Q1b Typo: if there is a directed from $v_\ell$ to $v_{\overline{\ell}}$ AND from $v_{\overline{\ell}}$ to $v_\ell$ then the 2-SAT instance is not satisfiable.
8.4 NP-Completeness
NP-Complete

NP-complete. A problem $Y$ in NP with the property that for every problem $X$ in NP, $X \leq_p Y$.

Theorem. Suppose $Y$ is an NP-complete problem. Then $Y$ is solvable in polynomial time iff $P = NP$.

Fundamental question. Do there exist "natural" NP-complete problems?
Circuit Satisfiability

CIRCUIT-SAT. Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is 1?

yes: 1 0 1
The "First" NP-Complete Problem

**Theorem.** CIRCUIT-SAT is NP-complete. [Cook 1971, Levin 1973]

**Pf.** (sketch)

- Any algorithm that takes a fixed number of bits $n$ as input and produces a yes/no answer can be represented by such a circuit. Moreover, if algorithm takes poly-time, then circuit is of poly-size.

  sketchy part of proof; fixing the number of bits is important, and reflects basic distinction between algorithms and circuits

- Consider some problem $X$ in NP. It has a poly-time certifier $C(s, t)$. To determine whether $s$ is in $X$, need to know if there exists a certificate $t$ of length $p(|s|)$ such that $C(s, t) = \text{yes}$.

- View $C(s, t)$ as an algorithm on $|s| + p(|s|)$ bits (input $s$, certificate $t$) and convert it into a poly-size circuit $K$.
  - first $|s|$ bits are hard-coded with $s$
  - remaining $p(|s|)$ bits represent bits of $t$

- Circuit $K$ is satisfiable iff there exists a length $p(|s|)$ bit input string $t$ s.t. $C(s, t) = \text{yes}$. 
Example

Ex. Construction below creates a circuit $K$ whose inputs can be set so that $K$ outputs true iff graph $G$ has an independent set of size 2.

Example diagram:

$G = (V, E), n = 3$

$\left( \begin{array}{c} n \\ 2 \end{array} \right)$ hard-coded inputs (graph description)  $n$ inputs (nodes in independent set)
Establishing NP-Completeness

Remark. Once we establish first "natural" NP-complete problem, others fall like dominoes.

Recipe to establish NP-completeness of problem $Y$.

- Step 1. Show that $Y$ is in NP.
- Step 2. Choose a known NP-complete problem $X$.
- Step 3. Prove that $X \leq_p Y$.

Justification. If $X$ is an NP-complete problem, and $Y$ is a problem in NP with the property that $X \leq_p Y$ then $Y$ is NP-complete.

Pf. Let $W$ be any problem in NP. Then $W \leq_p X \leq_p Y$.

- By transitivity, $W \leq_p Y$.
- Hence $Y$ is NP-complete. ·
Theorem. 3-SAT is NP-complete.

Pf. Suffices to show that CIRCUIT-SAT $\leq P$ 3-SAT since we previously showed 3-SAT is in NP.

- Let K be any circuit we build a 3-SAT formula $\Phi_K = f(K)$.
- Create a 3-SAT variable $x_i$ for each circuit element $i$.
- Make circuit compute correct values at each node:
  - $x_2 = \neg x_3 \implies$ add 2 clauses: $x_2 \lor x_3$, $\overline{x_2} \lor \overline{x_3}$
  - $x_1 = x_4 \lor x_5 \implies$ add 3 clauses: $x_1 \lor \overline{x_4}$, $x_1 \lor \overline{x_5}$, $\overline{x_1} \lor x_4 \lor x_5$
  - $x_0 = x_1 \land x_2 \implies$ add 3 clauses: $\overline{x_0} \lor x_1$, $\overline{x_0} \lor x_2$, $x_0 \lor \overline{x_1} \lor \overline{x_2}$

- Hard-coded input values and output value.
  - $x_5 = 0 \implies$ add 1 clause: $\overline{x_5}$
  - $x_0 = 1 \implies$ add 1 clause: $x_0$
- Final step: turn clauses of length < 3 into clauses of length exactly 3.
  - Add new variable $z$ and new clause $(\overline{z} \lor \overline{z} \lor \overline{z})$
  - Pad short clauses (len < 3) with $z$ e.g. $\overline{x_5} \lor z \lor z$. 

output

\[ x_0 \]
\[ x_1 \]
\[ x_2 \]
\[ x_3 \]
\[ x_4 \]
\[ x_5 \]
Theorem. 3-SAT is NP-complete.

Pf. Suffices to show that CIRCUIT-SAT ≤ₚ 3-SAT since we previously showed 3-SAT is in NP.

- Let K be any circuit we built a 3-SAT formula Φₖ = f(K).

1. Explain why Φₖ = f(K) can be constructed in polynomial time.
2. Show that if K has a satisfying assignment then Φₖ = f(K) has a satisfying assignment.
3. Show that if K does not have a satisfying assignment then Φₖ = f(K) does not have a satisfying assignment

(Easier to show contrapositive of step 3)
Assume Φₖ = f(K) has a satisfying assignment and then build satisfying assignment for circuit K.
**Observation.** All problems below are NP-complete and polynomial reduce to one another!

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**NP-Completeness**

- **CIRCUIT-SAT**
  - 3-SAT
    - INDEPENDENT SET
      - VERTEX COVER
        - SET COVER
    - DIR-HAM-CYCLE
    - GRAPH 3-COLOR
    - SUBSET-SUM
      - PLANAR 3-COLOR
      - SCHEDULING

*by definition of NP-completeness*
Jane is excited! She thinks she has proved that $P=NP$. In particular, she claims to have proved that 2-SAT is NP-Complete by showing that $2\text{SAT} \leq_p 3\text{SAT}$. She has a reduction $f$ which maps 2-SAT instances ($\Phi_{2\text{SAT}}$) to 3-SAT instances ($\Phi_{3\text{SAT}}$) in polynomial time. She proved that if $\Phi_{2\text{SAT}}$ is satisfiable then $\Phi_{3\text{SAT}} = f(\Phi_{2\text{SAT}})$ is satisfiable. What mistakes (if any) did Jane make?

A. Proving that 2-SAT is NP-Complete would not imply $P=NP$.
B. Jane still needs to show that if $\Phi_{3\text{SAT}}$ is satisfiable then $\Phi_{2\text{SAT}}$ is satisfiable to conclude that $2\text{SAT} \leq_p 3\text{SAT}$.
C. The reduction is in the wrong direction to conclude that $P=NP$.
D. Claims B and C are both true.
E. Jane’s logic is sound! She can claim the $1,000,000 prize!
Jane is excited! She thinks she has proved that P=NP. In particular, she claims to have proved that 2-SAT is NP-Complete by showing that $2\text{SAT} \leq_p 3\text{SAT}$. She has a reduction $f$ which maps 2-SAT instances ($\Phi_{2\text{SAT}}$) to 3-SAT instances ($\Phi_{3\text{SAT}}$) in polynomial time. She proved that if $\Phi_{2\text{SAT}}$ is satisfiable then $\Phi_{3\text{SAT}} = f(\Phi_{2\text{SAT}})$ is satisfiable. What mistakes (if any) did Jane make?

**A.** Proving that 2-SAT is NP-Complete would not imply P=NP.
   - 2SAT is in P (see homework 7). This would imply P=NP.

**B.** Jane still needs to show that if $\Phi_{3\text{SAT}}$ is satisfiable then $\Phi_{2\text{SAT}}$ is satisfiable to conclude that $2\text{SAT} \leq_p 3\text{SAT}$.
   - Otherwise $f(\Phi_{2\text{SAT}})$ could ignore the input $\Phi_{2\text{SAT}}$ and always output a satisfiable 3SAT formula.

**C.** The reduction is in the wrong direction to conclude that P=NP.
   - Jane would need to prove that $3\text{SAT} \leq_p 2\text{SAT}$

**D.** Claims B and C are both true.

**E.** Jane’s logic is sound! She can claim the $1,000,000 prize!
Some NP-Complete Problems

Six basic genres of NP-complete problems and paradigmatic examples.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Practice. Most NP problems are either known to be in P or NP-complete.

Notable exceptions. Factoring, graph isomorphism, Nash equilibrium.