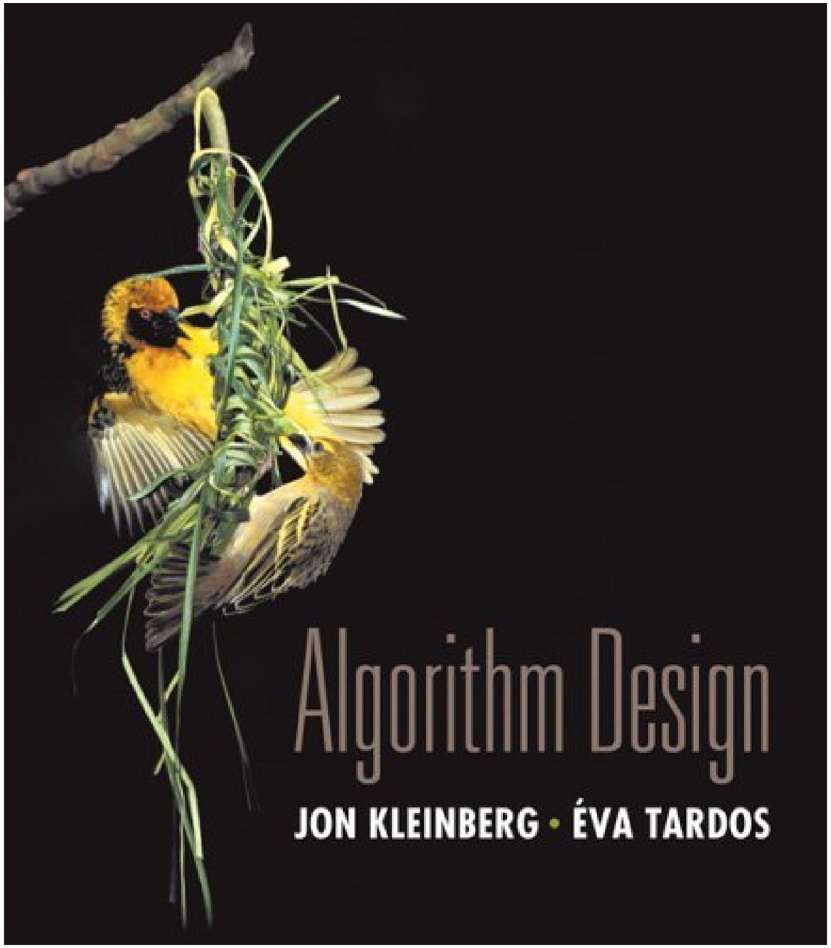
CS 381 – FALL 2019

Week 8.3, Friday, Oct 11

Homework 4 Due: October 14th @ 11:59PM (Gradescope)
Reminder: You must include a collaborator/resource (CR) statement for every problem.



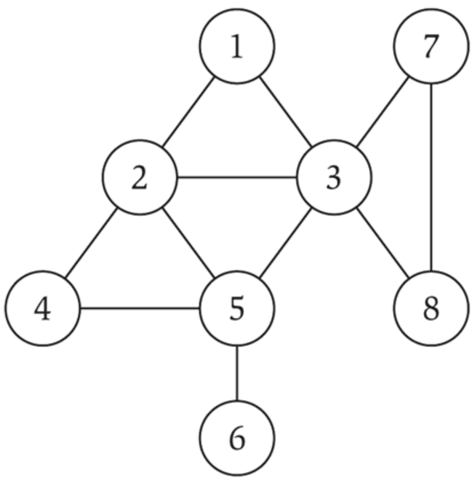
Graphs

3.1 Basic Definitions and Applications

Undirected Graphs

Undirected graph. G = (V, E)

- V = nodes.
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: n = |V|, m = |E|.



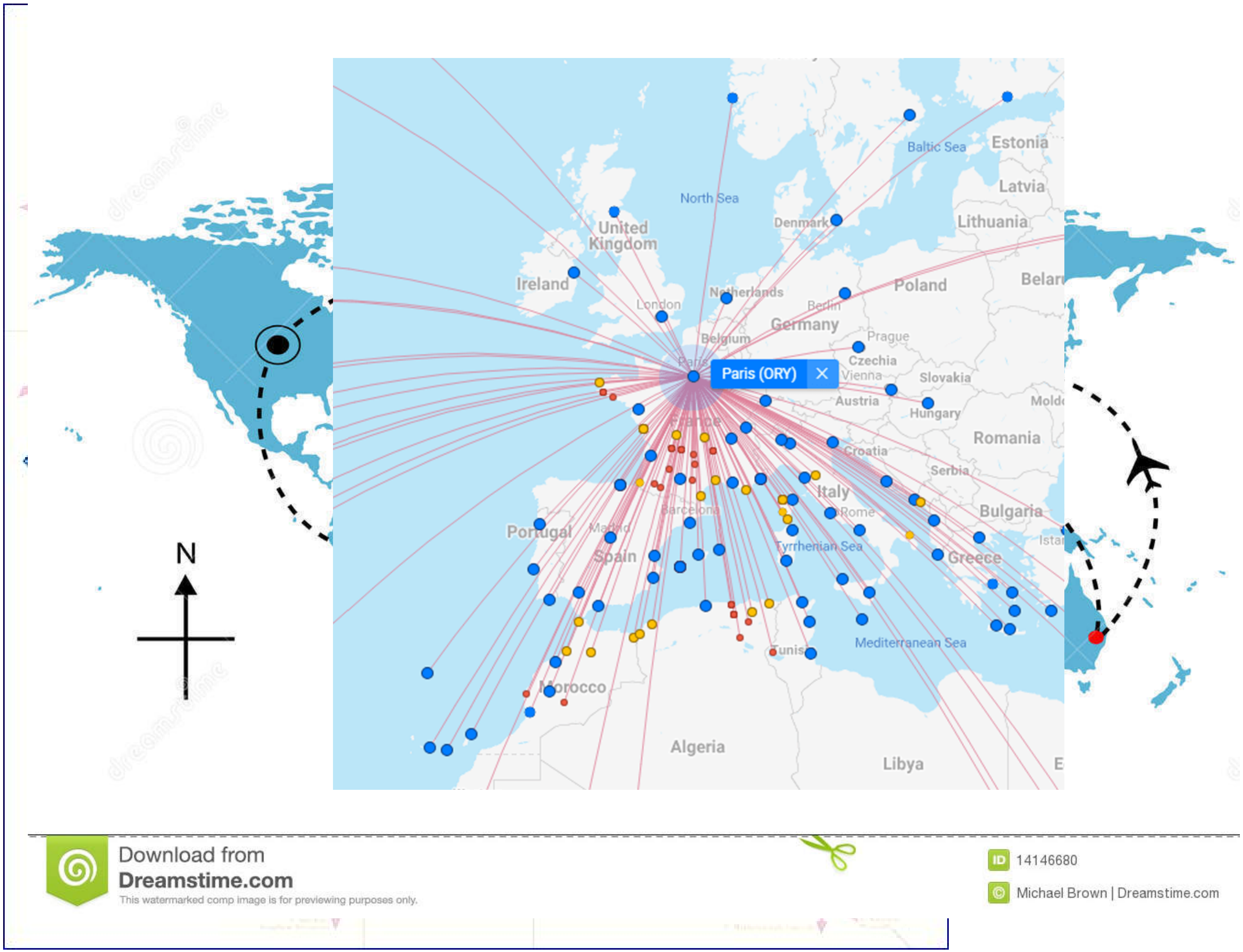
V = { 1, 2, 3, 4, 5, 6, 7, 8 }

E = { 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6 }

n = 8

m = 11

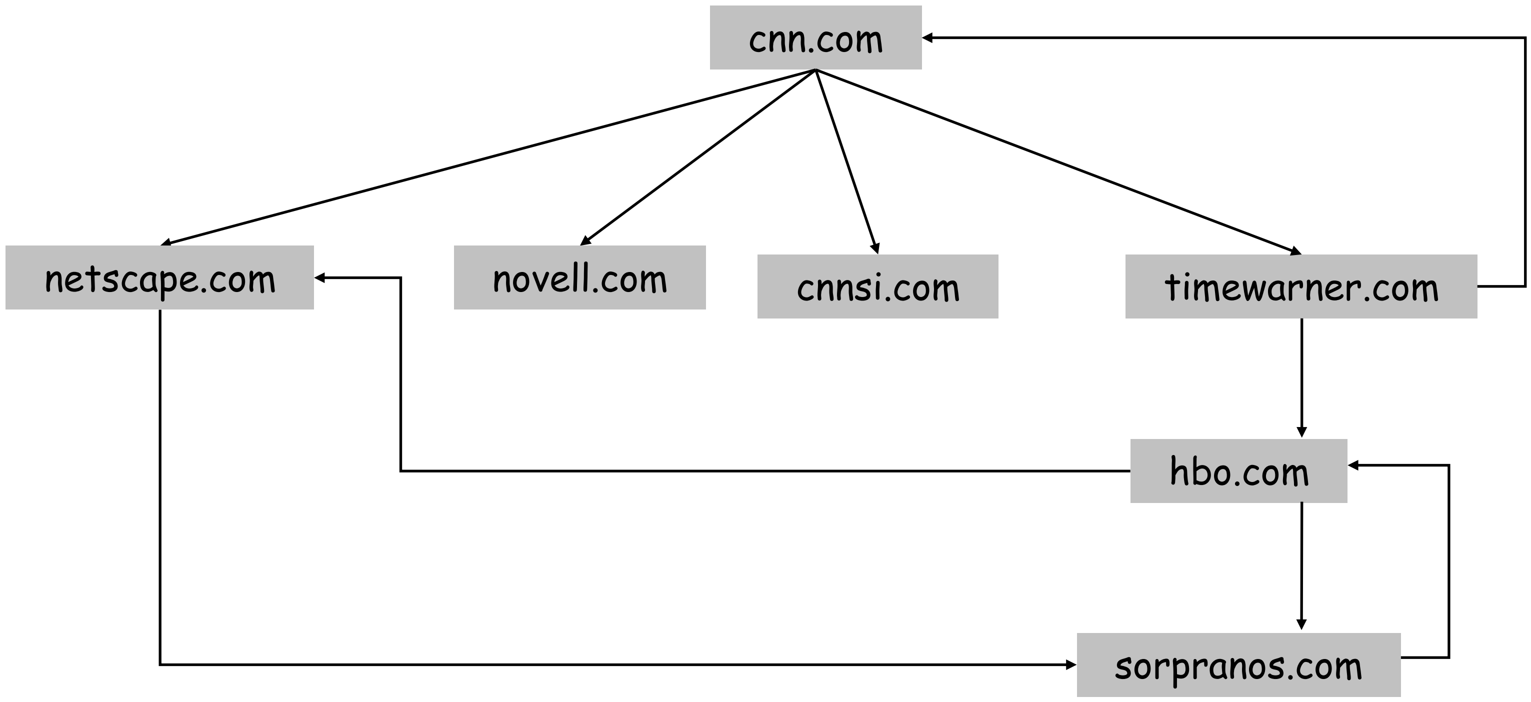
Transportation Networks (Planes, Trains, Highways etc…)



World Wide Web

Web graph.

- Node: web page.
- Edge: hyperlink from one page to another.



Social Network

9-11 Terrorist Network

Social network graph.

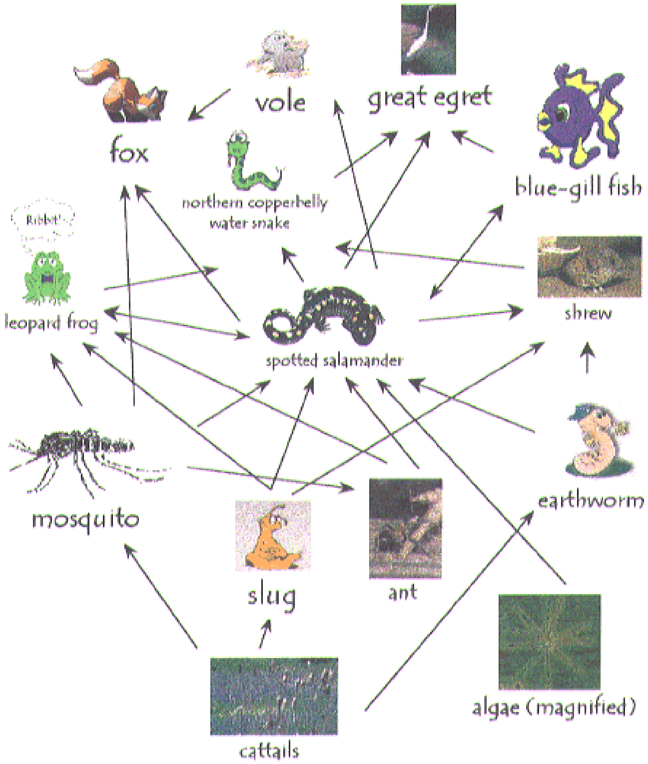
- Node: people.
- Edge: relationship between two people.

Reference: Valdis Krebs, http://www.firstmonday.org/issues/issue7_4/krebs

Ecological Food Web

Food web graph.

- Node = species.
- Edge = from prey to predator.

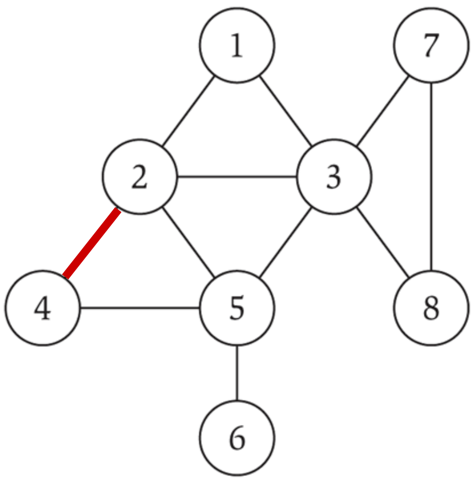


Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff

Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with A_{uv} = 1 if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



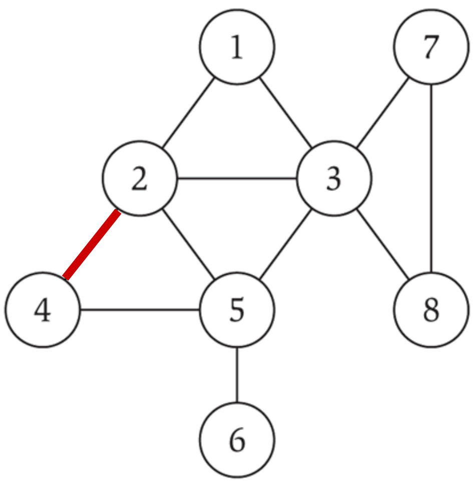
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

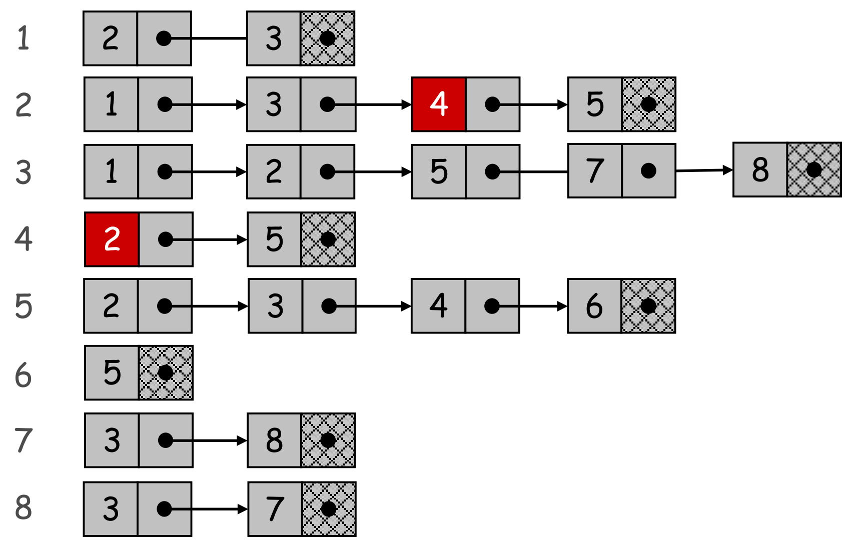
Graph Representation: Adjacency List

Adjacency list. Node indexed array of lists.

- Two representations of each edge.
- Space proportional to m + n.
- Checking if (u, v) is an edge takes O(deg(u)) time.
- Identifying all edges takes Θ(m + n) time.

degree = number of neighbors of u



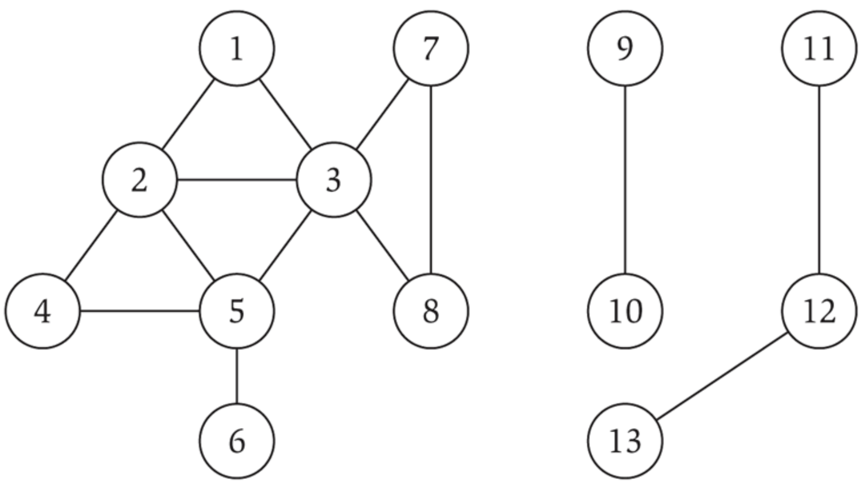


Paths and Connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.

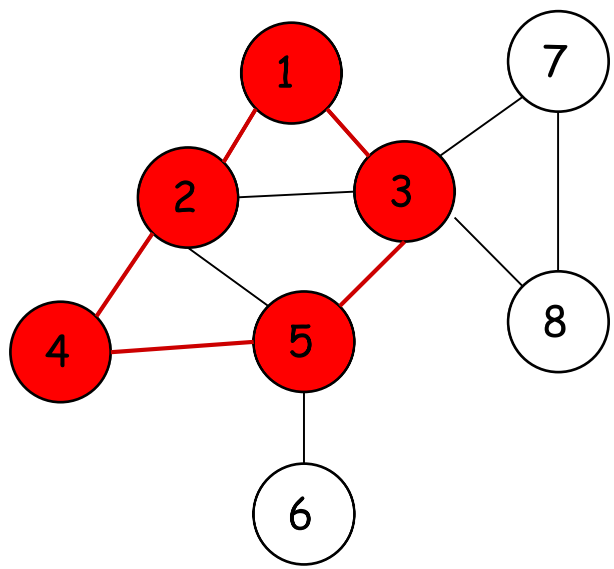
Def. A path is simple if all nodes are distinct (e.g., 1,2,5,6 below)

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first k-1 nodes are all distinct.



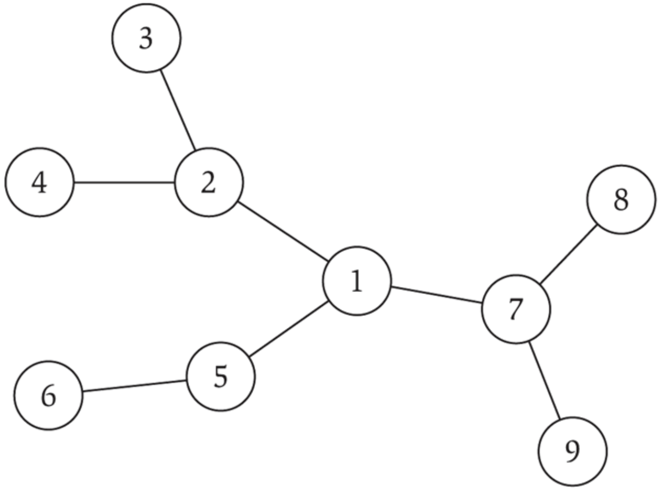
cycle C = 1-2-4-5-3-1

Trees

Def. An undirected graph is a **tree** if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

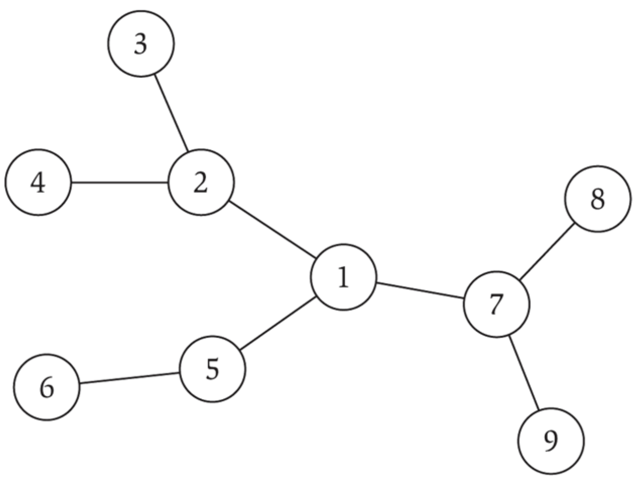
- G is connected.
- G does not contain a cycle.
- G has n-1 edges.



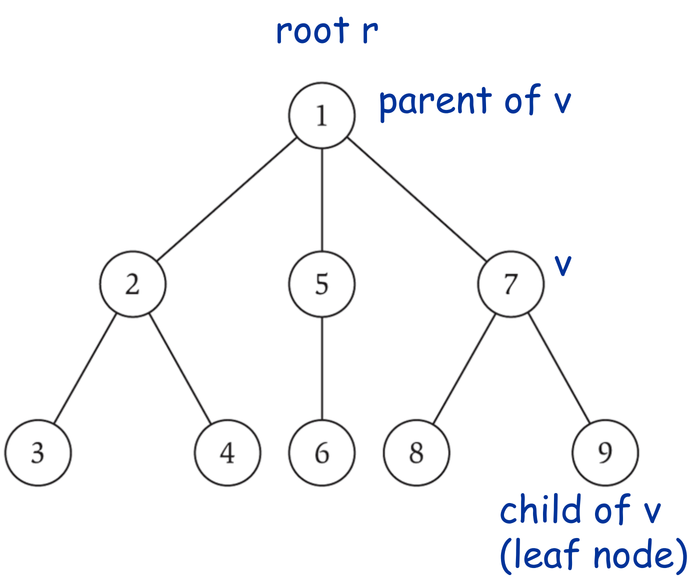
Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.

Importance. Models hierarchical structure.



a tree

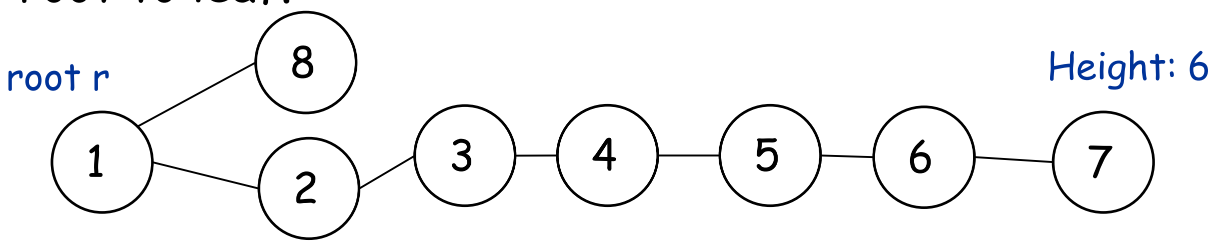


the same tree, rooted at 1

Binary Tree

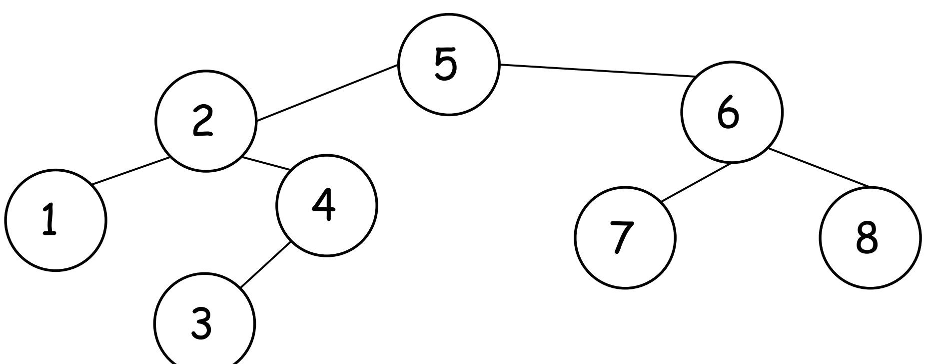
Def. A rooted tree in which each node has at most 2 children

Def. Height of a tree is the number of edges in the longest path from root to leaf.



Thm. Number of nodes in binary tree of height h is $n \le 2^{h+1} - 1 \ \ (= 2^0 + 2^1 + 2^2 + \cdots + 2^h)$.

Balanced Binary Tree. Height $h = O(\log n)$



iClicker Quiz

Let $G=(V,E)$ be an simple undirected graph. Which of the following claims are <u>false</u>?

A. If $|E| > n-1$ then G contains a cycle

B. If $|E|=n-1$ then G is a tree

C. If $|E| = n(n+1)/2$ then G is connected

D. If G is represented as an adjacency matrix then we can test whether or not a particular edge (u,v) exists in time $O(1)$

E. We can store G using $O(m+n)$ space using adjacency list representation.

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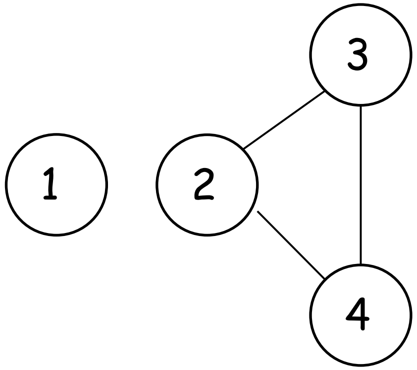
Let G=(V,E) be an simple undirected graph. Which of the following claims are false?



A. If |E|> n-1 then G contains a cycle

B. If |E|=n-1 then G is a tree

C. If |E| = n(n+1)/2 then G is connected
(G=Kn is complete graph & contains all possible edges)

D. If G is represented as an adjacency matrix then we can test whether or not a particular edge (u,v) exists in time O(1)

E. We can store G using O(m+n) space using adjacency list representation.

3.2 Graph Traversal

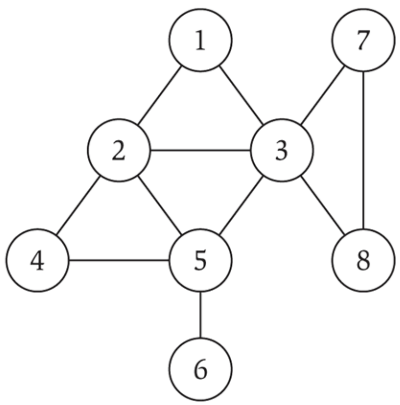
Connectivity

s-t connectivity problem. Given two node s and t, is there a path between s and t?

s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

Applications.

- Navigation (Google Maps).
- Maze traversal.
- Kevin Bacon number (or Erdős Number).
- Fewest number of hops in a communication network.



Breadth First Search: **BFS**(v, G=(V,E))

Input: Start node v, graph G (adjacency list) with n node V={1,…,n}
Output: Array Level[u] = distance from v to u

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 For each node u in V
    Explored[u]=0
    Level[u] = ∞          // No v→u path found yet
 Q.Enqueue(v)             // Queue Q (First In, First Out)
 Level[v]=0
 while(Q is not empty)
   u = Q.Dequeue()
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     foreach node w in u.AdjList
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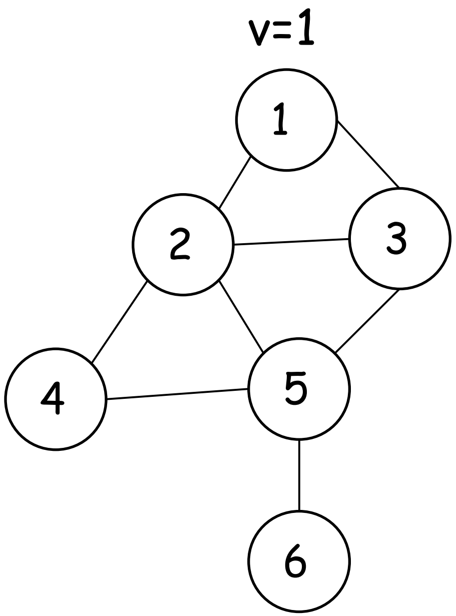

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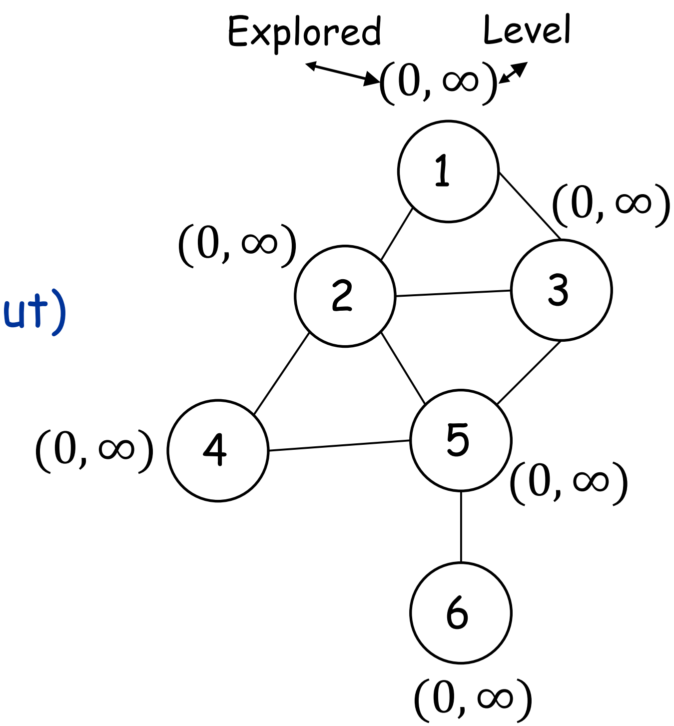

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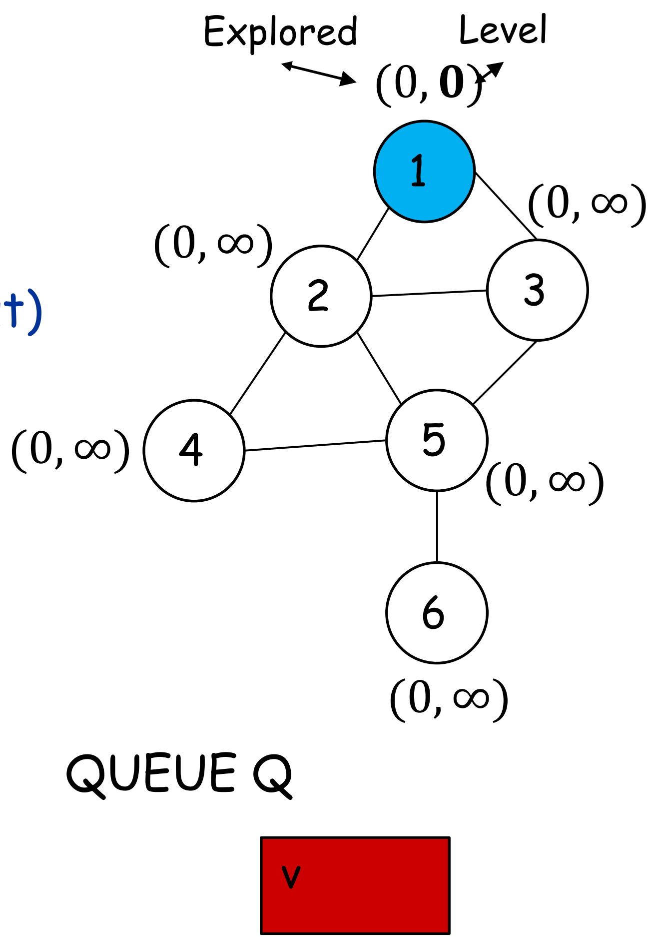

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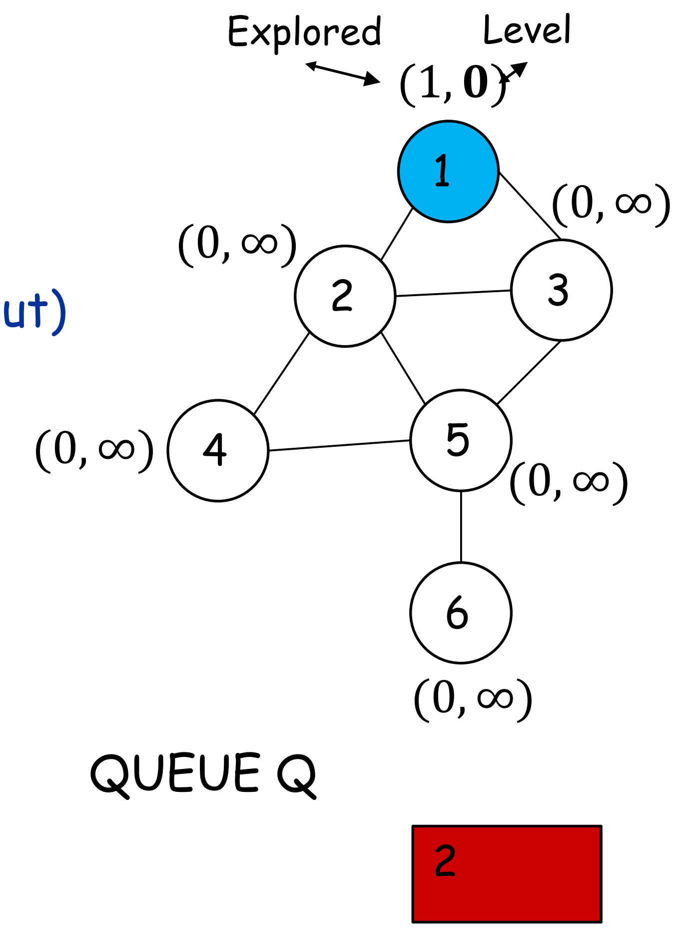

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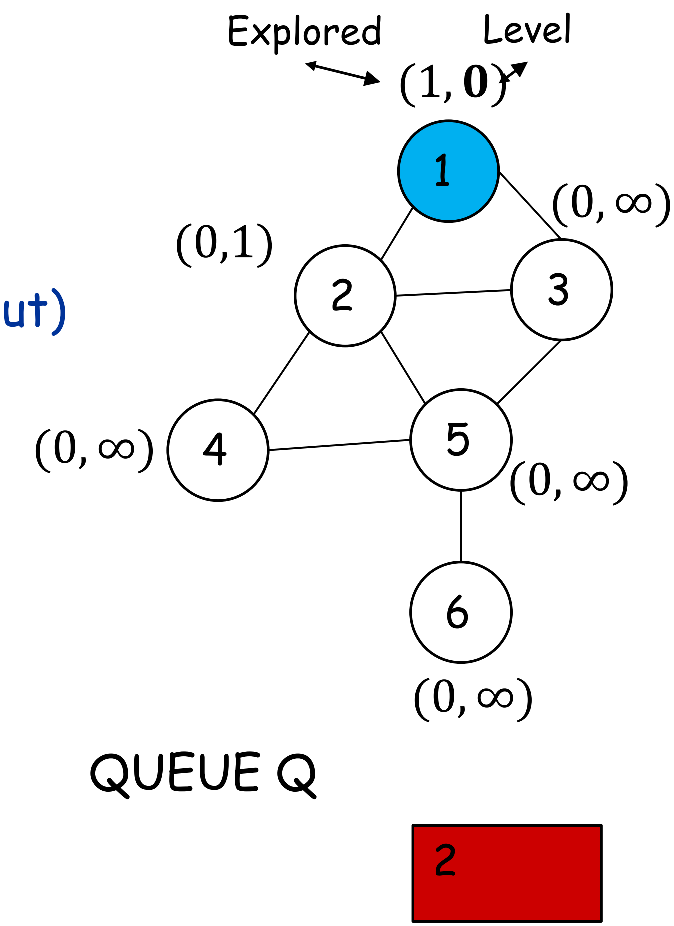

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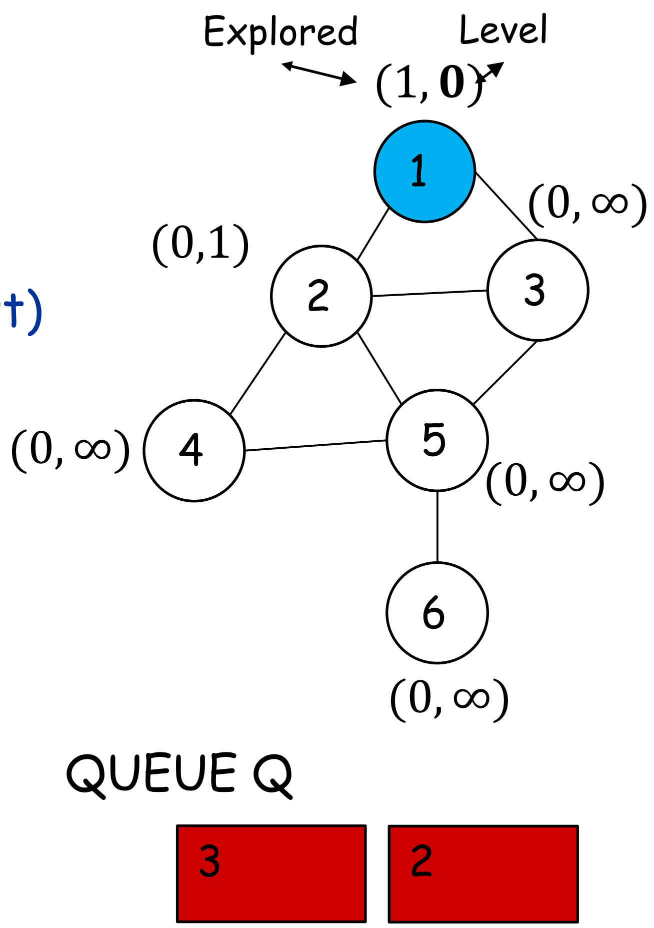

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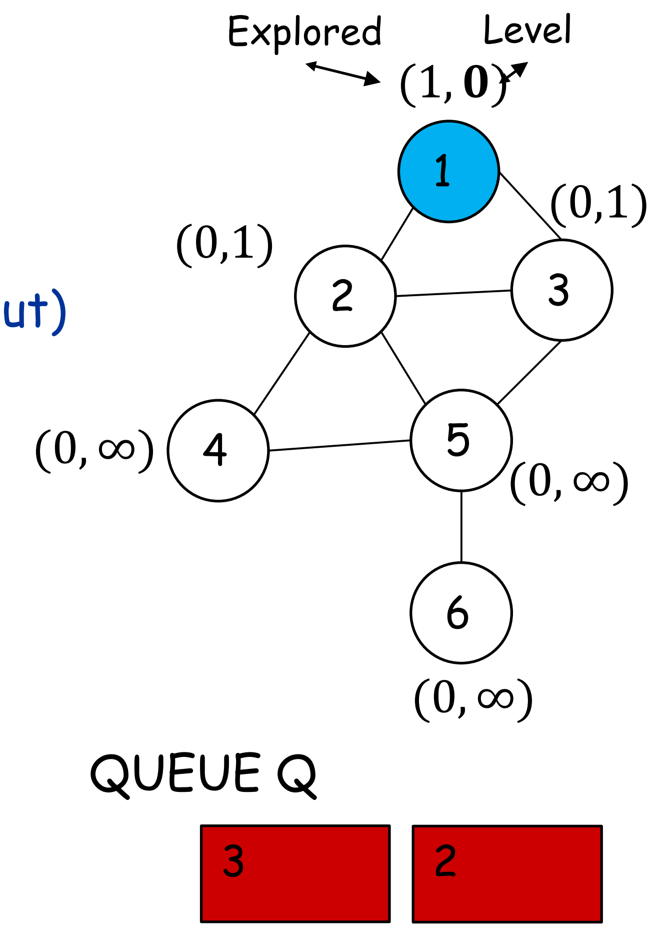

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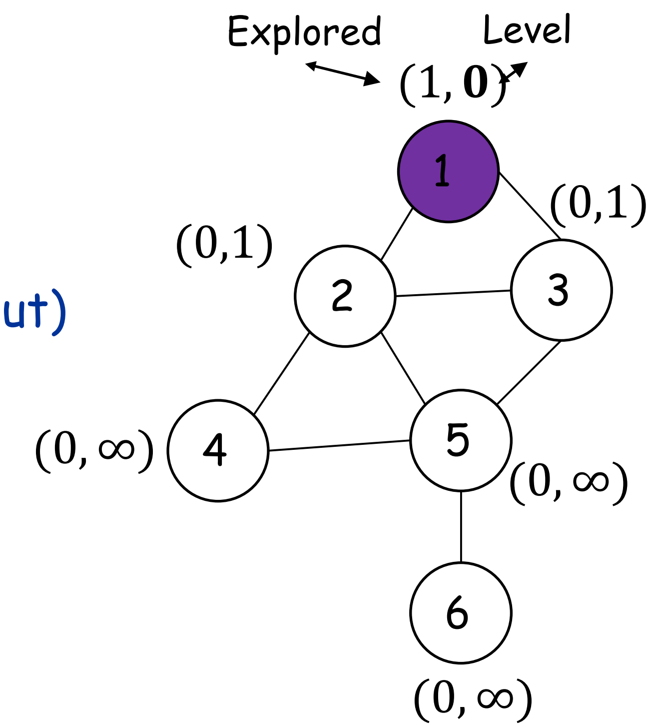
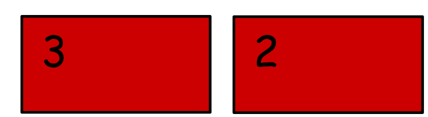


QUEUE Q



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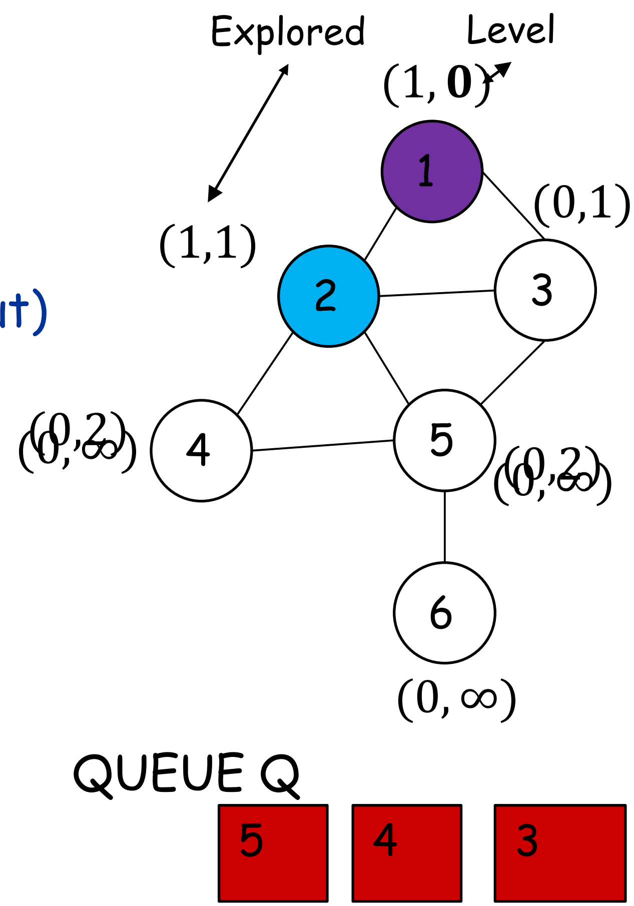

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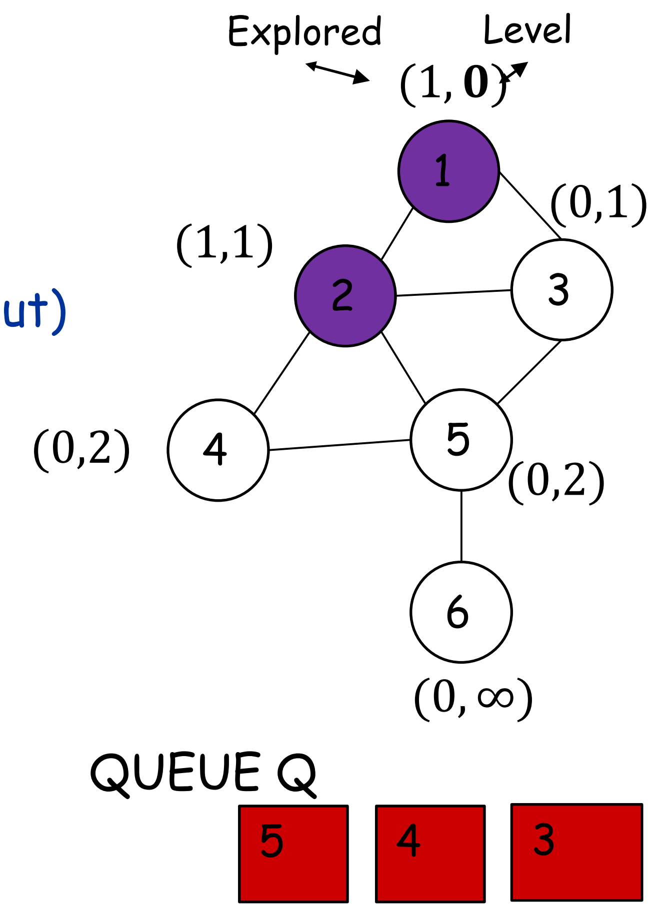

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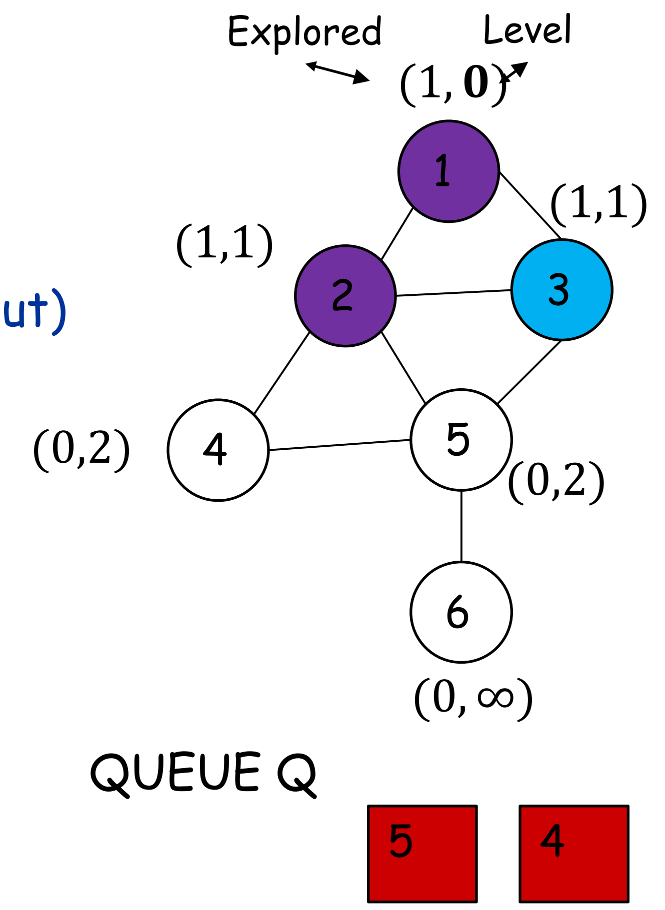

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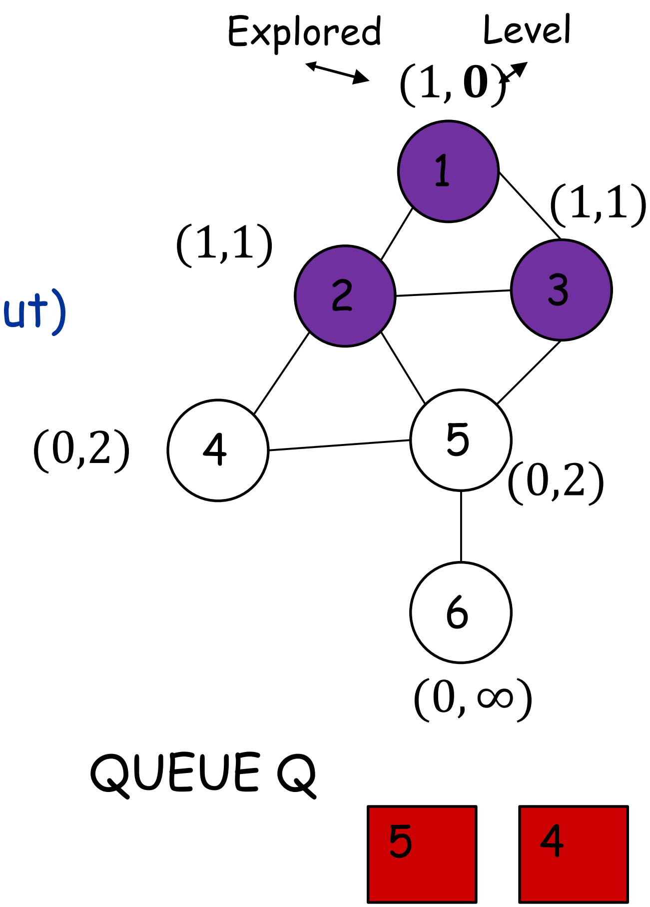

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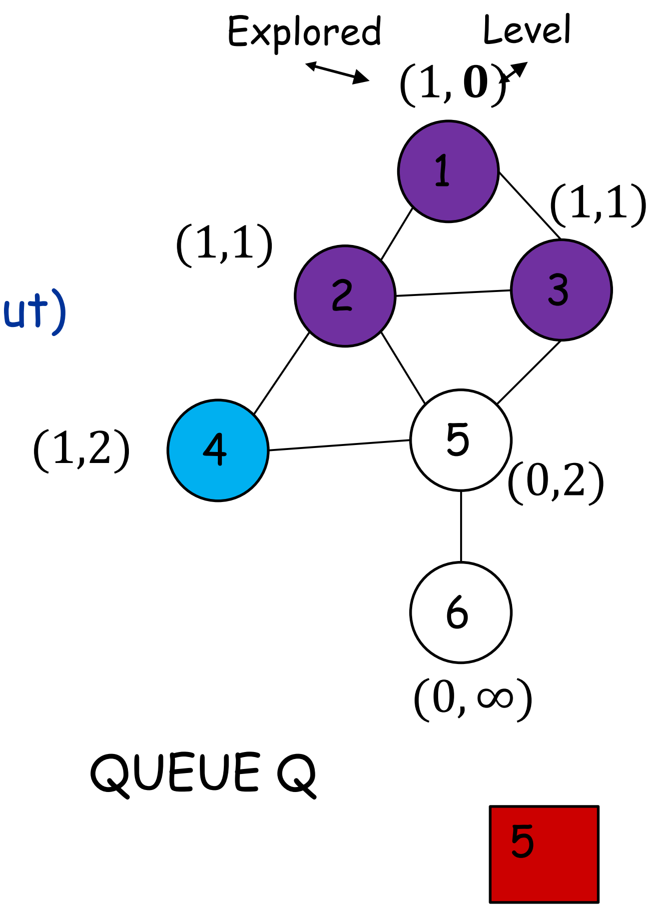

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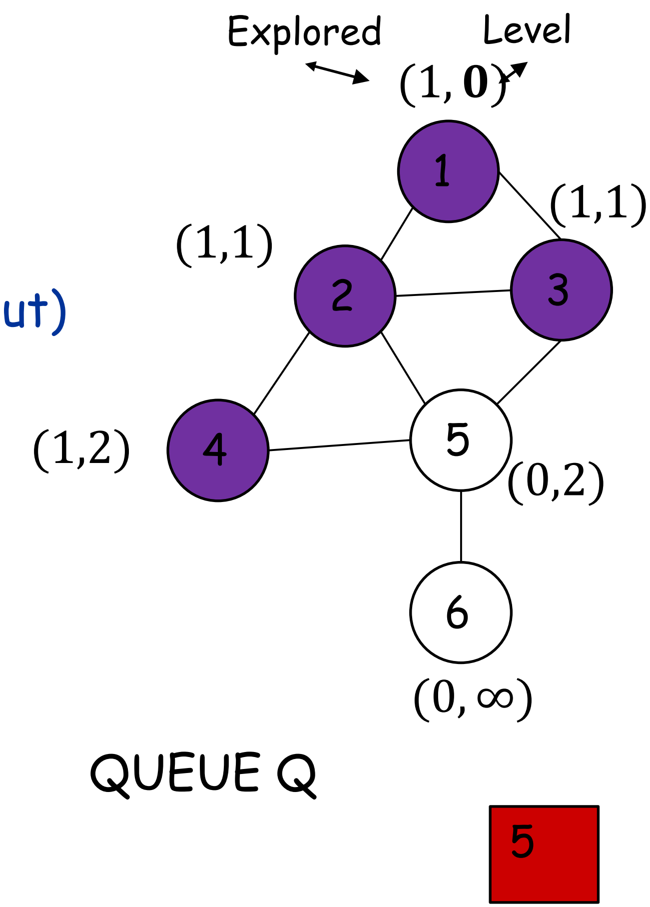

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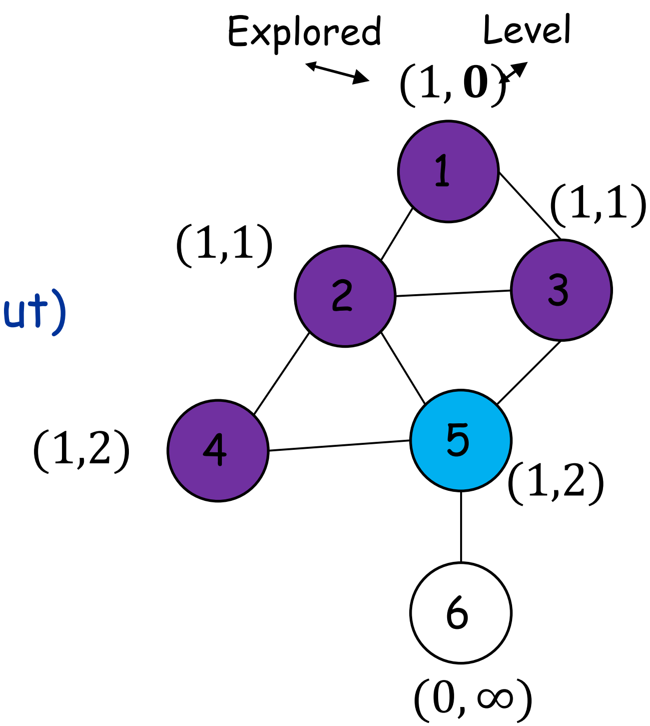
    end if
```



QUEUE Q

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Input: Start node v, graph G (adjacency list) with n node V={1,…,n}

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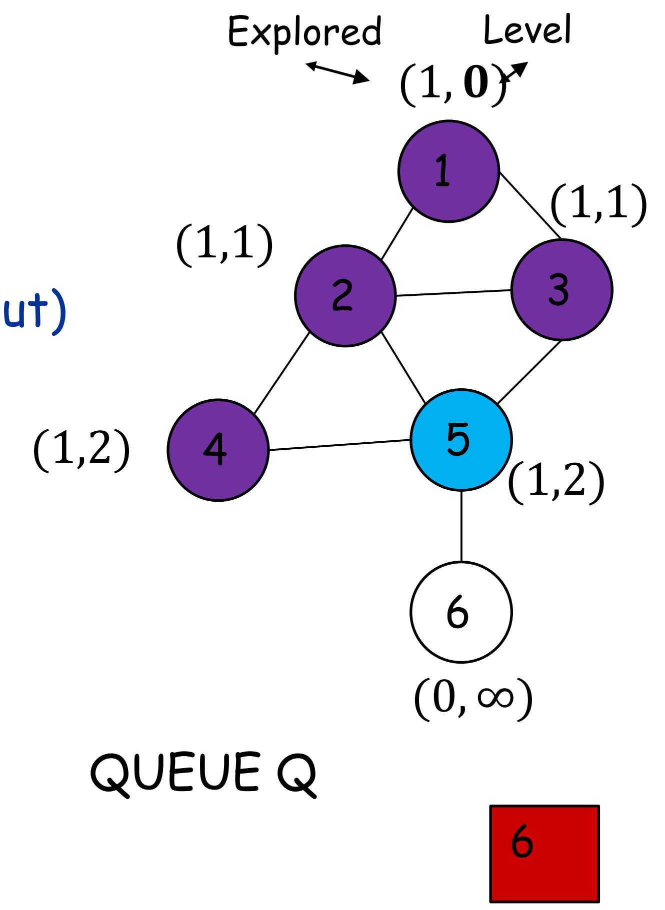

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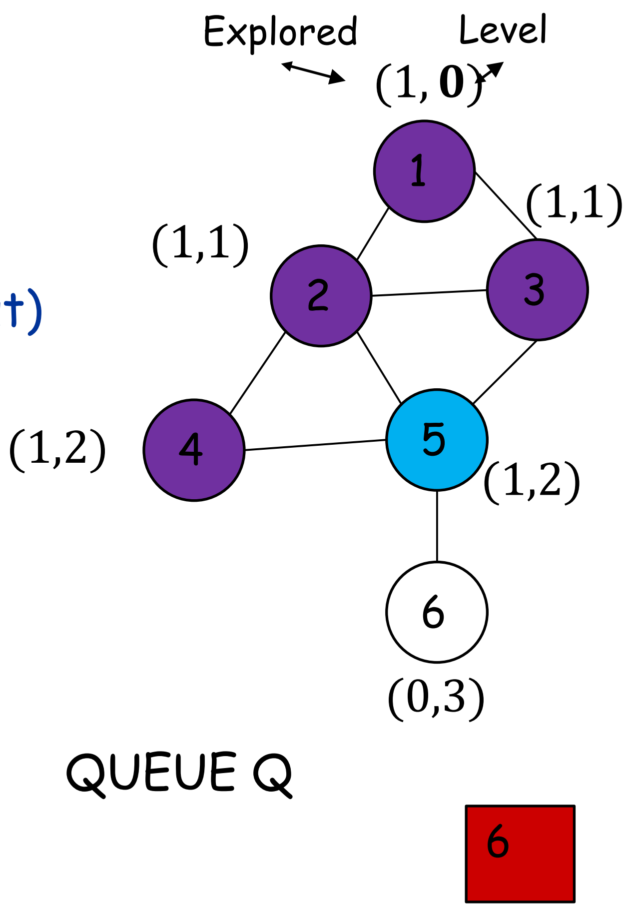

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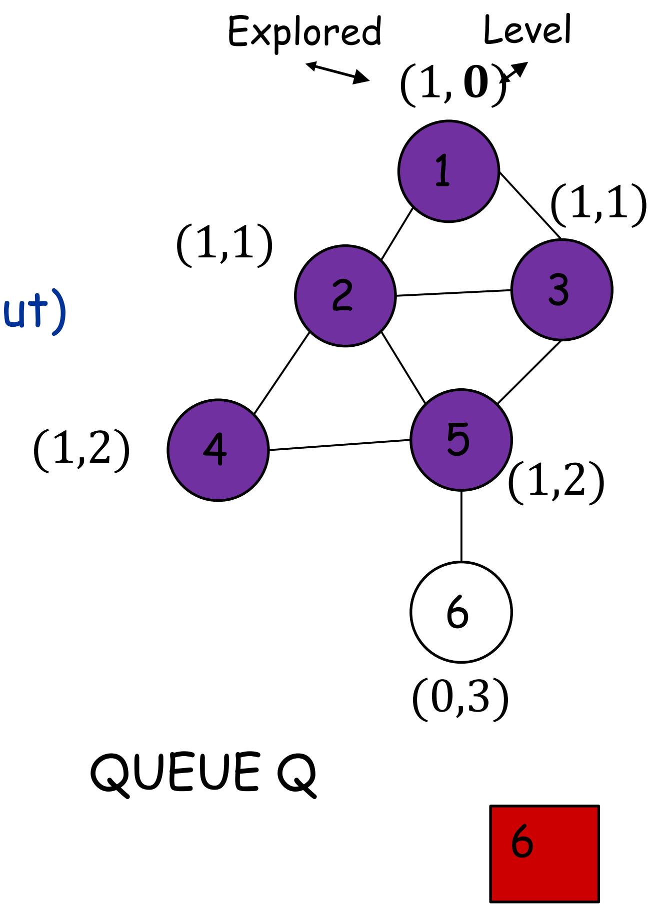

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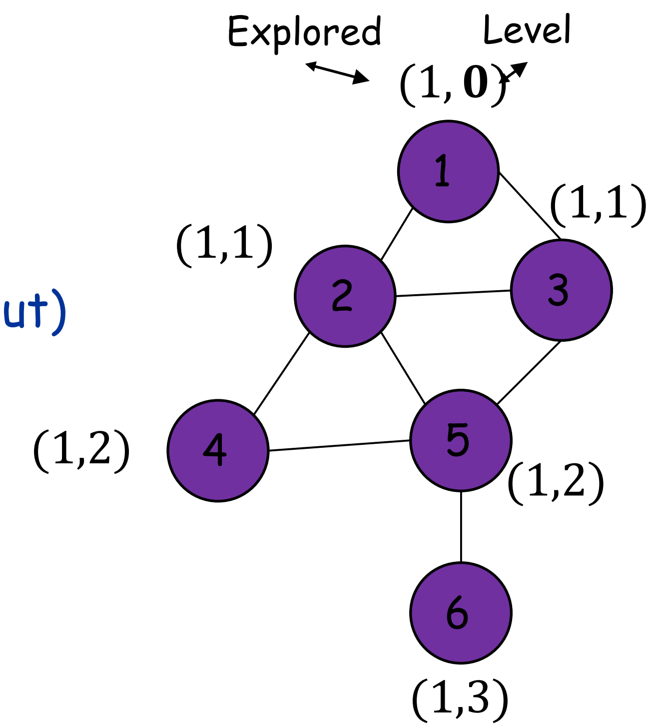
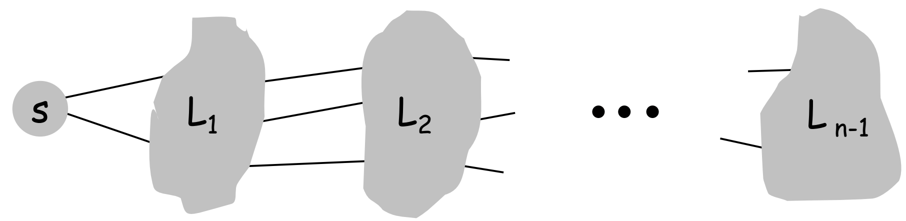


QUEUE Q

Done!

Breadth First Search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



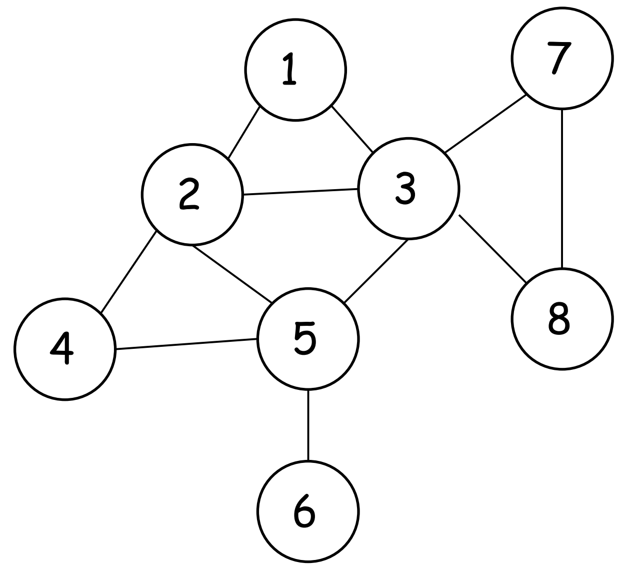
BFS algorithm.

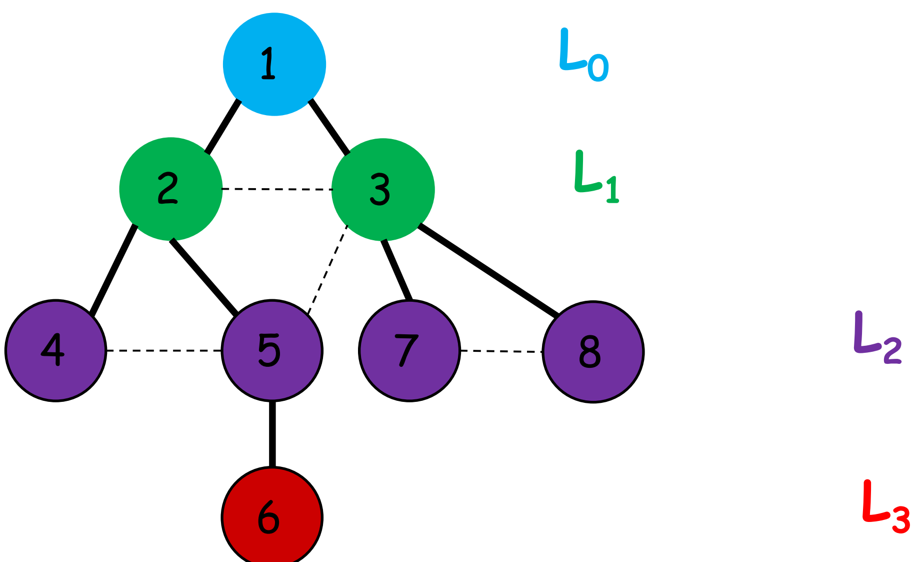
- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

Breadth First Search

Property. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G. Then the level of x and y differ by at most 1.





Breadth First Search: Analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency representation.

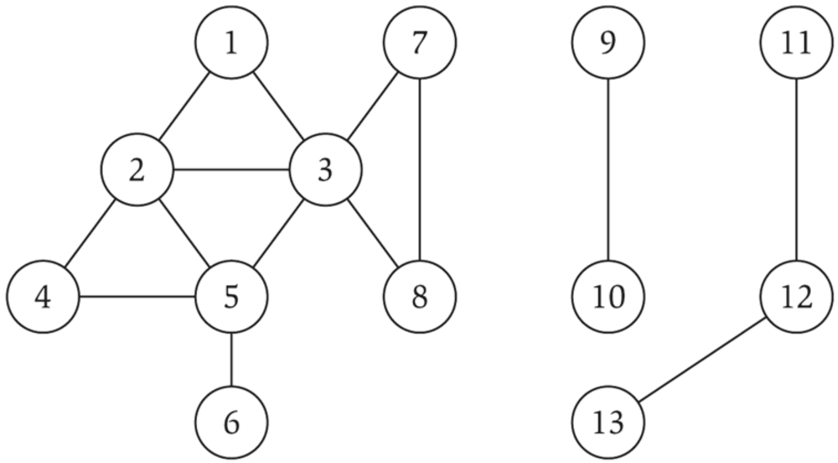
Pf.

- Easy to prove $O(n^2)$ running time:
  - at most n lists L[i]
  - each node occurs on at most one list; for loop runs $\leq$ n times
  - when we consider node u, there are $\leq$ n incident edges (u, v), and we spend $O(1)$ processing each edge
- Actually runs in $O(m + n)$ time:
  - when we consider node u, there are deg(u) incident edges (u,v)
  - total time processing edges is $\Sigma_{u \in V} \deg(u) = 2m$ ▪

each edge (u, v) is counted exactly twice
in sum: once in deg(u) and once in deg(v)

Connected Component

Connected component. Find all nodes reachable from s.



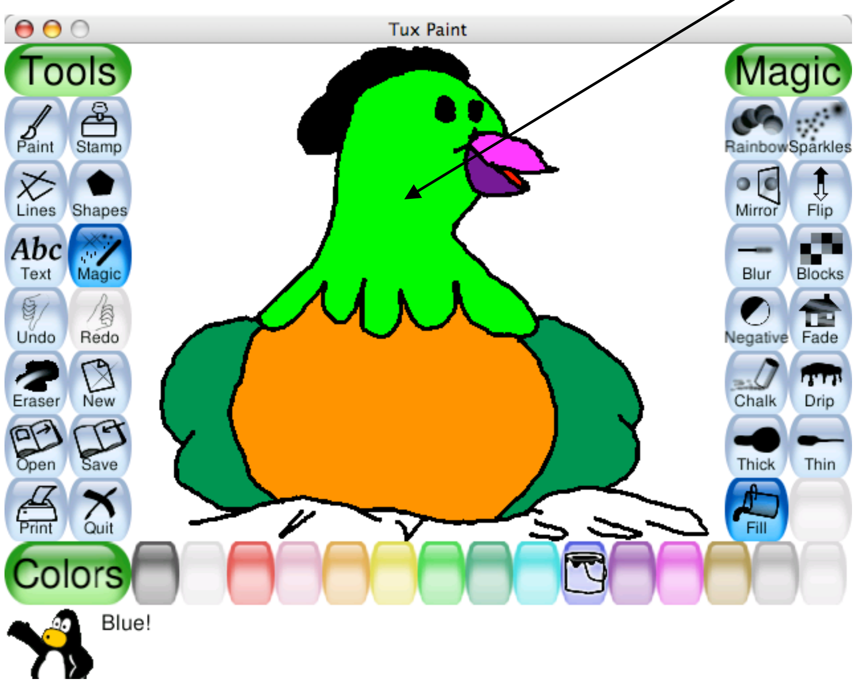
Connected component containing node 1 = { 1, 2, 3, 4, 5, 6, 7, 8 }.

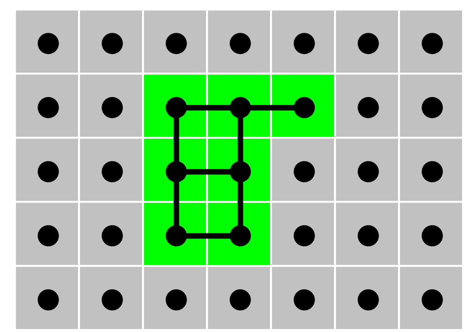
Flood Fill

Flood fill. Given lime green pixel in an image, change color of entire blob of neighboring lime pixels to blue.

- Node: pixel.
- Edge: two neighboring lime pixels.
- Blob: connected component of lime pixels.

recolor lime green blob to blue





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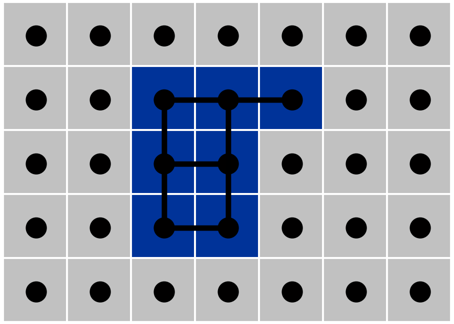
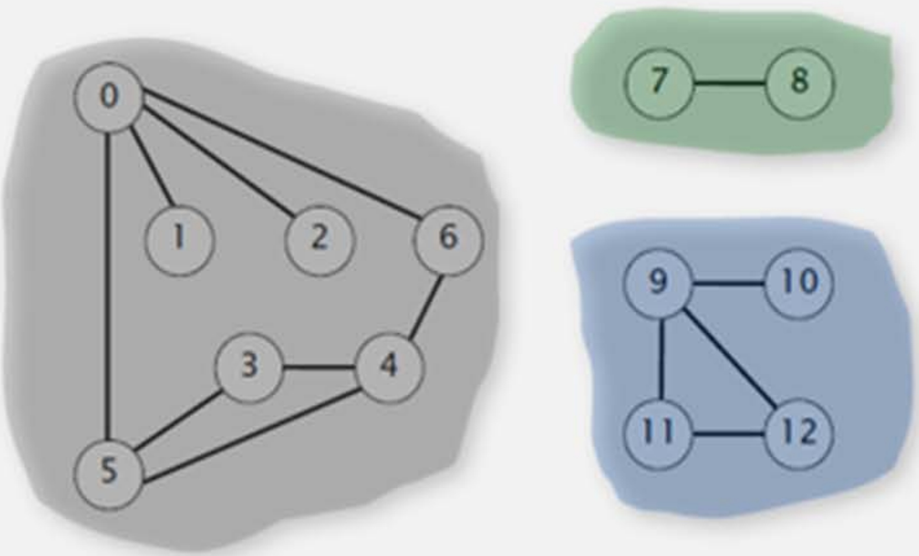


What problems can BFS solve?

- Check if an undirected graph is connected
- Determine *connected components* in an undirected graph
- Identify shortest path from source node to a destination node
- Is G a tree? (Does G have a cycle)

Finding connected components

Def. A connected component is a maximal set of connected vertices.



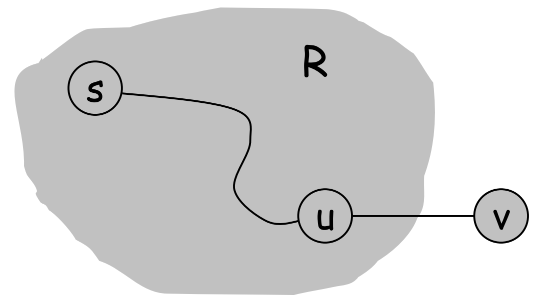
3 connected components

v	id[]
0	0
1	0
2	0
3	0
4	0
5	0
6	0
7	1
8	1
9	2
10	2
11	2
12	2

Sedgewick-Wayne 251 text

Connected component. Find all nodes reachable from s.

```
R will consist of nodes to which s has a path
Initially R = {s}
While there is an edge (u, v) where u ∈ R and v ∉ R
  Add v to R
Endwhile
```



it's safe to add v

Theorem. Upon termination, R is the connected component containing s.

- BFS = explore in order of distance from s.
- DFS = explore in a different way.

Challenge Puzzle

Suppose the shortest path from u to v has length t > n/2. Show that there exists some node $\mathbf{w} \neq \boldsymbol{u}, \boldsymbol{v}$ s.t. deleting w from G disconnects u and v.

Hint: Consider the BFS tree!

Solution: Run BFS(u) to obtain levels $L_0, L_1, \ldots$

Observation 1: v is in level L_t

Observation 2: $|L_0|+\ldots+|L_t| <= n$

Observation 3: Must have some $0 < k < t$ with $|L_k|=1$ by observation 2

Otherwise $|L_0|+|L_t|+(|L_1|+\ldots+|L_{t-1}|) \geq 2 + 2(t-1) > n$

Observation 4: Removing level L_k disconnects u and v

(BFS tree property → if $i < k < j$ then no edge connects L_i and L_j)

Depth First Search (22.3+22.5; 251 text)

- **DFS performs a recursive exploration of a graph**
  - DFS follows a path until it is forced to back up – “backtracking”
- DFS operates on an adjacency list representation
- Most uses of DFS result in O(n+m) time
- DFS be used on directed and undirected graphs
- **Undirected graph**
  - DFS partitions the edges into tree and back edges
  - Assigns numbers to the vertices during exploration (e.g. DFS number, discovery number, finish number)