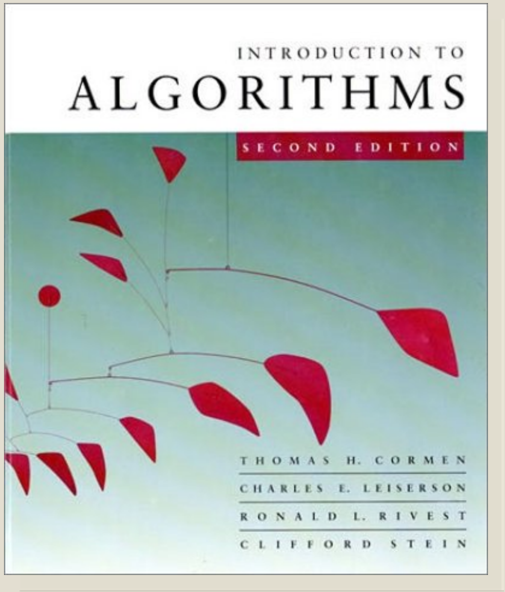
CS 381 – FALL 2019

Week 7.1, Monday, Sept 30

Tentative Date for Final Exam: Thursday, December 12
(7-9PM STEW 130)

Announcements

- Midterm grading in progress…



Dynamic Programming

- Longest Common Subsequence
- Optimal substructure
- Overlapping subproblems
- Sequence alignment (Edit Dist.)

Based on slides by Erik D. Demaine, Charles E. Leiserson and Kevin Wayne

Problem 4: ***Longest Common Subsequence (LCS)***

Longest Common Subsequence (LCS)

- Given two sequences $x[1\,.\,.\,m]$ and $y[1\,.\,.\,n]$, find a longest subsequence common to them both.

Dynamic programming

Example: ***Longest Common Subsequence (LCS)***

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Dynamic programming

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x: A B C B D A B

y: B D C A B A

BCBA = LCS(x, y)

functional notation, but not a function

Brute-force LCS algorithm

Check every subsequence of $x[1 \mathinner{.\,.} m]$ to see if it is also a subsequence of $y[1 \mathinner{.\,.} n]$.

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Analysis

- Checking $= O(n)$ time per subsequence.
- 2^m subsequences of x (each bit-vector of length m determines a distinct subsequence of x).

Worst-case running time $= O(n2^m)$

$= \text{exponential time.}$

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Notation: Denote the length of a sequence s by $|s|$.

Clicker Question

Longest Common Subsequence (LCS)

- Input: two sequences $x[1\,.\,.\,m]$ and $y[1\,.\,.\,n]$
- Output: length of longest common subsequence

Let OPT(k) be longest subsequence common to both $x[1\,.\,.\,k]$ and $y[1\,.\,.\,n]$,

Which statement is true?

A. OPT(k)= $OPT(k-1)+1$ if x[k] = y[n]
B. OPT(k)= $OPT(k-1)$ if x[k] ≠ y[n]
C. Both A and B are true
D. Neither claim is true
E. Don’t pick this choice it is incorrect! ☺

Clicker Question

Let OPT(k) be the (length of) longest subsequence common to both $x[1\,.\,.\,k]$ and $y[1\,.\,.\,n]$,

Which statement is true?

A. OPT(k)= $OPT(k-1)+1$ if x[k] = y[n]
Counterexample: x="aa", y="bbba" ➔Opt(2) = 1 and Opt(1) = 1
$\mathrm{Opt}(2) \neq \mathrm{Opt}(1) + 1$
B. OPT(k)= $OPT(k-1)$ if x[k] ≠ y[n]
Counterexample: x="bb", y="bba" ➔Opt(2) = 2 and Opt(1) = 1
$\mathrm{Opt}(2) \neq \mathrm{Opt}(1)$
C. Both A and B are true
D. Neither claim is true
E. Don't pick this choice it is incorrect! ☺

Stuck?

Let OPT(k) be (length of) longest subsequence common to both
$x[1..k]$ and $y[1..n]$

We can try to develop a recurrence for Opt(k) but we will fail!

Why? There are not enough sub-problems to develop a recurrence…
(Sub-problems always try to match $x[1..i]$ with *all* of $y[1..n]$)

Solution: Introduce more sub-problems!

Let OPT(i,j) be (length of) longest subsequence common to both
$x[1..i]$ and $y[1..j]$

Stuck?

Let OPT(i,j) be (length of) longest subsequence common to both $x[1..i]$ and $y[1..j]$

Case 1: x[i]=y[j]

➔ OPT(i,j) = 1+OPT(i-1,j-1)

Case 2: x[i]≠ *y*[j]

➔OPT(i,j) = **max{OPT(i-1,j), OPT(i,j-1)}**

Base Cases: OPT(0,j)=0 and OPT(i,0)=0

Towards a better algorithm

Simplification:

1. Look at the *length* of a longest-common subsequence.
2. Extend the algorithm to find the LCS itself.

Notation: Denote the length of a sequence s by $|s|$.

Strategy: Consider ***prefixes*** of x and y.

- Define $\text{OPT}[i, j] = |\text{LCS}(x[1\,.\,.\,i], y[1\,.\,.\,j])|$.
- Then, $\text{OPT}[m, n] = |\text{LCS}(x, y)|$.

Recursive formulation

Theorem.

$$\text{OPT}[i, j] = \begin{cases} \text{OPT}[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\left\{\text{OPT}[i-1, j], \text{OPT}[i, j-1]\right\} & \text{otherwise.} \end{cases}$$

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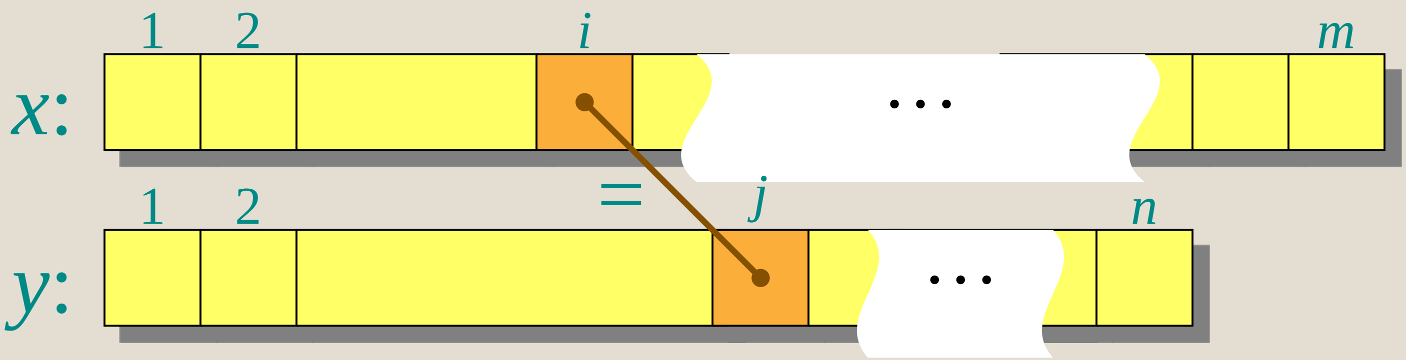
Proof. Case $x[i] = y[j]$:



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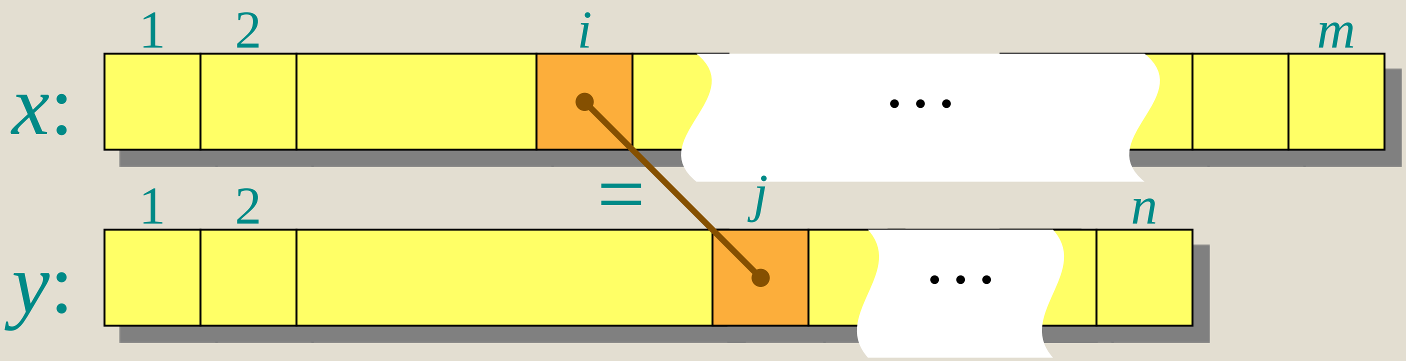

Let $z[1 \,.\,.\, k] = \mathrm{LCS}(x[1 \,.\,.\, i], y[1 \,.\,.\, j])$. Then, $z[k] = x[i]$, or else z could be extended. Thus, $z[1 \,.\,.\, k-1]$ is Common Substring of $x[1 \,.\,.\, i-1]$ and $y[1 \,.\,.\, j-1]$.

Proof (continued)

Claim: $z[1\,..\,k-1] = \mathrm{LCS}(x[1\,..\,i-1], y[1\,..\,j-1])$.

Suppose w is a longer CS of $x[1\,..\,i-1]$ and $y[1\,..\,j-1]$, that is, $|w| > k-1$. Then, ***cut and paste***: $w \,\|\, z[k]$ (w concatenated with $z[k]$) is a common subsequence of $x[1\,..\,i]$ and $y[1\,..\,j]$ with $\big|w \,\|\, z[k]\big| > k$. Contradiction, proving the claim.

Proof (continued)

Claim 1: $z[1\ .\ .\ k-1] = \mathrm{LCS}(x[1\ .\ .\ i-1], y[1\ .\ .\ j-1])$.

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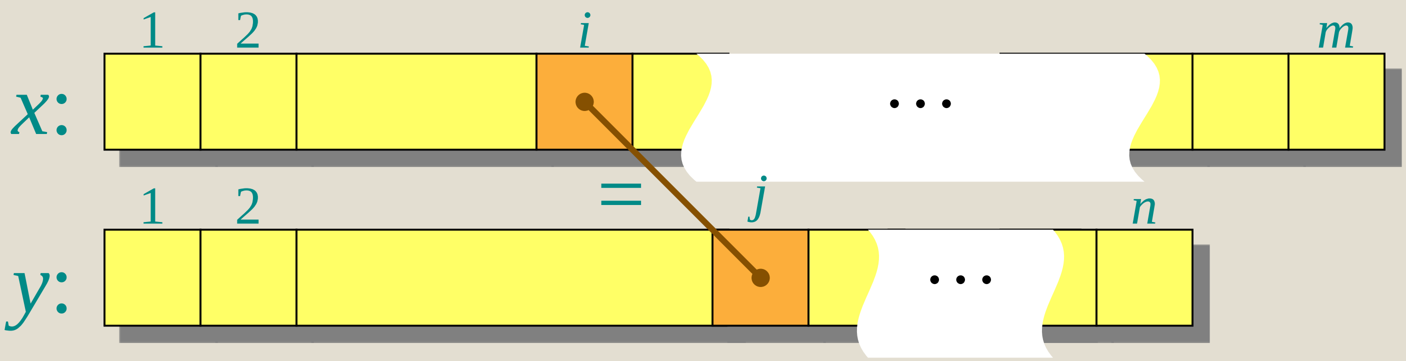
Thus, if $x[i] = y[j]$, we have $\mathrm{OPT}[i-1, j-1] = k-1$, which implies that $\mathrm{OPT}[i, j] = \mathrm{OPT}[i-1, j-1] + 1$.

Recursive formulation

Theorem.

$$\mathrm{OPT}[i, j] = \begin{cases} \mathrm{OPT}[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{\mathrm{OPT}[i-1, j], \mathrm{OPT}[i, j-1]\} & \text{otherwise.} \end{cases}$$

Proof. Case $x[i] \neq y[j]$:



Let $z[1 \,..\, k] = \mathrm{LCS}(x[1 \,..\, i], y[1 \,..\, j])$. If $z[k] \neq y[j]$ then $z[1 \,..\, k]$ is a Common Substring of $x[1 \,..\, i]$ and $y[1 \,..\, j-1]$.

Proof (continued)

Claim 3: $z[1\,.\,.\,k] = \mathrm{LCS}(x[1\,.\,.\,i], y[1\,.\,.\,j–1])$.

Suppose w is a longer CS of $x[1\,.\,.\,i]$ and $y[1\,.\,.\,j–1]$, that is, $|w| > k$. Then w is a common subsequence of $x[1\,.\,.\,i]$ and $y[1\,.\,.\,j]$ with $|w| > k$. Contradiction, proving the claim. $\square$

Thus, $\mathrm{OPT}[i, j–1] = k$, which implies that $\mathrm{OPT}[i, j] = \mathrm{OPT}[i, j–1]$. Similarly, if $z[k] = y[j]$, then $z[k] \neq x[i]$ and

$z = \mathrm{LCS}(x[1\,.\,.\,i\text{-}1], y[1\,.\,.\,j])$. It follows that

$$\mathrm{OPT}[i, j] = \max\{\mathrm{OPT}[i–1, j], \mathrm{OPT}[i, j–1]\}$$ $\square$

Finding the LCS

Base Case: OPT[0,j] =OPT[i,0]= 0

$$\text{OPT}[i,j] = \begin{cases} \text{OPT}[i-1, j-1] + 1 & \text{if } x[i] = y[j], \\ \max\{\text{OPT}[i-1, j], \text{OPT}[i, j-1]\} & \text{otherwise.} \end{cases}$$

Step 1: Fill in the DP table and compute OPT[i, j] for all $i \le m$ and $j \le n$ using above recurrence

Step 2: Backtrack to construct a common subsequence z of length k=OPT[m,n]

- If OPT[m,n]=OPT[m-1, n–1] + 1 then set z[k] = x[m] and recurse with x[1,…,m-1], y[1,….,n-1]
- If OPT[m,n]= OPT[m-1, n] then recurse with x[1,…,m-1], y[1,…,n]
- If OPT[m,n]= OPT[m, n-1] then recurse with x[1,…,m], y[1,…,n-1]

Dynamic-programming hallmark #1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

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An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If $z = \text{LCS}(x, y)$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

6.4 Knapsack Problem

Knapsack Problem

Knapsack problem.

- Given n objects and a "knapsack."
- Item i weighs $w_i > 0$ kilograms and has value $v_i > 0$.
- Knapsack has capacity of W kilograms.
- Goal: fill knapsack so as to maximize total value.

Ex: { 3, 4 } has value 40.

#	value	weight
1	1	1
2	6	2
3	18	5
4	22	6
5	28	7

W = 11

Greedy: repeatedly add item with maximum ratio v_i / w_i.
Ex: { 5, 2, 1 } achieves only value = 35 $\Rightarrow$ greedy not optimal.

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```
W = 4-2
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```

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Dynamic Programming: False Start

Def. OPT(i) = max profit subset of items 1, …, i.

- Case 1: OPT does not select item i.
  - OPT selects best of { 1, 2, …, i-1 }
- Case 2: OPT selects item i.
  - accepting item i does not immediately imply that we will have to reject other items
  - without knowing what other items were selected before i, we don't even know if we have enough room for i

Conclusion. Need more sub-problems!