- Applications of DDH
- Factoring Algorithms, Discrete Log Attacks + NIST Recommendations for Concrete Security Parameters

**Readings:** Katz and Lindell Chapter 8.4 & Chapter 9

# Recap: Cyclic Group

- $\mathbb{G} = \langle g \rangle = \{g^0, g^1, g^2, \dots\}$ (g is generator)
- If $m = |\mathbb{G}|$ then for each $\mathrm{h} \in \mathbb{G}$ and each integer $x \geq 0$ we have

$$h^x = h^{x \, mod \, m}$$

**Fact 1:** Let p be a prime then $\mathbb{Z}_p^*$ is a cyclic group of order p-1.

**Fact 2**: Number of generators g s.t. of $\langle g \rangle = \mathbb{Z}_p^*$ is $\frac{|\phi(p-1)|}{p-1}$

**Example (generator)**: p=7, g=5

<2>={1,5,4,6,2,3}

# Recap: Cyclic Group

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- If $m = |\mathbb{G}|$ then for each $\mathrm{h} \in \mathbb{G}$ and each integer $x \geq 0$ we have
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**Fact 1:** Let p be a prime then $\mathbb{Z}_p^*$ is a cyclic group of order p-1.

**Fact 2**: Number of generators g s.t. of $\langle g \rangle = \mathbb{Z}_p^*$ is $\frac{|\phi(p-1)|}{p-1}$

**Proof:** Suppose that $\langle g \rangle = \mathbb{Z}_p^*$ and let $\mathrm{h} = g^i$ then

$\langle h \rangle = \left\{g^0, g^i, g^{2i \bmod (p-1)}, g^{3i \bmod (p-1)}, \dots\right\}$

**Recall:** $\{ij \bmod (p-1) : j \geq 0\} = \{0, \dots, p-1\}$ if and only if gcd(i,p-1)=1.

# Recap Diffie-Hellman Problems

**Computational Diffie-Hellman Problem (CDH)**

- Attacker is given $\mathrm{h}_1 = g^{x_1} \in \mathbb{G}$ and $\mathrm{h}_2 = g^{x_2} \in \mathbb{G}$.
- Attackers goal is to find $g^{x_1 x_2} = (\mathrm{h}_1)^{x_2} = (\mathrm{h}_2)^{x_1}$
- **CDH Assumption**: For all PPT A there is a negligible function negl such that A succeeds with probability at most negl(n).

**Decisional Diffie-Hellman Problem (DDH)**

- Let $\mathrm{z}_0 = g^{x_1 x_2}$ and let $\mathrm{z}_1 = g^{r}$, where $\mathrm{x}_1$,$\mathrm{x}_2$ and r are random
- Attacker is given $g^{x_1}$, $g^{x_2}$ and $z_b$ (for a random bit b)
- Attackers goal is to guess b
- **DDH Assumption**: For all PPT A there is a negligible function negl such that A succeeds with probability at most ½ + negl(n).

# Can we find a cyclic group where DDH holds?

- **Example 1:** $\mathbb{Z}_p^*$ where p is a random n-bit prime.
  - CDH is believed to be hard
  - DDH is *not* hard (You will prove this in homework 4 ☺)

- **Theorem:** $Let$ p=rq+1 be a random n-bit prime where q is a large $\lambda$-bit prime then the set of r$^{th}$ residues modulo p is a cyclic subgroup of order q. Then $\mathbb{G}_r = \left\{[h^r mod\ p] \middle| h \in \mathbb{Z}_p^*\right\}$ is a cyclic subgroup of $\mathbb{Z}_p^*$ of order q.
  - **Remark 1:** DDH is believed to hold for such a group
  - **Remark 2:** It is easy to generate uniformly random elements of $\mathbb{G}_r$
  - **Remark 3:** Any element (besides 1) is a generator of $\mathbb{G}_r$

# Can we find a cyclic group where DDH holds?

- **Theorem:** $Let$ p=rq+1 be a random n-bit prime where q is a large $\lambda$-bit prime then the set of rth residues modulo p is a cyclic subgroup of order q. Then $\mathbb{G}_r = \{[h^r mod\ p] | h \in \mathbb{Z}_p^*\}$ is a cyclic subgroup of $\mathbb{Z}_p^*$ of order q.
  - **Closure**: $h^r g^r = (hg)^r$
  - **Inverse** of $h^r$ is $(h^{-1})^r \in \mathbb{G}_r$
  - **Size** $(h^r)^x = h^{[rx\ mod\ rq]} = (h^r)^x = h^{r[x\ mod\ q]} = (h^r)^{[x\ mod\ q]} mod\ p$

**Remark**: Two known attacks on Discrete Log Problem for $\mathbb{G}_r$(Section 9.2).

- First runs in time $O(\sqrt{q}) = O(2^{\lambda/2})$
- Second runs in time $2^{O(\sqrt[3]{n}(\log n)^{2/3})}$

# Can we find a cyclic group where DDH holds?

**Remark**: Two known attacks (Section 9.2).

- First runs in time $O(\sqrt{q}) = O(2^{\lambda/2})$
- Second runs in time $2^{O(\sqrt[3]{n}(\log n)^{2/3})}$, where n is bit length of p

**Goal**: Set $\lambda$ and n to balance attacks

$$\lambda = O(\sqrt[3]{n}(\log n)^{2/3})$$

How to sample p=rq+1?

- First sample a random $\lambda$-bit prime q and
- Repeatedly check if rq+1 is prime for a random n- $\lambda$ bit value r

# More groups where DDH holds?

**Elliptic Curves Example**: Let p be a prime (p > 3) and let A, B be constants. Consider the equation

$$y^2 = x^3 + Ax + B \ mod \ p$$

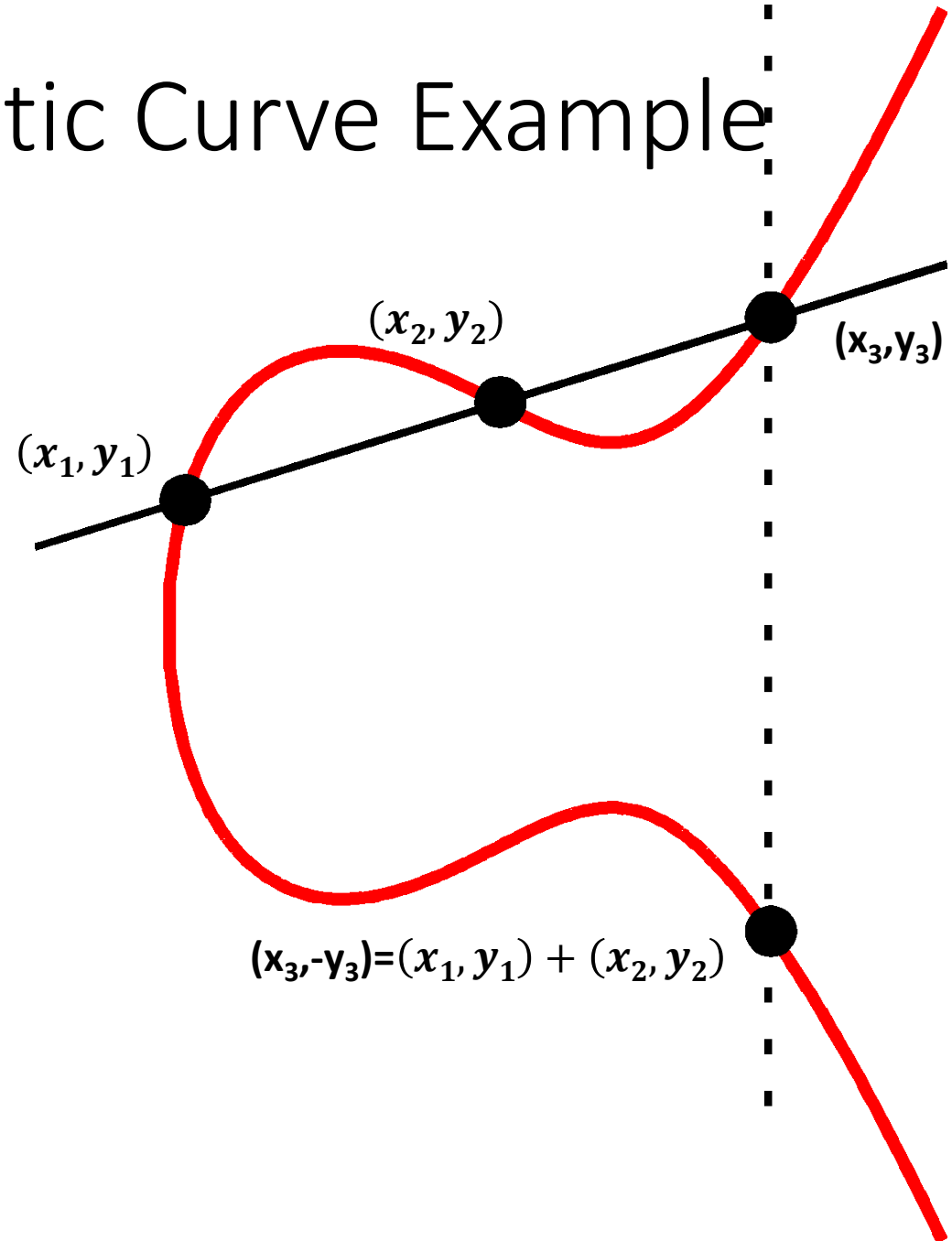
And let

$$E(\mathbb{Z}_p) = \{(x, y) \in \mathbb{Z}_p^2 | y^2 = x^3 + Ax + B \ mod \ p\} \cup \{\mathcal{O}\}$$

**Note**: $\mathcal{O}$ is defined to be an additive identity $(x, y) + \mathcal{O} = (x, y)$

What is $(x_1, y_1) + (x_2, y_2)$?

# Elliptic Curve Example



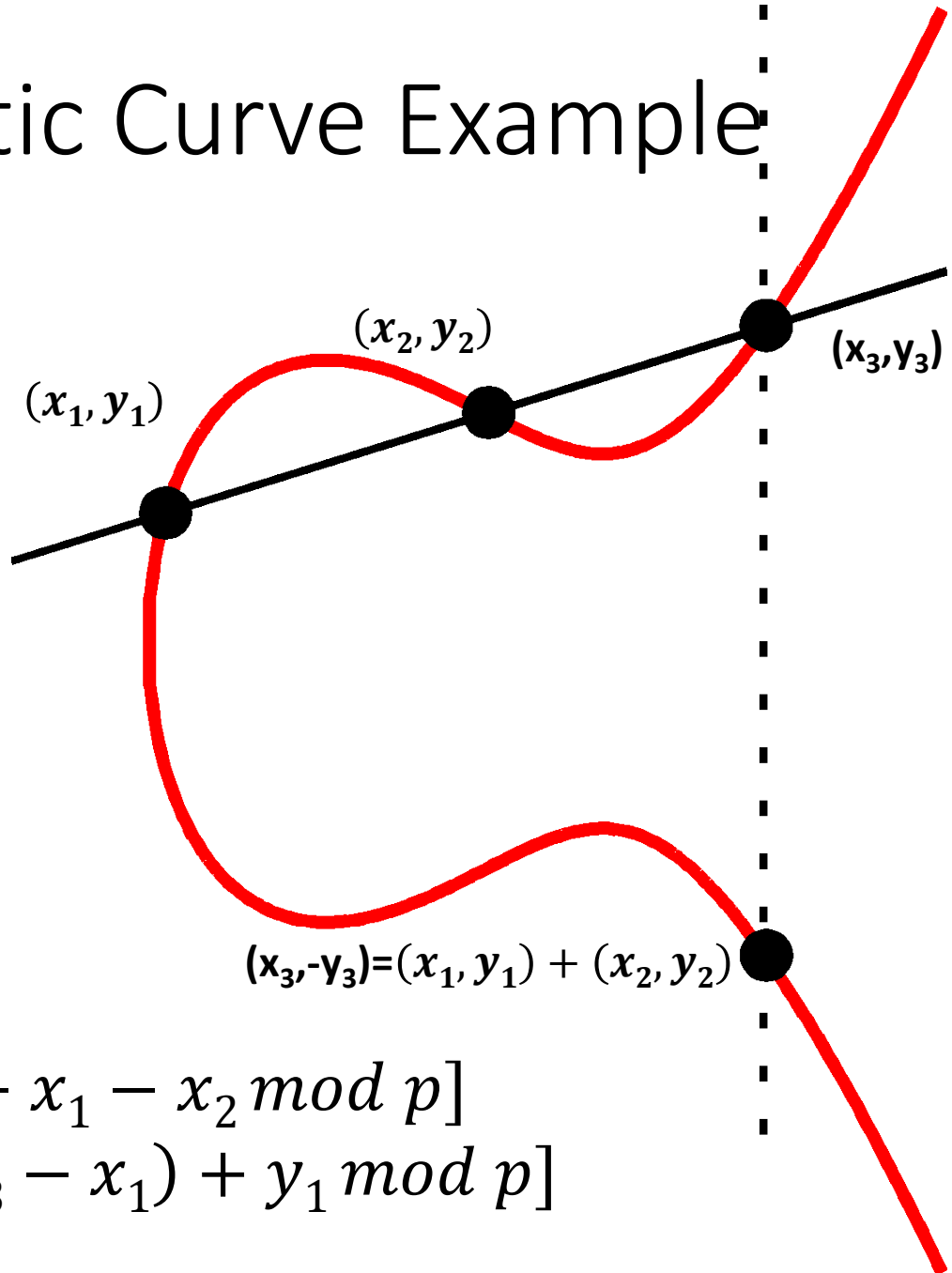

The line passing through $(\boldsymbol{x_1}, \boldsymbol{y_1})$ and $(\boldsymbol{x_2}, \boldsymbol{y_2})$ has the equation

$$y = m(x - x_1) + y_1 \; mod \; P$$

Where the slope

$$m = \left[\frac{y_1 - y_2}{x_1 - x_2} \; mod \; p\right]$$

# Elliptic Curve Example




Formally, let

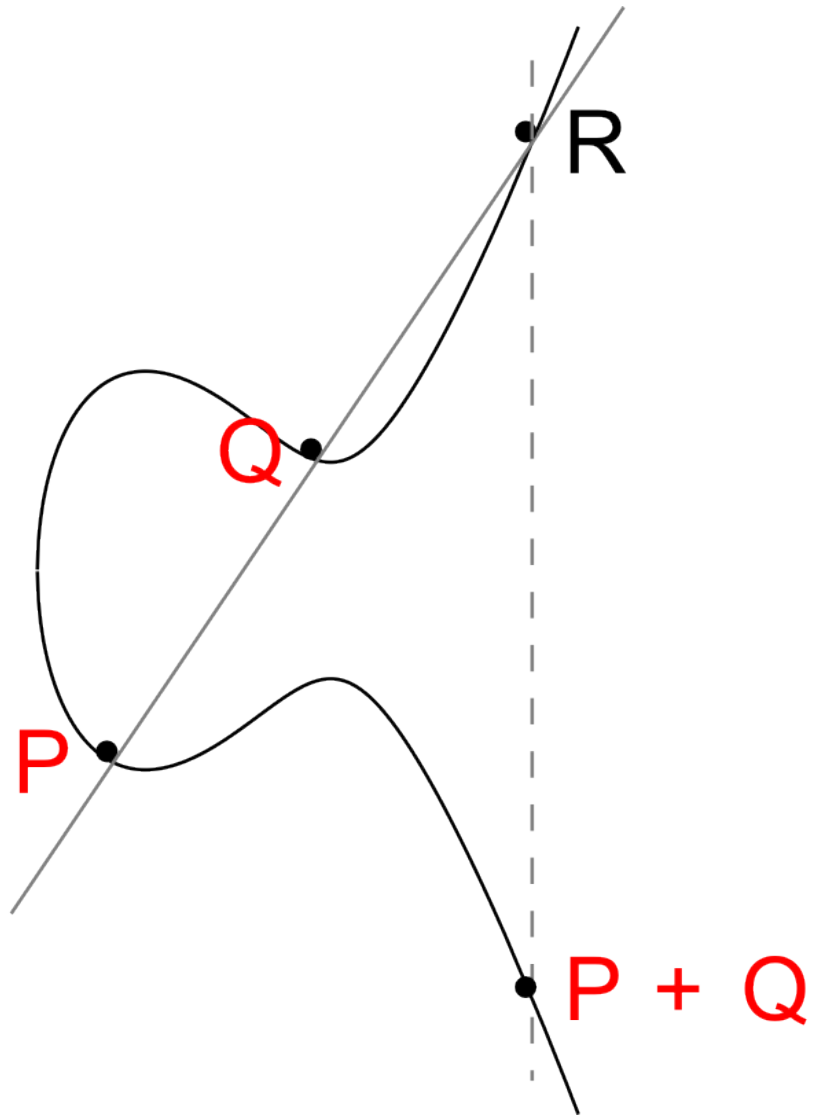
$$m = \left[\frac{y_1 - y_2}{x_1 - x_2} \; mod \; p\right]$$

Be the slope. Then the line passing through $(\boldsymbol{x_1, y_1})$ and $(\boldsymbol{x_2, y_2})$ has the equation

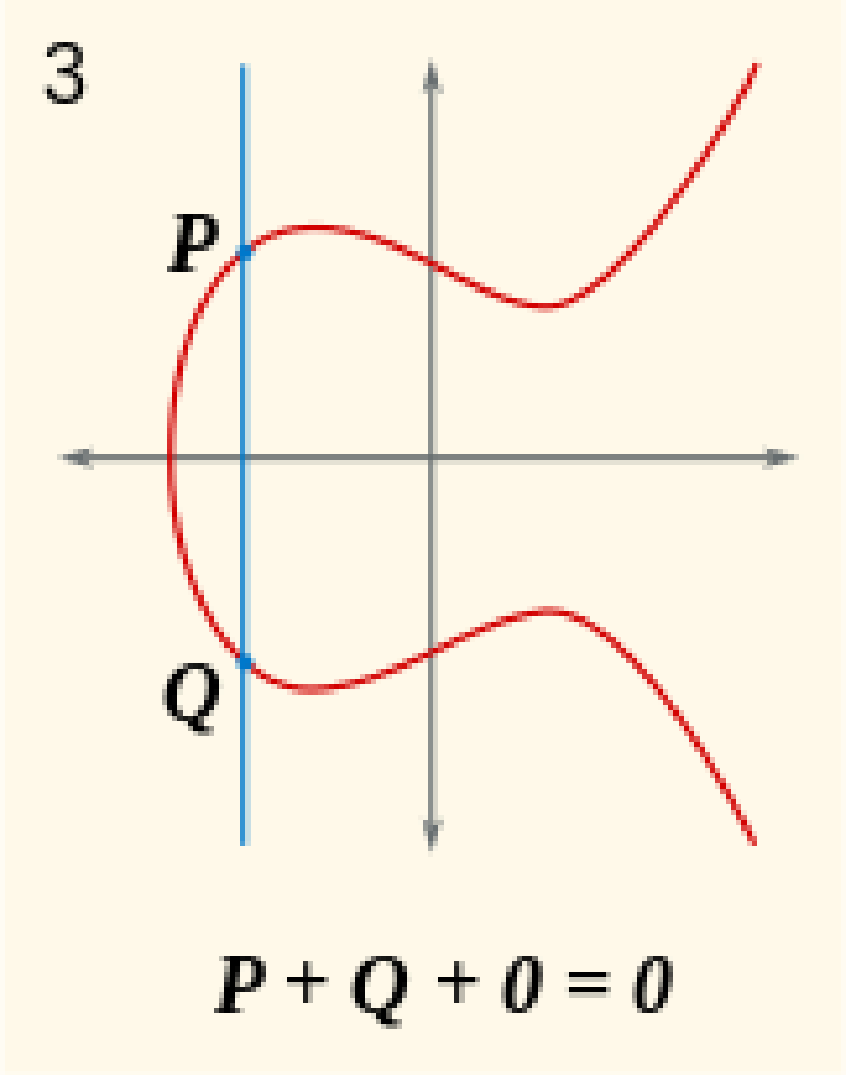
$$y = m(x - x_1) + y_1 \; mod \; P$$

$$x_3 = [m^2 - x_1 - x_2 \, mod \; p]$$
$$y_3 = [m(x_3 - x_1) + y_1 \, mod \; p]$$

$$(m(x - x_1) + y_1)^2 = x^3 + Ax + B \; mod \; p$$


R
Q
P
P + Q

# Elliptic Curve Example




- No third point R on the line intersects our elliptic curve.
- Thus,

$$P + Q = \mathcal{O}$$

# Summary: Elliptic Curves

**Elliptic Curves Example**: Let p be a prime (p > 3) and let A, B be constants. Consider the equation

$$y^2 = x^3 + Ax + B \ mod \ p$$

And let

$$E(\mathbb{Z}_p) = \{(x, y) \in \mathbb{Z}_p^2 | y^2 = x^3 + Ax + B \ mod \ p\} \cup \{\mathcal{O}\}$$

**Fact**: $E(\mathbb{Z}_p)$ defines an abelian group

- For *appropriate curves* the DDH assumption is believed to hold
- If you make up your own curve there is a good chance it is broken…
- NIST has a list of recommendations

# Week 11: Topic 1: Discrete Logarithm Applications

Diffie-Hellman Key Exchange

Collision Resistant Hash Functions

Password Authenticated Key Exchange

# Diffie-Hellman Key Exchange

1. Alice picks $x_A$ and sends $g^{x_A}$ to Bob
2. Bob picks $x_B$ and sends $g^{x_B}$ to Alice
3. Alice and Bob can both compute $K_{A,B} = g^{x_B\, x_A}$

# Key-Exchange Experiment $KE_{A,\Pi}^{eav}(n)$:

- Two parties run $\Pi$ to exchange secret messages (with security parameter $1^n$).
- Let **trans** be a transcript which contains all messages sent and let k be the secret key output by each party.
- Let b be a random bit and let $\mathbf{k_b}$ = k if b=0; otherwise $\mathbf{k_b}$ is sampled uniformly at random.
- Attacker A is given **trans** and $\mathbf{k_b}$ (passive attacker).
- Attacker outputs b' ($KE_{A,\Pi}^{eav}(n)$=1 if and only if b=b')

Security of $\Pi$ against an eavesdropping attacker: For all PPT A there is a negligible function **negl** such that

$$\Pr\left[KE_{A,\Pi}^{eav}(n)\right] = \frac{1}{2} + \mathbf{negl}(n).$$

# Diffie-Hellman Key-Exchange is Secure

**Theorem:** If the decisional Diffie-Hellman problem is hard relative to group generator $\mathcal{G}$ then the Diffie-Hellman key-exchange protocol $\Pi$ is secure in the presence of a (passive) eavesdropper (*).

(*) Assuming keys are chosen uniformly at random from the cyclic group $\mathbb{G}$

Protocol $\Pi$

1. Alice picks $x_A$ and sends $g^{x_A}$ to Bob
2. Bob picks $x_B$ and sends $g^{x_B}$ to Alice
3. Alice and Bob can both compute $K_{A,B} = g^{x_B\, x_A}$

# Diffie-Hellman Assumptions

Computational Diffie-Hellman Problem (CDH)

- Attacker is given $\mathrm{h}_1 = g^{x_1} \in \mathbb{G}$ and $\mathrm{h}_2 = g^{x_2} \in \mathbb{G}$.
- Attackers goal is to find $g^{x_1 x_2} = (\mathrm{h}_1)^{x_2} = (\mathrm{h}_2)^{x_1}$
- **CDH Assumption**: For all PPT A there is a negligible function negl upper bounding the probability that A succeeds

Decisional Diffie-Hellman Problem (DDH)

- Let $\mathrm{z}_0 = g^{x_1 x_2}$ and let $\mathrm{z}_1 = g^r$, where $\mathrm{x}_1$,$\mathrm{x}_2$ and r are random
- Attacker is given $g^{x_1}$, $g^{x_2}$ and $z_b$ (for a random bit b)
- Attackers goal is to guess b
- **DDH Assumption**: For all PPT A there is a negligible function negl such that A succeeds with probability at most ½ + negl(n).

# Diffie-Hellman Key Exchange

1. Alice picks $x_A$ and sends $g^{x_A}$ to Bob
2. Bob picks $x_B$ and sends $g^{x_B}$ to Alice
3. Alice and Bob can both compute $K_{A,B} = g^{x_B\, x_A}$

**Intuition:** Decisional Diffie-Hellman assumption implies that a passive attacker who observes $g^{x_A}$ and $g^{x_B}$ still cannot distinguish between $K_{A,B} = g^{x_B\, x_A}$ and a random group element**.**

**Remark:** Modified protocol sets $K_{A,B} = H(g^{x_B\, x_A})$**.** You will prove that this protocol is secure under the weaker CDH assumption in homework 4.

# Diffie-Hellman Key-Exchange is Secure

**Theorem:** If the decisional Diffie-Hellman problem is hard relative to group generator $\mathcal{G}$ then the Diffie-Hellman key-exchange protocol $\Pi$ is secure in the presence of an eavesdropper (*).

**Proof:**

$$\Pr\left[KE_{A,\Pi}^{eav}(n) = 1\right]$$
$$= \tfrac{1}{2}\Pr\left[KE_{A,\Pi}^{eav}(n) = 1 | b = 1\right] + \tfrac{1}{2}\Pr\left[KE_{A,\Pi}^{eav}(n) = 1 | b = 0\right]$$
$$= \tfrac{1}{2}\Pr[A(\mathbb{G}, g, q, g^x, g^y, g^{xy}) = 1] + \tfrac{1}{2}\Pr[A(\mathbb{G}, g, q, g^x, g^y, g^z) = 1]$$
$$= \tfrac{1}{2} + \tfrac{1}{2}(\Pr[A(\mathbb{G}, g, q, g^x, g^y, g^{xy}) = 1] - \Pr[A(\mathbb{G}, g, q, g^x, g^y, g^z) = 1]).$$
$$\leq \tfrac{1}{2} + \tfrac{1}{2}\text{negl(n) (by DDH)}$$

(*) Assuming keys are chosen uniformly at random from the cyclic group $\mathbb{G}$

# Diffie-Hellman Key Exchange

1. Alice picks $x_A$ and sends $g^{x_A}$ to Bob
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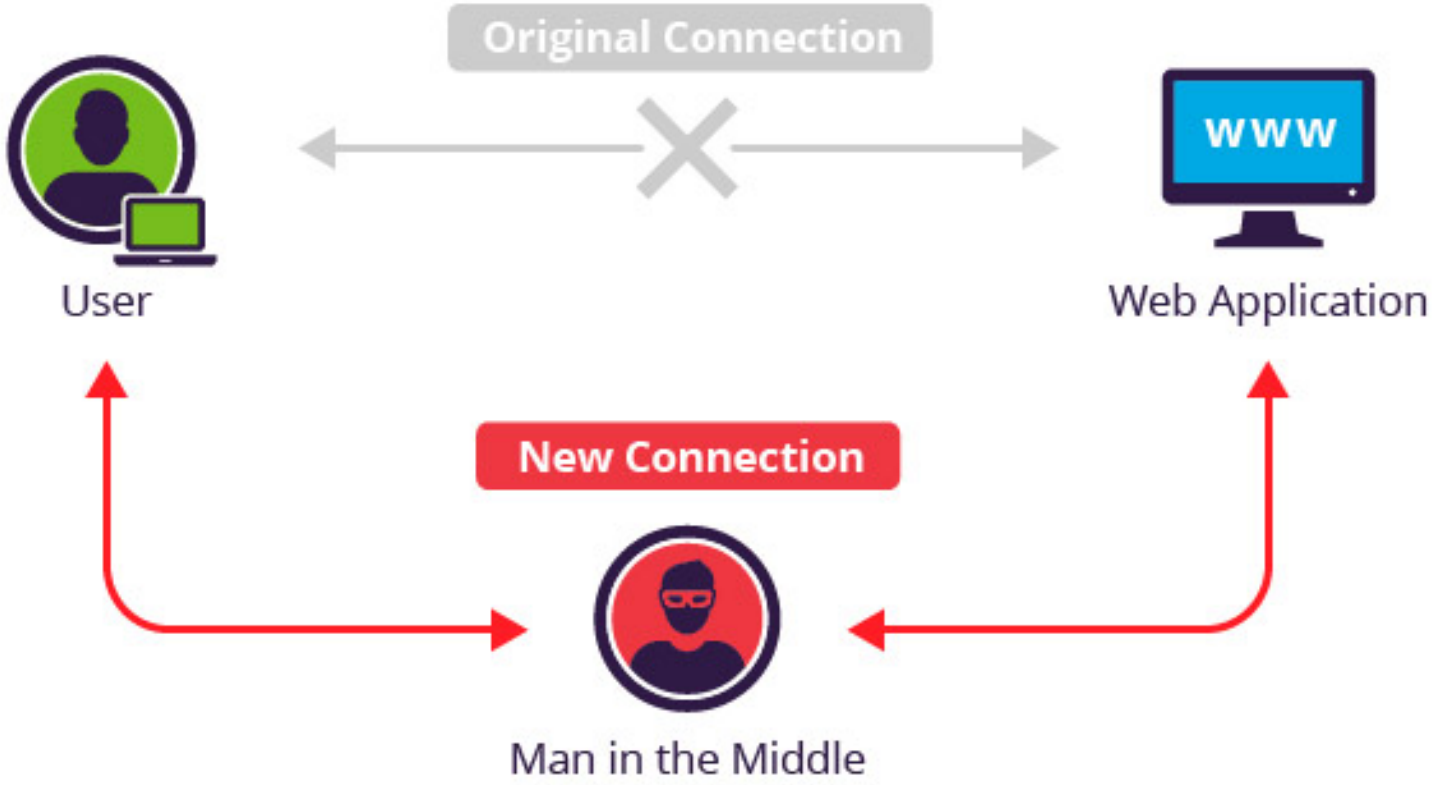
3. Alice and Bob can both compute $K_{A,B} = g^{x_B\, x_A}$

**Intuition:** Decisional Diffie-Hellman assumption implies that a passive attacker who observes $g^{x_A}$ and $g^{x_B}$ still cannot distinguish between $K_{A,B} = g^{x_B\, x_A}$ and a random group element**.**

**Remark**: The protocol is vulnerable against active attackers who can tamper with messages.

# Man in the Middle Attack (MITM)



# Man in the Middle Attack (MITM)

1. Alice picks $x_A$ and sends $g^{x_A}$ to Bob
   - Mallory intercepts $g^{x_A}$ , picks $x_E$ and sends $g^{x_E}$ to Bob instead
2. Bob picks $x_B$ and sends $g^{x_B}$ to Alice
   1. Mallory intercepts $g^{x_B}$, picks $x_{E'}$ and sends $g^{x_{E'}}$ to Alice instead
3. Eve computes $g^{x_{E'}x_A}$ and $g^{x_E x_B}$
   1. Alice computes secret key $g^{x_{E'}x_A}$ (shared with Eve not Bob)
   2. Bob computes $g^{x_E x_B}$(shared with Eve not Alice)
4. Mallory forwards messages between Alice and Bob (tampering with the messages if desired)
5. Neither Alice nor Bob can detect the attack

# Discrete Log Experiment $\text{DLog}_{A,G}(n)$

1. Run $\mathcal{G}(1^n)$ to obtain a cyclic group $\mathbb{G}$ of order q (with $\|q\| = n$) and a generator g such that $< \text{g} >= \mathbb{G}$.
2. Select $\text{h} \in \mathbb{G}$ uniformly at random.
3. Attacker A is given $\mathbb{G}$, q, g, h and outputs integer x.
4. Attacker wins ($\text{DLog}_{A,G}(n)=1$) if and only if $g^x=h$.

We say that the discrete log problem is hard relative to generator $\mathcal{G}$ if

$$\forall PPT\ A\ \exists \mu\ (\text{negligible})\ \text{s.t}\ \Pr[\text{DLog}_{A,n} = 1] \leq \mu(n)$$

# Collision Resistant Hash Functions (CRHFs)

- Recall: not known how to build CRHFs from OWFs
- Can build collision resistant hash functions from Discrete Logarithm Assumption
- Let $\mathcal{G}(1^n)$ output $(\mathbb{G}, q, g)$ where $\mathbb{G}$ is a cyclic group of order $q$ and g is a generator of the group.
- Suppose that discrete log problem is hard relative to generator $\mathcal{G}$.

$$\forall PPT\ A\ \exists \mu\ (\text{negligible})\ \text{s.t}\ \Pr[\text{DLog}_{\text{A,n}} = 1] \leq \mu(n)$$

# Collision Resistant Hash Functions

- Let $\mathcal{G}(1^n)$ output $(\mathbb{G}, q, g)$ where $\mathbb{G}$ is a cyclic group of order $q$ and g is a generator of the group.

Collision Resistant Hash Function (Gen,H):

- $Gen(1^n)$
  1. $(\mathbb{G}, q, g) \leftarrow \mathcal{G}(1^n)$
  2. Select random $\mathrm{h} \leftarrow \mathbb{G}$
  3. Output $\mathrm{s} = (\mathbb{G}, q, g, h)$
- $H^s(x_1, x_2) = g^{x_1} h^{x_2}$ (where, $x_1, x_2 \in \mathbb{Z}_q$ )

**Claim**: (Gen,H) is collision resistant if the discrete log assumption holds for $\mathcal{G}$

# Collision Resistant Hash Functions

- $H^s(x_1, x_2) = g^{x_1} h^{x_2}$ (where, $x_1, x_2 \in \mathbb{Z}_q$ )

**Claim**: (Gen,H) is collision resistant

**Proof:** Suppose we find a collision $H^s(x_1, x_2) = H^s(y_1, y_2)$ then we have $g^{x_1} h^{x_2} = g^{y_1} h^{y_2}$ which implies

$$h^{x_2 - y_2} = g^{y_1 - x_1}$$

Use extended GCD to find $(x_2 - y_2)^{-1}$ mod q then

$$h = h^{(x_2 - y_2)(x_2 - y_2)^{-1} \, mod \, q} = g^{(y_1 - x_1)(x_2 - y_2)^{-1} \, mod \, q}$$

Which means that $(y_1 - x_1)(x_2 - y_2)^{-1} \, mod \, q$ is the discrete log of h.

# Password Authenticated Key-Exchange

- Suppose Alice and Bob share a low-entropy password pwd and wish to communicate securely
  - (without using any trusted party)
  - Assuming an active attacker may try to mount a man-in-the-middle attack
- Can they do it?

**Tempting Approach:**

- Alice and Bob both compute K= KDF(pwd)=$H^n$(pwd) and communicate with using an authenticated encryption scheme.
- **Practice Midterm Exam:** Secure in random oracle model if attacker cannot query random oracle H(.) too many times.

# Password Authenticated Key-Exchange

**Tempting Approach:**

- Alice and Bob both compute K= KDF(pwd)=$H^n$(pwd) and communicate with using an authenticated encryption scheme.
- **Midterm Exam:** Secure in random oracle model if attacker cannot query random oracle too many time.
- **Problems:**
  - In practice the attacker can (and will) query the random oracle many times.
  - In practice people tend to pick very weak passwords
  - Brute-force attack: Attacker enumerates over a dictionary of passwords and attempts to decrypt messages with $K_{pwd'}$=KDF(pwd') (only succeeds if $K_{pwd'}$=K).
  - An offline attack (brute-force) will almost always succeed

# Attempt 2

1. Alice picks $x_A$ and sends $g^{H(pwd)+xA}$ to Bob
2. Bob picks $x_B$ and sends $g^{H(pwd)+x_B}$ to Alice
3. Alice and Bob can both compute $K_{A,B} = H(g^{x_B\, x_A})$
4. Alice picks random nonce $r_A$ and sends $Enc_{K_{A,B}}(r_A)$ to Bob
    1. Enc is an authentication encryption scheme
5. Bob decrypts and sends $r_A$ to Alice

Advantage: MITM Attacker cannot establish connection without password

Disadvantage: Mallory could mount a brute-force attack after attempted MITM attack

# Attempt 2: MITM Attack

1. Alice picks $x_A$ and sends $g^{H(pwd)+xA}$ to Bob
2. Bob picks $x_B$ and sends $g^{H(pwd)+x_B}$ to Alice
    1. Mallory intercepts $g^{H(pwd)+xB}$, picks $x_E$ and sends $g^{x_E}$ to Alice instead
3. Bob can both compute $K_{A,B} = H(g^{x_B\, x_A})$
    1. Allice computes $K_{A,B}{}' = H\left(g^{(x_E - H(pwd))\, x_A}\right)$ instead
4. Alice picks random nonce $r_A$ and sends c $= Enc_{K_{A,B}{}'}(r_A)$ to Bob
    1. Mallory intercepts $Enc_{K_{A,B}{}'}(r_A)$ and proceeds to mount brute-force attack on password
5. For each password guess y
    1. let $K_y = H\left(g^{(x_E - H(y))\, x_A}\right)$ and
    2. if $Dec_{K_{A,B}{}'}(c) \neq \perp$ then output y

Advantage: MITM Attacker cannot establish connection without password

Disadvantage: Mallory could mount a brute-force attack on password after attempted MITM attack

# Password Authenticated Key-Exchange (PAKE)

**Better Approach (PAKE):**

1. Alice and Bob both compute $\mathrm{W} = g^{pwd}$
2. Alice picks $x_A$ and sends "Alice", $X = g^{x_A}$ to Bob
3. Bob picks $x_B$ computes $\mathrm{r} = \mathrm{H}(1, Alice, Bob, X)$ and $Y = (X \times (W)^r)^{x_B}$ and sends Alice the following message: "$Bob$," $Y$
4. Alice computes $\mathrm{K} = Y^Z = g^{x_B}$ where $z = 1/\big((pwd \times r) + x_A\big)\ mod\ p$. Alice sends the message $V_A$= H(2,Alice,Bob,X,Y,K) to Bob.
5. Bob verifies that $V_A$== H(2,Alice,Bob,X,Y,K) where K = $g^{x_B}$. Bob generates $V_B$= H(3,Alice,Bob,X,Y,K) and sends $V_B$ to Alice.
6. Alice verifies that $V_B$==H(3,Alice,Bob,X,Y, $Y^Z$ ) where $z = 1/\big((pwd \times r) + x_A\big)$.
7. If Alice and Bob don't terminate the session key is H(4,Alice,Bob,X,Y, $K$)

**Security:**

- No offline attack (brute-force) is possible. Attacker get's one password guess per instantiation of the protocol.
- If attacker is incorrect and he tampers with messages then he will cause the Alice & Bob to quit.
- If Alice and Bob accept the secret key K and the attacker did not know/guess the password then K is "just as good" as a truly random secret key.



See RFC 6628

# Week 11: Topic 2: Factoring Algorithms, Discrete Log Attacks + NIST Recommendations for Concrete Security Parameters

# Pollard's p-1 Algorithm (Factoring)

- Let $N = pq$ where (p-1) has only "small" prime factors.
- Pollard's p-1 algorithm can factor N.
  - **Remark 1**: This happens with very small probability if p is a random n bit prime.
  - **Remark 2**: One convenient/fast way to generate big primes it to multiply many small primes, add 1 and test for primality.
    - Example: $2 \times 3 \times 5 \times 7 + 1 = 211$ is prime

**Claim**: Suppose we are given an integer B such that (p-1) divides B but (q-1) does not divide B then we can factor N.

# Pollard's p-1 Algorithm (Factoring)

**Claim**: Suppose we are given an integer B such that (p-1) divides B but (q-1) does not divide B then we can factor N.

**Proof**: Suppose B=c(p-1) for some integer c and let

$$y = [x^B - 1 \; mod \; N]$$

Applying the Chinese Remainder Theorem we have

$$y \leftrightarrow (x^B - 1 \bmod \mathrm{p}, x^B - 1 \bmod \mathrm{q})$$
$$= (0, x^B - 1 \bmod \mathrm{q})$$

This means that p divides y, but q does not divide y (unless $x^B = 1 \bmod \mathrm{q}$, which is very unlikely).

Thus, GCD(y,N) = p

# Pollard's p-1 Algorithm (Factoring)

- Let $N = pq$ where (p-1) has only "small" prime factors.
- Pollard's p-1 algorithm can factor N.

**Claim**: Suppose we are given an integer B such that (p-1) divides B but (q-1) does not divide B then we can factor N.

- **Goal**: Find B such that (p-1) divides B but (q-1) does not divide B.
- **Remark**: This is difficult if (p-1) has a large prime factor.

$$B = \prod_{i=1}^{k} p_i^{[n/\log p_i]}$$

# Pollard's p-1 Algorithm (Factoring)

- **Goal**: Find B such that (p-1) divides B but (q-1) does not divide B.
- **Remark**: This is difficult if (p-1) has a large prime factor.

$$B = \prod_{i=1}^{k} p_i^{[n/\log p_i]}$$

Here $p_1$=2,$p_2$=3,…

**Fact**: If (q-1) has prime factor larger than $p_k$ then (q-1) does not divide B.

**Fact**: If (p-1) does not have prime factor larger than $p_k$ then (p-1) does divide B.

# Pollard's p-1 Algorithm (Factoring)

- **Option 1:** To defeat this attack we can choose strong primes p and q
  - A prime p is strong if (p-1) has a large prime factor
- **Drawback:** It takes more time to generate (provably) strong primes

- **Option 2:** A random prime is strong with high probability

- **Current Consensus:** Just pick a random prime

# Pollard's Rho Algorithm

- General Purpose Factoring Algorithm
  - Doesn't assume (p-1) has no large prime factor
  - **Goal**: factor N=pq (product of two n-bit primes)
- Running time: $O\left(\sqrt[4]{N}\ \mathrm{poly}log(N)\right)$
  - Naïve Algorithm takes time $O\left(\sqrt{N}\ \mathrm{poly}log(N)\right)$ to factor

- **Core idea:** find distinct $\mathrm{x}, \mathrm{x}' \in \mathbb{Z}_N^*$ such that $x = x' \bmod p$
  - Implies that x-x' is a multiple of p and, thus, GCD(x-x',N)=p (whp)

# Pollard's Rho Algorithm

- General Purpose Factoring Algorithm
  - Doesn't assume (p-1) has no large prime factor

- Running time: $O\left(\sqrt[4]{N}\ \mathrm{poly}log(N)\right)$

- **Core idea:** find distinct $\mathrm{x}, \mathrm{x}' \in \mathbb{Z}_N^*$ such that $x = x' \mathrm{mod}\ p$
  - Implies that x-x' is a multiple of p and, thus, GCD(x-x',N)=p (whp)

- **Question**: If we pick $\mathrm{k} = \mathrm{O}\left(\sqrt{p}\right)$ random $x^{(1)}, \ldots, x^{(k)} \in \mathbb{Z}_N^*$ then what is the probability that we can find distinct $i$ and $j$ such that $x^{(i)} = x^{(j)} \mathrm{mod}\ \mathrm{p}?$

# Pollard's Rho Algorithm

- **Question**: If we pick $\mathrm{k} = \mathrm{O}\left(\sqrt{p}\right)$ random $x^{(1)}, \ldots, x^{(k)} \in \mathbb{Z}_N^*$ then what is the probability that we can find distinct $i$ and $j$ such that $x^{(i)} = x^{(j)} \mathrm{mod}\ \mathrm{p}$?
- **Answer:** $\geq {}^{1}/_{2}$
- **Proof (sketch):** Use the Chinese Remainder Theorem + Birthday Bound

$$x^{(i)} = \left(x^{(i)}\ mod\ p, x^{(i)}\ mod\ q\right)$$

**Note**: We will also have $x^{(i)} \neq x^{(j)}\ \mathrm{mod}\ \mathrm{q}$ (whp)

# Pollard's Rho Algorithm

- **Question**: If we pick $\mathrm{k} = \mathrm{O}\left(\sqrt{p}\right)$ random $x^{(1)}, \ldots, x^{(k)} \in \mathbb{Z}_N^*$ then what is the probability that we can find distinct $i$ and $j$ such that $x^{(i)} = x^{(j)} \mathrm{mod}\ \mathrm{p}$?
- **Answer:** $\geq {}^{1}/_{2}$
- **Challenge:** We do not know p or q so we cannot sort the $x^{(i)}$'s using the Chinese Remainder Theorem Representation

$$x^{(i)} = \left(x^{(i)}\ mod\ p, x^{(i)}\ mod\ q\right)$$

How can we identify the pair $i$ and $j$ such that $x^{(i)} = x^{(j)} \mathrm{mod}\ \mathrm{p}$?

# Pollard's Rho Algorithm

- Pollard's Rho Algorithm is similar the low-space version of the birthday attack

**Input**: N (product of two n bit primes)

$x^{(0)} \leftarrow \mathbb{Z}_N^*$, $\mathrm{x} = \mathrm{x}' = x^{(0)}$

**For** i=1 to $2^{n/2}$

$x \leftarrow F(x)$

$x' \leftarrow F\big(F(x)\big)$

p = **GCD**(x-x',N)

**if** 1< p < N **return** p

**Remark 1:** F should have the property that if x=x' mod p then F(x) = F(x') mod p.

**Remark 2:** $F(x) = [x^2 + 1 \; mod \; N]$ will work since

$$F(x) = [x^2 + 1 \; mod \; N]$$
$$\leftrightarrow (x^2 + 1 \; mod \; p, x^2 + 1 \; mod \; q)$$
$$\leftrightarrow (F([x \bmod p]) \; mod \; p, F([x \bmod q]) \; mod \; q)$$

# Pollard's Rho Algorithm

- Pollard's Rho Algorithm is similar the low-space version of the birthday attack

**Input**: N (product of two n bit primes)

$x^{(0)} \leftarrow \mathbb{Z}_N^*$, x = x′ = $x^{(0)}$

**For** i=1 to $2^{n/2}$

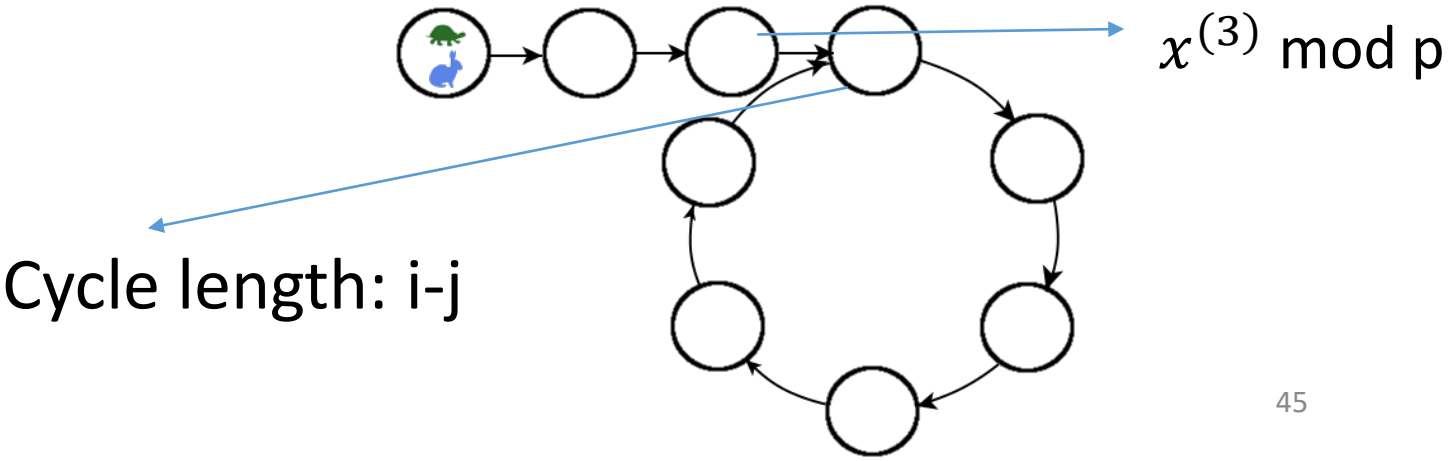
$x \leftarrow F(x)$

$x' \leftarrow F\left(F(x)\right)$

p = **GCD**(x-x',N)

**if** 1< p < N **return** p

**Claim**: Let $x^{(i+1)} = F\left(x^{(i)}\right)$ and suppose that for some distinct i, j $< 2^{n/2}$ we have $x^{(i)} = x^{(j)}$ mod p but $x^{(i)} \neq x^{(j)}$. Then the algorithm will find p.



# Pollard's Rho Algorithm (Summary)

- General Purpose Factoring Algorithm
  - Doesn't assume (p-1) has no large prime factor

- Expected Running Time: $O\left(\sqrt[4]{N}\ \mathrm{poly}log(N)\right)$
  - (Birthday Bound)
  - (still exponential in number of bits $\sim 2^{n/4}$)
- Required Space: $O(\log(N))$

# Quadratic Sieve Algorithm

- Runs in sub-exponential time $2^{O\left(\sqrt{\log N \ \log\log N}\right)} = 2^{O\left(\sqrt{n \log n}\right)}$
  - Still not polynomial time but $2^{\sqrt{n \log n}}$ grows much slower than $2^{n/4}$.

- **Core Idea**: Find $\mathrm{x}, \mathrm{y} \in \mathbb{Z}_N^*$ such that

$$x^2 = y^2 \bmod N$$

and

$$x \neq \pm y \bmod N$$

# Quadratic Sieve Algorithm

- **Core Idea**: Find x, y ∈ $\mathbb{Z}_N^*$ such that

$$x^2 = y^2 \bmod N \quad (1)$$

and

$$x \neq \pm y \bmod N \quad (2)$$

**Claim**: gcd(x-y,N)∈ $\{p, q\}$

→N=pq divides $x^2 - y^2 = (x - y)(x + y)$. (by (1)).

→$(x - y)(x + y) \neq 0$ (by (2)).

→N does not divide $(x - y)$ (by (2)).

→N does not divide $(x + y)$. (by (2)).

→*p is a factor of exactly one of the terms* $(x - y)$ *and* $(x + y)$.

→*(q is a factor of the other term)*

# Quadratic Sieve Algorithm

- **Core Idea**: Find $x, y \in \mathbb{Z}_N^*$ such that

$$x^2 = y^2 \bmod N$$

and

$$x \neq \pm y \bmod N$$

- **Key Question**: How to find such an $x, y \in \mathbb{Z}_N^*$?
- **Step 1:**

j=0;

For $x = \sqrt{N} + 1, \sqrt{N} + 2, \ldots, \sqrt{N} + i, \ldots$

$$q \leftarrow \left[\left(\sqrt{N} + i\right)^2 \, mod \, N\right] = \left[2i\sqrt{N} + i^2 \, mod \, N\right]$$

Check if q is B-smooth (all prime factors of q are in $\{p_1,\ldots,p_k\}$ where $p_k < B$).

If q is B smooth then factor q, increment j and define

$$q_j \leftarrow q = \prod_{i=1}^{k} p_i^{e_{j,i}},$$

# Quadratic Sieve Algorithm

- **Core Idea**: Find $x, y \in \mathbb{Z}_N^*$ such that

$$x^2 = y^2 \bmod N$$

and

$$x \neq \pm y \bmod N$$

- **Key Question**: How to find such an $x, y \in \mathbb{Z}_N^*$?
- **Step 2:** Once we have $\ell > k$ equations of the form

$$q_j \leftarrow q = \prod_{i=1}^{k} p_i^{e_{j,i}},$$

We can use linear algebra to find S such that for each $i \leq k$ we have

$$\sum_{j \in S} e_{j,i} = 0 \; mod \; 2.$$

# Quadratic Sieve Algorithm

- **Key Question**: How to find x, y $\in \mathbb{Z}_N^*$ such that $x^2 = y^2 \bmod N$ and $x \neq \pm y \bmod N$?
- **Step 2:** Once we have $\ell > k$ equations of the form

$$q_j \leftarrow q = \prod_{i=1}^{k} p_i^{e_{j,i}},$$

We can use linear algebra to find a subset S such that for each i ≤ k we have

$$\sum_{j \in S} e_{j,i} = 0 \; mod \; 2.$$

Thus,

$$\prod_{j \in S} q_j = \prod_{i=1}^{k} p_i^{\sum_{j \in S} e_{j,i}} = \left( \prod_{i=1}^{k} p_i^{\frac{1}{2} \sum_{j \in S} e_{j,i}} \right)^2 = y^2$$

# Quadratic Sieve Algorithm

- **Key Question**: How to find x, y $\in \mathbb{Z}_N^*$ such that $x^2 = y^2 \bmod N$ and $x \neq \pm y \bmod N$?

Thus,

$$\prod_{j \in S} q_j = \prod_{i=1}^{k} p_i^{\sum_{j \in S} e_{j,i}} = \left( \prod_{i=1}^{k} p_i^{\frac{1}{2} \sum_{j \in S} e_{j,i}} \right)^2 = y^2$$

But we also have

$$\prod_{j \in S} q_j = \prod_{j \in S} (x_j^2) = \left( \prod_{j \in S} x_j \right)^2 = x^2 \, mod \, N$$

# Quadratic Sieve Algorithm (Summary)

- Appropriate parameter tuning yields sub-exponential time algorithm $2^{O\left(\sqrt{\log N \ \log\log N}\right)} = 2^{O\left(\sqrt{n \log n}\right)}$
  - Still not polynomial time but $2^{\sqrt{n \log n}}$ grows much slower than $2^{n/4}$.

# Discrete Log Attacks

- Pohlig-Hellman Algorithm
  - Given a cyclic group $\mathbb{G}$ of non-prime order q=| $\mathbb{G}$ |=rp
  - Reduce discrete log problem to discrete problem(s) for subgroup(s) of order p (or smaller).
  - Preference for prime order subgroups in cryptography
- Baby-step/Giant-Step Algorithm
  - Solve discrete logarithm in time $O\left(\sqrt{q}\, polylog(q)\right)$
- Pollard's Rho Algorithm
  - Solve discrete logarithm in time $O\left(\sqrt{q}\, polylog(q)\right)$
  - Bonus: Constant memory!
- Index Calculus Algorithm
  - Similar to quadratic sieve
  - Runs in sub-exponential time $2^{O\left(\sqrt{\log q \log\log q}\right)}$
  - Specific to the group $\mathbb{Z}_p^*$ (e.g., attack doesn't work elliptic-curves)

# Discrete Log Attacks

- **Pohlig-Hellman Algorithm**
  - Given a cyclic group $\mathbb{G}$ of non-prime order q=| $\mathbb{G}$ |=rp
  - Reduce discrete log problem to discrete problem(s) for subgroup(s) of order p (or smaller).
  - Preference for prime order subgroups in cryptography
- Let $\mathbb{G} = \langle g \rangle$ and $\mathrm{h} = g^x \in \mathbb{G}$ be given. For simplicity assume that r is prime and r < p.
- Observe that $\langle g^r \rangle$ generates a subgroup of size p and that $\mathrm{h}^{\mathrm{r}} \in \langle g^r \rangle$.
  - Solve discrete log problem in subgroup $\langle g^r \rangle$ with input $\mathrm{h}^{\mathrm{r}}$.
  - Find z such that $\mathrm{h}^{\mathrm{rz}} = g^{rz}$.
- Observe that $\langle g^p \rangle$ generates a subgroup of size p and that $\mathrm{h}^{\mathrm{p}} \in \langle g^p \rangle$.
  - Solve discrete log problem in subgroup $\langle g^p \rangle$ with input $\mathrm{h}^{\mathrm{p}}$.
  - Find y such that $\mathrm{h}^{\mathrm{yp}} = g^{yp}$.
- Chinese Remainder Theorem $\mathrm{h} = g^x$ where $\mathrm{x} \leftrightarrow ([z \bmod p], [y \bmod r])$

# Baby-step/Giant-Step Algorithm

- Input: $\mathbb{G} = \langle g \rangle$ of order q, generator g and $\mathrm{h} = g^x \in \mathbb{G}$
- Set $t = \lfloor \sqrt{q} \rfloor$

**For** i =0 to $\left\lfloor \frac{q}{t} \right\rfloor$

$$g_i \leftarrow g^{it}$$

**Sort** the pairs ($i,g_i$) by their second component

**For** i =0 to $t$

$$h_i \leftarrow hg^i$$

if $h_i = gk \in \{g_0, \dots, g_t\}$ then

return [kt-i mod q]

$$h_i = hg^i = g^{kt}$$

$$\rightarrow h = g^{kt-i}$$

# Discrete Log Attacks

- Baby-step/Giant-Step Algorithm
  - Solve discrete logarithm in time $O\left(\sqrt{q}\, polylog(q)\right)$
  - Requires memory $O\left(\sqrt{q}\, polylog(q)\right)$
- Pollard's Rho Algorithm
  - Solve discrete logarithm in time $O\left(\sqrt{q}\, polylog(q)\right)$
  - Bonus: Constant memory!
- **Key Idea:** Low-Space Birthday Attack (*) using our collision resistant hash function

$$H_{g,h}(x_1, x_2) = g^{x_1} h^{x_2}$$

$$H_{g,h}(y_1, y_2) = H_{g,h}(x_1, x_2) \rightarrow h^{y_2 - x_2} = g^{x_1 - y_1}$$

$$\rightarrow h = g^{(x_1 - y_1)(y_2 - x_2)^{-1}}$$

(*) A few small technical details to address

# Discrete Log Attacks

- Baby-step/Giant-Step Algorithm
  - Solve discrete logarithm in time $O\left(\sqrt{q}\ polylog(q)\right)$
  - Requires memory $O\left(\sqrt{q}\ polylog(q)\right)$
- Pollard's Rho Algorithm
  - Solve discrete logarithm in time $O\left(\sqrt{q}\ pol$
  - Bonus: Constant memory!
- **Key Idea:** Low-Space Birthday Attack (*)

$$H_{g,h}(x_1, x_2) = g^{x_1} h^{x_2}$$
$$H_{g,h}(y_1, y_2) = H_{g,h}(x_1, x_2)$$

$$\rightarrow\ h^{y_2 - x_2} = g^{x_1 - y_1}$$
$$\rightarrow h = g^{(x_1 - y_1)(y_2 - x_2)^{-1}}$$

(*) A few small technical details to address

**Remark**: We used discrete-log problem to construct collision resistant hash functions.

Security Reduction showed that attack on collision resistant hash function yields attack on discrete log.

→Generic attack on collision resistant hash functions (e.g., low space birthday attack) yields generic attack on discrete log.

# Discrete Log Attacks

- Index Calculus Algorithm
  - Similar to quadratic sieve
  - Runs in sub-exponential time $2^{O(\sqrt{\log q \log\log q})}$
  - Specific to the group $\mathbb{Z}_p^*$ (e.g., attack doesn't work elliptic-curves)

- As before let $\{p_1,\ldots,p_k\}$ be set of prime numbers < B.
- **Step 1.A:** Find $\ell > k$ distinct values $x_1, \ldots, x_k$ such that $g_j = [g^{x_j} \; mod \; p]$ is B-smooth for each j. That is

$$g_j = \prod_{i=1}^{k} p_i^{e_{i,j}} .$$

# Discrete Log Attacks

- As before let $\{p_1,\ldots,p_k\}$ be set of prime numbers < B.
- **Step 1.A:** Find $\ell > k$ distinct values $x_1, \ldots, x_k$ such that $g_j = [g^{x_j} \, mod \, p]$ is B-smooth for each j. That is

$$g_j = \prod_{i=1}^{k} p_i^{e_{i,j}} .$$

- **Step 1.B:** Use linear algebra to solve the equations

$$x_j = \sum_{i=1}^{k} \left( \mathbf{\log_g p_i} \right) \times e_{i,j} \mod (p-1).$$

(Note: the $\mathbf{\log_g p_i}$'s are the unknowns)

# Discrete Log

- As before let $\{p_1,\ldots,p_k\}$ be set of prime numbers < B.
- **Step 1 (precomputation):** Obtain $y_1,\ldots,y_k$ such that $p_i = g^{y_i} \; mod \; p$.
- **Step 2:** Given discrete log challenge $h=g^x \bmod p$.
  - Find y such that $[g^y h \bmod p]$ is B-smooth

$$[g^y h \bmod p] = \prod_{i=1}^{k} p_i^{e_i}$$

$$= \prod_{i=1}^{k} (g^{y_i})^{e_i} = g^{\sum_i e_i y_i}$$

# Discrete Log

- As before let $\{p_1,\ldots,p_k\}$ be set of prime numbers < B.
- **Step 1 (precomputation):** Obtain $y_1,\ldots,y_k$ such that $p_i = g^{y_i} \; mod \; p$.
- **Step 2:** Given discrete log challenge $h=g^x \bmod p$.
  - Find z such that $[g^z h \bmod p]$ is B-smooth

$$[g^z h \bmod p] = g^{\sum_i e_i y_i} \rightarrow h = g^{\sum_i e_i y_i - z}$$

$$\rightarrow x = \sum_i e_i y_i - z$$

- **Remark:** Precomputation costs can be amortized over many discrete log instances
  - In practice, the same group $\mathbb{G} = \langle g \rangle$ and generator g are used repeatedly.

Reference: https://www.weakdh.org/

# NIST Guidelines (Concrete Security)

Best known attack against 1024 bit RSA takes time (approximately) $2^{80}$

| Symmetric Key Size (bits) | RSA and Diffie-Hellman Key Size (bits) | Elliptic Curve Key Size (bits) |
|---|---|---|
| 80 | 1024 | 160 |
| 112 | 2048 | 224 |
| 128 | 3072 | 256 |
| 192 | 7680 | 384 |
| 256 | 15360 | 521 |

Table 1: NIST Recommended Key Sizes

# NIST Guidelines (Concrete Security)

Diffie-Hellman uses subgroup of $\mathbb{Z}_p^*$ size q

| Symmetric Key Size (bits) | RSA and Diffie-Hellman Key Size (bits) | | Elliptic Curve Key Size (bits) |
|---|---|---|---|
| 80 | 1024 | | 160 |
| 112 | 2048 | q=224 bits | 224 |
| 128 | 3072 | q=256 bits | 256 |
| 192 | 7680 | q=384 bits | 384 |
| 256 | 15360 | q=512 bits | 521 |

Table 1: NIST Recommended Key Sizes

| Security Strength | | 2011 through 2013 | 2014 through 2030 | 2031 and Beyond |
|---|---|---|---|---|
| 80 | Applying | Deprecated | Disallowed | |
| | Processing | Legacy use | | |
| 112 | Applying | Acceptable | Acceptable | Disallowed |
| | Processing | | | Legacy use |
| 128 | Applying/Processing | Acceptable | Acceptable | Acceptable |
| 192 | | Acceptable | Acceptable | Acceptable |
| 256 | | Acceptable | Acceptable | Acceptable |

NIST's security strength guidelines, from Specialist Publication SP 800-57
*Recommendation for Key Management – Part 1: General (Revision 3)*