CS 381 – FALL 2019

Week 2.1, Monday, August 26

Homework 1 available on course web page
(**Due:** September 3 at 11:59PM on Gradescope)

Office Hours (Piazza)

- Monday
  - 9-10:30 (Hai Nguyen --- HAAS G050)
  - 2:30-3:30 PM (Prof. Blocki --- LWSN 1165)
  - 3:30-4:30 PM (Utkarsh Jain --- HAAS G050)
- Tuesday
  - 8-9:30AM (Noah Franks --- HAAS G050)
  - 11:45AM-1:15PM (Himanshi Mehta --- HAAS G050)
- Wednesday
  - 9:30 -11AM (Mike Cinkoske --- HAAS G050)
  - 11:30AM-1PM (Kevin Xia --- HAAS G050)
  - 2:30-3:30PM (Prof. Blocki --- LWSN 1165)
  - 4-5:30PM (Ahammed Ullah --- HAAS G050)

Office Hours (Piazza)

- Thursday
  - 8-9:30AM (Noah Franks --- HAAS G050)
  - 11:45-1:15AM (Himanshi Mehta --- HAAS G050)
- Friday
  - 10-11:30AM (Abhishek Sharma --- LWSN 3rd floor lobby*)
  - 2:30-3:30PM (Hiten Rathod --- HAAS G050)
  - 5-6:30PM (Tunaz Islam --- HAAS G050)

* If available; otherwise HAAS G050

What do we count?

- Time and space
  - time in terms of number of basic operations on basic data types
- Ignore machine dependent factors, but remain realistic
- Random Access Model (RAM)
  - no concurrency
  - count instructions (arithmetic operation, comparison, data movement)
  - each instruction takes constant time
  - realistic assumption on the size of the numbers (to represent n, it takes log n bits)

Asymptotic notation: Big-O

$O(g(n)) = \{ f(n) \mid$ there exist positive constants c and n_0 such that $0 \le f(n) \le c\, g(n)$ or all $n \ge n_0\}$

We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

$4n + 23\log n - 28 = O(n)$

- Drops low-order terms
- Ignores leading constants
- May not hold for small values of n

$f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$

such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

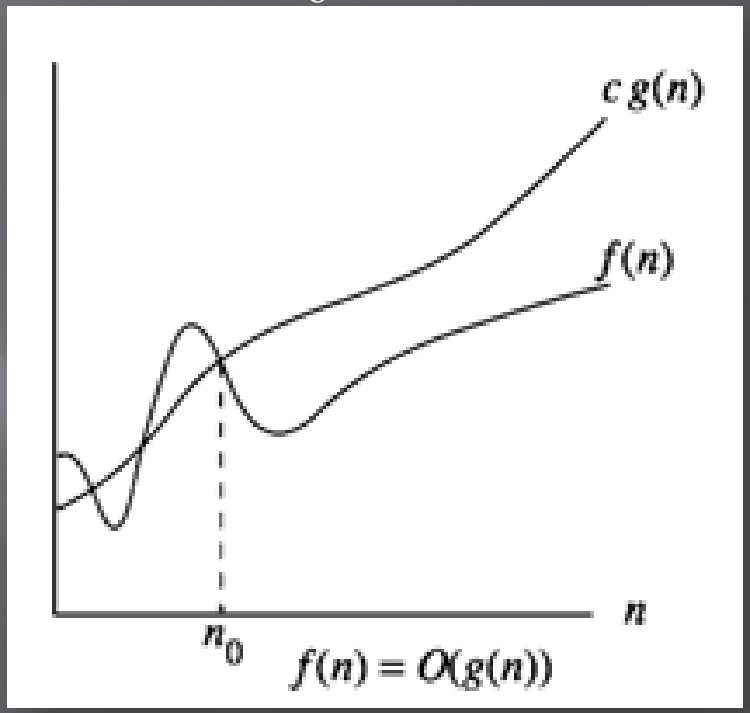
$f(n) = 3n^2 - 4n + 512$

$\le 3n^2 + 512$

$\le 4n^2$ for $n \ge 23$

(set $c=4$, $n_0 = 23$)

- $f(n) = O(n^2)$
- $f(n) = O(n^3)$ also holds
- $f(n) = O(n)$ is false



CLRS text Figure 3.1

Which statements are true?

$3n^3 + 90n^2 - 5n = O(n^3)$

$3n^3 + 90n^2 - 5n = O(2^n)$

$3n^3 + 90n^2 - 5n = O(n^2)$

$5 \log n = O(n)$

$\sqrt{n} = O(\log n^8)$

$n \log n = O(n)$

$4n = O(n \log n)$

$n / \log n = O(\sqrt{n})$

Which statements are true?

$3n^3 + 90n^2 - 5n = O(n^3)$	true
$3n^3 + 90n^2 - 5n = O(2^n)$	true
$3n^3 + 90n^2 - 5n = O(n^2)$	false
$5 \log n = O(n)$	true
$\sqrt{n} = O(\log n^8)$	false
$n \log n = O(n)$	false
$4n = O(n \log n)$	true
$n/\log n = O(\sqrt{n})$	false

Consider two running times: $4n\log n$ and $8nn^{1/8}$

Which relationships hold?

1. $4n\log n = O(8nn^{1/8})$
2. $8nn^{1/8} = O(4n\log n)$
3. $4n\log n = \Theta(8nn^{1/8})$
4. $8nn^{1/8} = \Theta(4n\log n)$

A. None
B. 1
C. 2
D. 1 and 3
E. 4

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4. $8nn^{1/8} = \Theta(4n\log n)$

A. None
B. 1
C. 2
D. 1 and 3
E. 4

Clicker: Question 2

Suppose that f,g, and h are positive functions (i.e., $f(n), g(n), h(n) \geq 1$ for all $n \geq 1$). Which of the following claims are necessarily true?

1. f(n)=O(g(n)) → f(n) = O(f(n)+g(n))
2. f(n) = O(g(n)) and g(n) = O(h(n))→ f(n) = O(h(n))
3. f(n) = O(f(n/2))

A. All of the above
B. 1
C. 2
D. 1 and 3
E. 1 and 2

Clicker: Question 2

Suppose that f,g, and h are positive functions (i.e., $f(n), g(n), h(n) \geq 1$ for all $n \geq 1$). Which of the following claims are necessarily true?

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3. f(n) = O(f(n/2))

A. All of the above
B. 1
C. 2
D. 1 and 3
E. 1 and 2

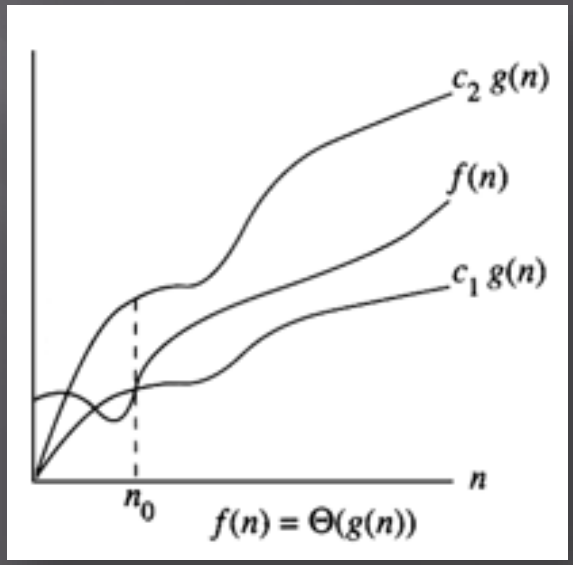
Asymptotic Bounds

$O(g(n)) = \{ f(n) \mid$ there exist positive constants c and n_0 such that $0 \leq f(n) \leq c\, g(n)$ for all $n \geq n_0\}$

O captures upper bounds

$\Theta(g(n)) = \{ f(n) \mid$ there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1\, g(n) \leq f(n) \leq c_2\, g(n)$ for all $n \geq n_0\}$

Θ captures upper and lower bounds



Examples

- $3n^3 + 90n^2 – 5n$ is $O(n^3)$ and $\Theta(n^3)$ is true
- $3n^3 + 90n^2 – 5n$ is $O(2^n)$ true, but $\Theta(2^n)$ false
- 5 log n is O(n) true, but $\Theta(n)$ false
- 4n = O(n log n) is true, but $\Theta(n \log n)$ false

Asymptotic Bounds

$O(g(n)) = \{ f(n) \mid$ there exist positive constants c and n_0 such that $0 \le f(n) \le c\, g(n)$ for all $n \ge n_0\}$

O captures upper bounds

$\Theta(g(n)) = \{ f(n) \mid$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0\}$

Θ captures upper and lower bounds

$\Omega(g(n)) = \{ f(n) \mid$ there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$

Ω captures lower bounds

$4n \log n = \Omega(n)$

Note

- We will generally assume that n is "nice"
  - E.g., power of 2
  - We are not implementing the algorithms and only need to consider crucial the boundary/special cases
- When asked to design an efficient algorithm
  - sometimes you will be given a target asymptotic bound
  - other times you need to find the "best" one
- You can use known data structures
  - State how they are implemented and give time bounds of operations

How many times is F called?

Assume n is a power of 4 ($n=4^k$)

```
while n > 1 do
    for i = 1 to n do
        F(i,n)
    n = n/4
```

$O(n \log n)$ $\Theta(n \log n)$

$O(n^2)$ $\Theta(n^2)$

$O(n)$ $\Theta(n)$

$O(\log n)$ $\Theta(\log n)$

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How many times is F called?

Assume n is a power of 4 ($n=4^k$)

$$\sum_{i=1}^{k} 4^k$$

```
while n > 1 do
    for i = 1 to n do
        F(i,n)
    n = n/4
```

O(n log n) Θ(n log n)
O(n^2) Θ(n^2)
O(n) Θ(n)
O(log n) Θ(log n)

Useful Fact: Geometric Series

Fact: Suppose $x \neq 1$ then $\sum_{i=0}^{k} x^i = \frac{x^{k+1}-1}{x-1}$

Proof: $\sum_{i=0}^{k} x^i = \frac{1-x}{1-x}\sum_{i=0}^{k} x^i$

$$= \frac{1}{1-x}\left(\sum_{i=0}^{k} x^i - \sum_{i=0}^{k} x^{i+1}\right)$$

$$= \frac{1}{1-x}\left(\sum_{i=0}^{k} x^i - \sum_{i=1}^{k+1} x^i\right) = \frac{x^{k+1}-1}{x-1}$$

Example: $x = 4$ we have $\sum_{i=0}^{k} 4^i = \frac{4^{k+1}-1}{3} = \frac{4n-1}{3}$

How many times is F called?

Assume n is a power of 4 ($n=4^k$)

```
while n > 1 do
    for i = 1 to n do
        F(i,n)
    n = n/4
```

$O(n \log n)$ ~~$\Theta(n \log n)$~~

$O(n^2)$ ~~$\Theta(n^2)$~~

$O(n)$ $\Theta(n)$

~~$O(\log n)$~~ ~~$\Theta(\log n)$~~

Common complexity classes

$O(1)$ – constant
$O(\log n)$ – logarithmic (any base)
$O(\log^k n)$ – poly log

$O(n)$ – linear
$O(n \log n)$- quasi-linear
$O(n^2)$ – quadratic
$O(n^3)$ – cubic
$O(n^k)$ – polynomial, k is a positive constant

$O(2^n)$, $O(c^n)$ – exponential, c is a constant > 1
$O(n!)$ – factorial
$O(n^n)$

Table 2.1 The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time.

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

The divide-and-conquer algorithm design paradigm

1. *Divide* the problem (instance) into subproblems.
2. *Conquer* the subproblems by solving them recursively.
3. *Combine* subproblem solutions.

Divide-and-Conquer

- Divide-and-conquer.
  - Break up problem into several parts.
  - Solve each part recursively.
  - Combine solutions to sub-problems into overall solution.

- Most common usage.
  - Break up problem of size n into two equal parts of size ½n.
  - Solve two parts recursively.
  - Combine two solutions into overall solution in linear time.

- Consequence.
  - Brute force: n^2.
  - Divide-and-conquer: n log n.

Divide et impera.
Veni, vidi, vici.
- Julius Caesar

Analysis: Divide and Conquer

- Running Time (Recurrences):
  - Let T(n) be the time to solve problem of size n (worst-case).
  - Suppose we split input X into 3 equal size parts A, B and C recursively solve smaller problems A, B and C and then merge the solutions.

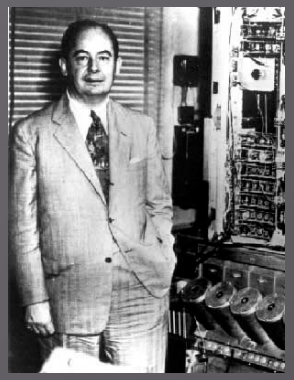
$$T(n) \leq 3T\left(\frac{n}{3}\right) + \#\text{Steps(Merge)}$$

- Correctness?
  - Induction!
  - Prove that algorithm is correct on small inputs (e.g., $n \leq 2$)
  - Prove that merge algorithm is correct (QED)

What you should learn?

- Solve Recurrences
- Identify recurrence associated with divide and conquer algorithm
- Prove that a divide and conquer algorithm is correct
- **Creative:** Design efficient divide and conquer algorithms
  - Build intuition about when the divide and conquer approach will work.

Mergesort



Jon von Neumann (1945)

- Mergesort.
  - Divide array into two halves.
  - Recursively sort each half.
  - Merge two halves to make sorted whole.

A	L	G	O	R	I	T	H	M	S

A	L	G	O	R

I	T	H	M	S

Divide O(1)

A	G	L	O	R

H	I	M	S	T

Sort 2T(n/2)

A	G	H	I	L	M	O	R	S	T

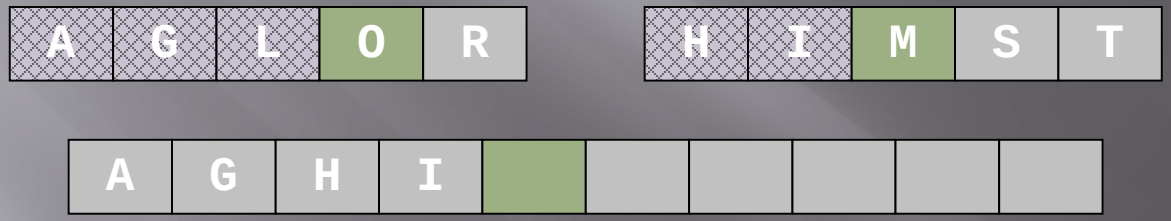
Merge O(n)

$$T(n) \leq 2\,T\left(\frac{n}{2}\right) + O(n)$$

Merging

- Merging. Combine two pre-sorted lists into a sorted whole.
- How to merge efficiently?
  - Linear number of comparisons.
  - Use temporary array.

05demo-merge.ppt



- Challenge for the bored. In-place merge. [Kronrud, 1969]

↑

using only a constant amount of extra storage

A Useful Recurrence Relation

- Def. T(n) = number of comparisons to mergesort an input of size n.

- Mergesort recurrence.

$$\mathrm{T}(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

- Solution. T(n) = O(n $\log_2$ n).

- Assorted proofs. We describe several ways to prove this recurrence. Initially we assume n is a power of 2 and replace ≤ with =.

Proof by Induction

- Claim. If T(n) satisfies this recurrence, then T(n) = n $\log_2$ n.

↑ assumes n is a power of 2

$$T(n) = \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{2T(n/2)}_{\text{sorting both halves}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

- Pf. (by induction on n)
  - Base case: n = 1.
  - Inductive hypothesis: T(n) = n $\log_2$ n.
  - Goal: show that T(2n) = 2n $\log_2$ (2n).

$$\begin{aligned} T(2n) &= 2T(n) + 2n \\ &= 2n\log_2 n + 2n \\ &= 2n(\log_2(2n) - 1) + 2n \\ &= 2n\log_2(2n) \end{aligned}$$

Analysis of Mergesort Recurrence

- Claim. If T(n) satisfies the following recurrence, then $T(n) \le n \lceil \lg n \rceil$.

$$T(n) \le \begin{cases} 0 & \text{if } n = 1 \\ \underbrace{T(\lceil n/2 \rceil)}_{\text{solve left half}} + \underbrace{T(\lfloor n/2 \rfloor)}_{\text{solve right half}} + \underbrace{n}_{\text{merging}} & \text{otherwise} \end{cases}$$

- Pf. (by induction on n)
  - Base case: n = 1.
  - Define $n_1 = \lfloor n / 2 \rfloor$, $n_2 = \lceil n / 2 \rceil$.
  - Induction step: assume true for 1, 2, ... , n–1.

Analysis of Mergesort Recurrence

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  - Induction step: assume true for 1, 2, ... , n–1.

$$
\begin{aligned}
T(n) &\le T(n_1) + T(n_2) + n \\
&\le n_1 \lceil \lg n_1 \rceil + n_2 \lceil \lg n_2 \rceil + n \\
&\le n_1 \lceil \lg n_2 \rceil + n_2 \lceil \lg n_2 \rceil + n \\
&= n \lceil \lg n_2 \rceil + n \\
&\le n(\lceil \lg n \rceil - 1) + n \\
&= n \lceil \lg n \rceil
\end{aligned}
$$

$$
\begin{aligned}
n_2 &= \lceil n/2 \rceil \\
&\le \lceil 2^{\lceil \lg n \rceil}/2 \rceil \\
&= 2^{\lceil \lg n \rceil}/2 \\
\Rightarrow \lg n_2 &\le \lceil \lg n \rceil - 1
\end{aligned}
$$

Problem: Compute a^n, n**>0**. Minimize number of multiplications.

Naive algorithm: $\Theta(n)$ multiplications

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

$$T(n) = T(n/2) + \Theta(1) \Rightarrow T(n) = \Theta(\log n)$$