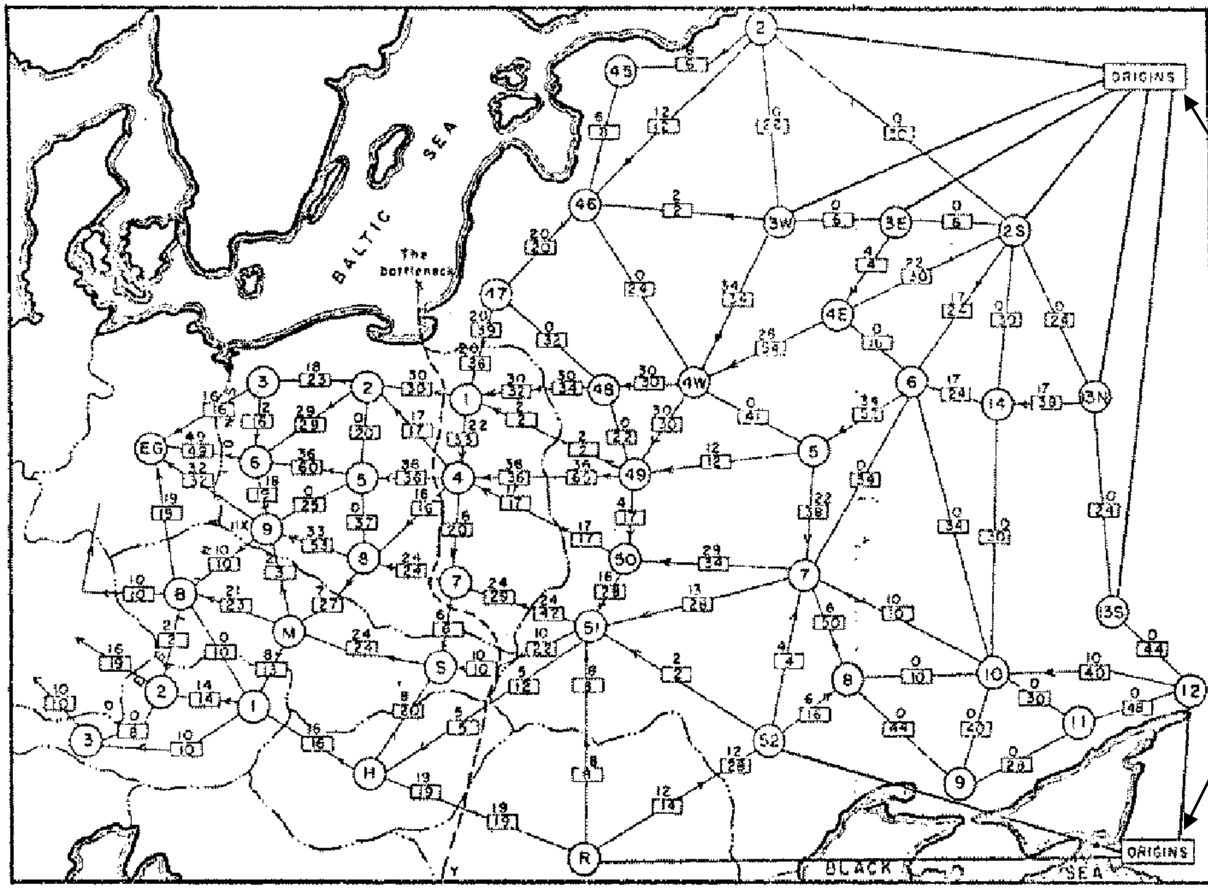
# CS 381 – FALL 2019

## Week 12.2, Wednesday, Nov 6

**Homework 6 Released: Due November 14th at 11:59PM (Gradescope)**
**Midterm 2 Graded**
**(Max: 122 Median: 80.5 Mean: 79.9 StdDev: 17.9)**

# 7.7 Extensions to Max Flow

**Multiple destinations?**




**Multiple sources?**

Reference: *On the history of the transportation and maximum flow problems.* Alexander Schrijver in Math Programming, 91: 3, 2002.

# Circulation with Demands

Circulation with demands.

- Directed graph $G = (V, E)$.
- Edge capacities $c(e)$, $e \in E$.
- Node supply and demands $d(v)$, $v \in V$.

demand if $d(v) > 0$; supply if $d(v) < 0$; transshipment if $d(v) = 0$

Def. A circulation is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ (capacity)
- For each $v \in V$: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

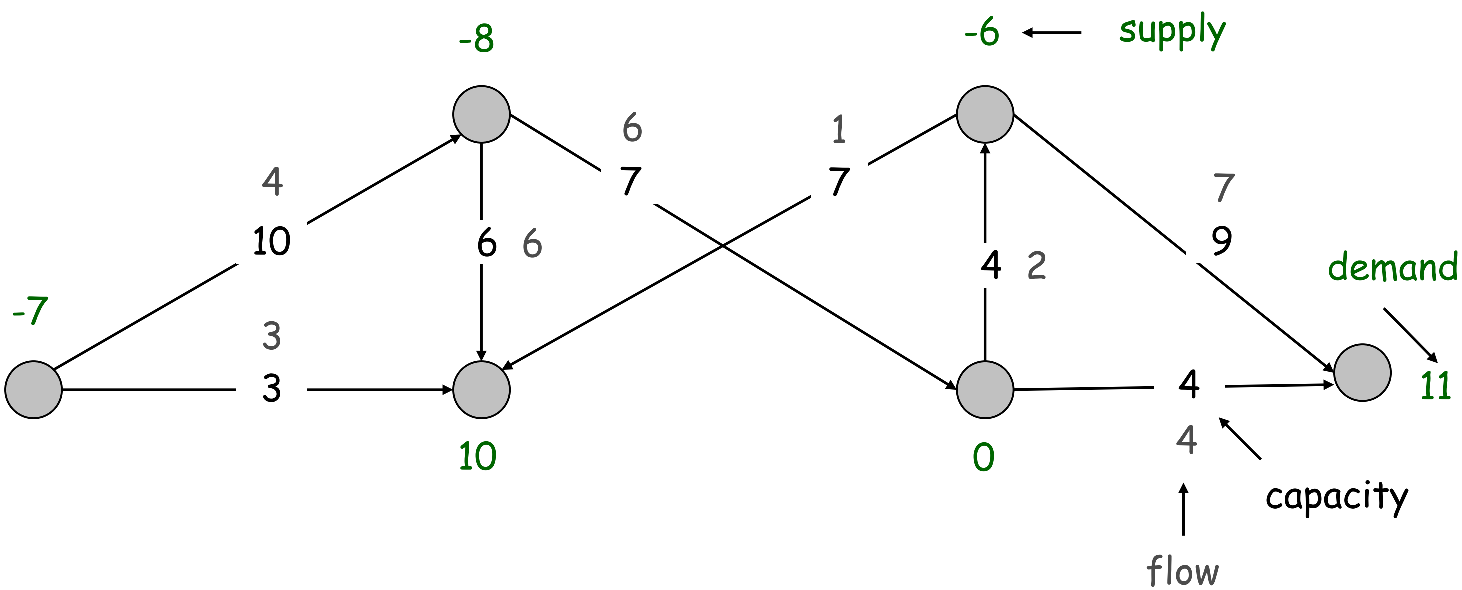
Circulation problem: given $(V, E, c, d)$, does there exist a circulation?

# Circulation with Demands

Necessary condition: sum of supplies = sum of demands.

$$\sum_{v\,:\,d(v)>0} d(v) \;=\; \sum_{v\,:\,d(v)<0} -d(v) \;=:\; D$$

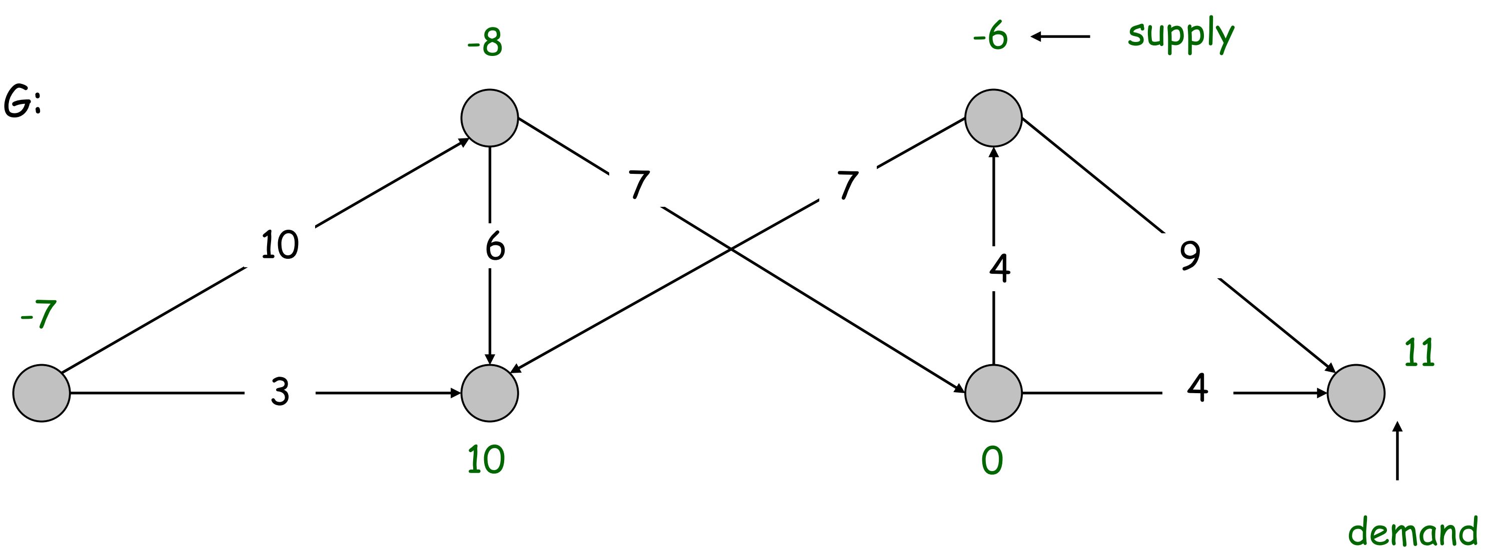
Pf. Sum conservation constraints for every demand node v.



# Circulation with Demands

Max flow formulation.

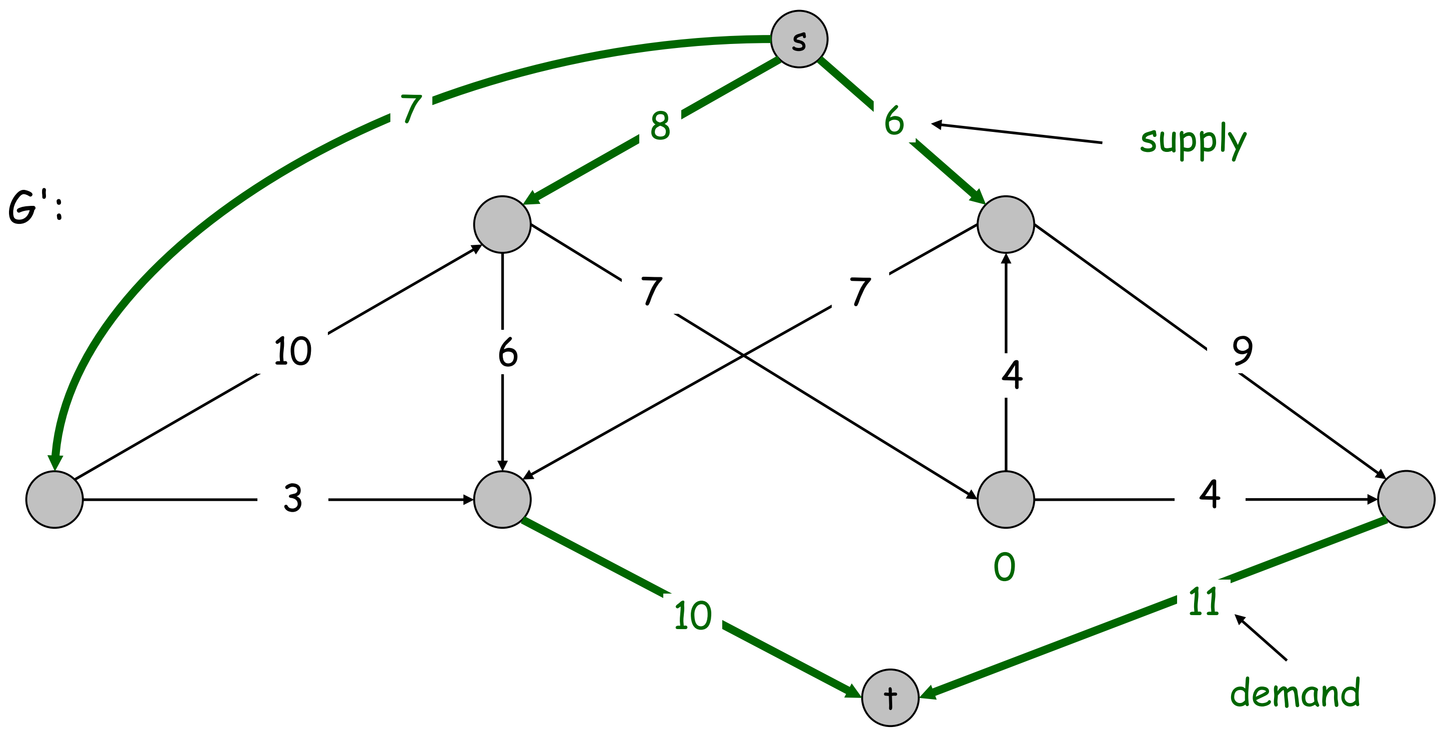
G:



Max flow formulation.

- Add new source s and sink t.
- For each v with $d(v) < 0$, add edge (s, v) with capacity $-d(v)$.
- For each v with $d(v) > 0$, add edge (v, t) with capacity $d(v)$.
- **Claim:** G has circulation iff G' has max flow of value D.

saturates all edges leaving s and entering t



# Circulation with Demands

## Max flow formulation.

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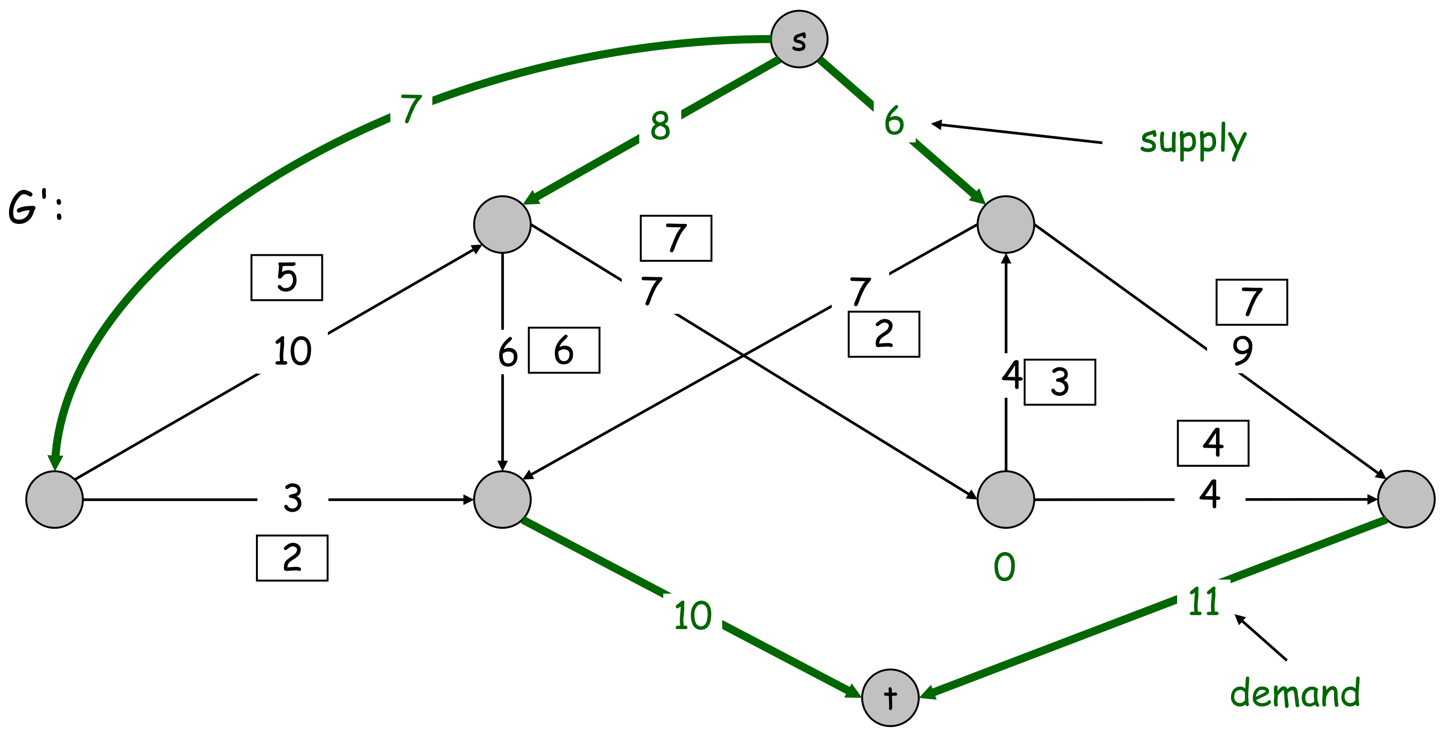


# Circulation with Demands

Integrality theorem. If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.

Pf. Follows from max flow formulation and integrality theorem for max flow.

Characterization. Given (V, E, c, d), there does not exists a circulation iff there exists a node partition (A, B) such that $\Sigma_{v \in B} d_v > cap(A, B)$

demand by nodes in B exceeds supply of nodes in B plus max capacity of edges going from A to B

Pf idea. Look at min cut (A', B') in G'.

Set $\mathrm{A} = \mathrm{A}' \backslash \{\mathrm{s}\}$

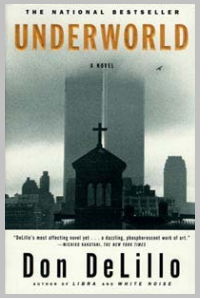
Set $\mathrm{B} = \mathrm{B}' \backslash \{\mathrm{t}\}$

How do we know that $\mathrm{A}' \neq \{s\}$ and $\mathrm{B}' \neq \{t\}$?

# 7.12 Baseball Elimination

"See that thing in the paper last week about Einstein? . . . Some reporter asked him to figure out the mathematics of the pennant race. You know, one team wins so many of their remaining games, the other teams win this number or that number. What are the myriad possibilities? Who's got the edge?"
"The hell does he know?"
"Apparently not much. He picked the Dodgers to eliminate the Giants last Friday."

*- Don DeLillo, Underworld*



# Baseball Elimination

| Team $i$ | Wins $w_i$ | Losses $l_i$ | To play $r_i$ | Against = $r_{ij}$ | | | |
|---|---|---|---|---|---|---|---|
| | | | | Atl | Phi | NY | Mon |
| Atlanta | 83 | 71 | 8 | - | 1 | 6 | 1 |
| Philly | 80 | 79 | 3 | 1 | - | 0 | 2 |
| New York | 78 | 78 | 6 | 6 | 0 | - | 0 |
| Montreal | 77 | 82 | 3 | 1 | 2 | 0 | - |

Which teams have a chance of finishing the season with most wins?

- Montreal eliminated since it can finish with at most 80 wins, but Atlanta already has 83.
- $w_i + r_i < w_j \Rightarrow$ team i eliminated.
- Only reason sports writers appear to be aware of.
- Sufficient, but not necessary!

# Baseball Elimination

| Team $i$ | Wins $w_i$ | Losses $l_i$ | To play $r_i$ | Against = $r_{ij}$ | | | |
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Which teams have a chance of finishing the season with most wins?

- Philly can win 83, but still eliminated . . .
- If Atlanta loses a game, then some other team wins one.

Remark. Answer depends not just on how many games already won and left to play, but also on whom they're against.

# Baseball Elimination

| Team $i$ | Wins $w_i$ | Losses $l_i$ | To play $r_i$ | Against = $r_{ij}$ | | | |
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| | | | | Atl | Phi | NY | Mon |
| Atlanta | **83** | 71 | 8 | - | 1 | **6** | 1 |
| Philly | 80 | 79 | 3 | 1 | - | 0 | 2 |
| New York | **78** | 78 | 6 | **6** | 0 | - | 0 |
| Montreal | 77 | 82 | 3 | 1 | 2 | 0 | - |

Which teams have a chance of finishing the season with most wins?

- Philly can win 83, but still eliminated . . .
- If Atlanta loses a game, then some other team wins one
  - → NY Wins 6 more games vs Atlanta (Total: 78+6=84 wins).

Remark. Answer depends not just on how many games already won and left to play, but also on whom they're against.

# Baseball Elimination

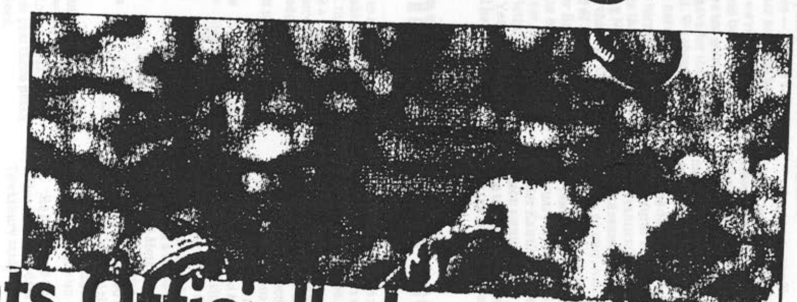
TUESDAY, SEPTEMBER 10, 1996

San Francisco Chronicle

The Gate
Sports Online
▸ http://www.sfgate.com

# SPORTING G

## 49ers, Young Get Big Brea



Quarterback m

By Gary Swan
Chronicle Staff Writer

The bye week has come at a perfect time for the 49ers and quarterback Steve Young. If they had a game next Sunday, there's a good chance Young would not play.

But the pulled groin muscle on his up-

## Giants Officially Leave the NL West Race

By Nancy Gay
Chronicle Staff Writer

With the smack of another National League West bat 500 miles away, the Giants' run at the division title ended last night, just as they were handing the visiting St. Louis Cardinals an even bigger lead in the NL Central.

| | |
|---|---|
| CARDINALS 6 | |
| GIANTS 2 | |

In San Diego, Greg Vaughn's three-run homer in the eighth pushed the Padres over the Pirates and officially shoved the rest of the Giants' season into the background. On the heels of their tedious 6-2 loss before an announced crowd of 10,307 at Candlestick Park, the Giants fell 19½ games off the lead.

As it is, the worst the Padres (80-65) can finish is 80-82. The Giants have fallen to 59-83 with 20 games left; they cannot win 80 games. Coming off a miserable 2-8 mark on a three-city road trip that saw their road record drop to 27-47, the Giants were hoping to get off on the right foot in their longest homestand of the year (15 games, 14 days).

**Financing in Place For Giants' New Stadium**
SEE PAGE B1, MAIN NEWS

"Where we are, you're going to be eliminated sooner or later," Baker said quietly. "But it doesn't alter the fact that we've still got to play ball. You've still got to play hard, the fans come out to watch you play. You've got to play for the fact of loving to play, no matter where you are in the standings.

"You've got to play the role of spoiler, to not make it easier on

GIANTS: Page D5 Col. 3

# Baseball Elimination

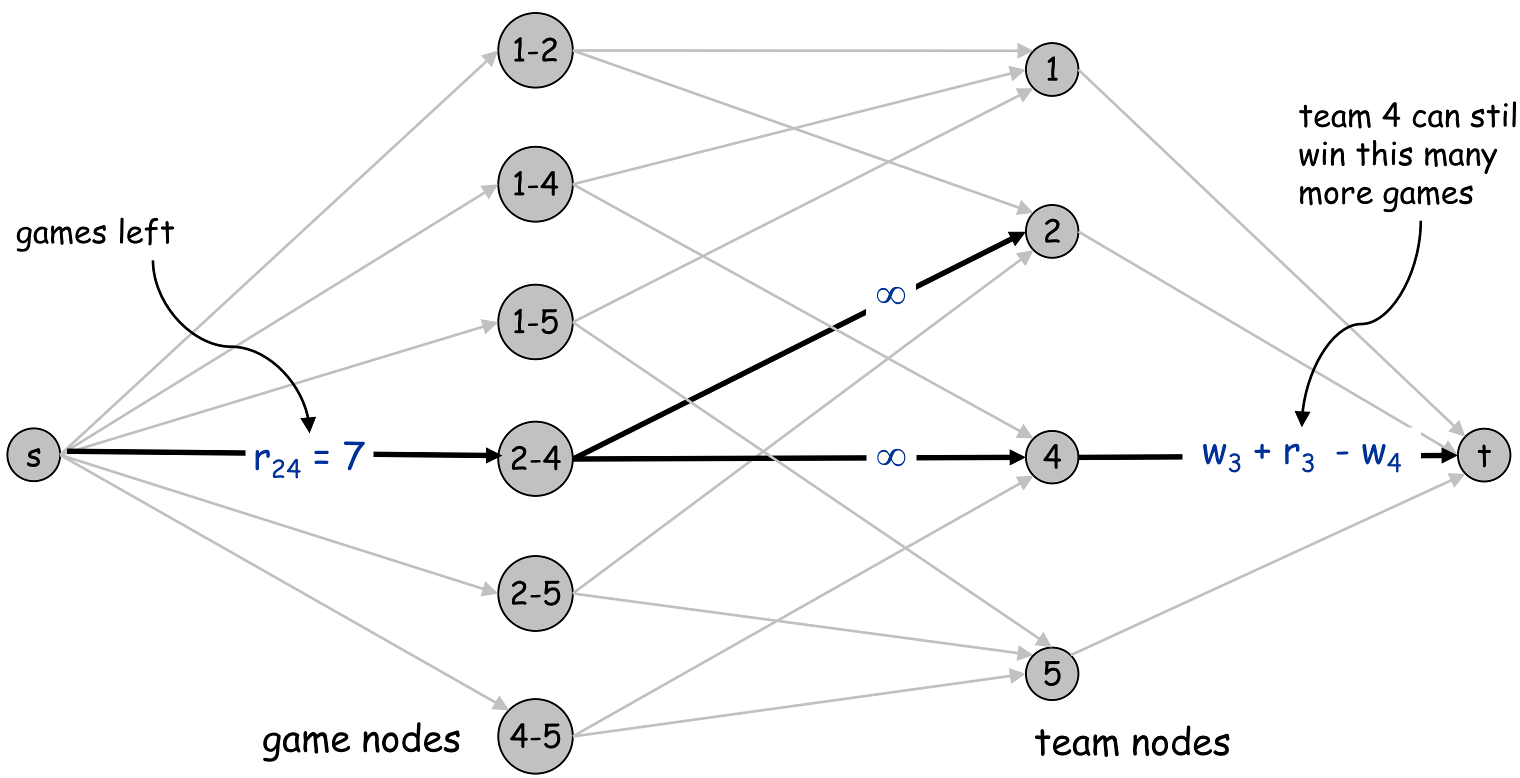
Baseball elimination problem.

- Set of teams $S$.
- Distinguished team $s \in S$.
- Team $x$ has won $w_x$ games already.
- Teams $x$ and $y$ play each other $r_{xy}$ additional times.
- Is there any outcome of the remaining games in which team $s$ finishes with the most (or tied for the most) wins?

# Baseball Elimination: Max Flow Formulation

Can team 3 finish with most wins?

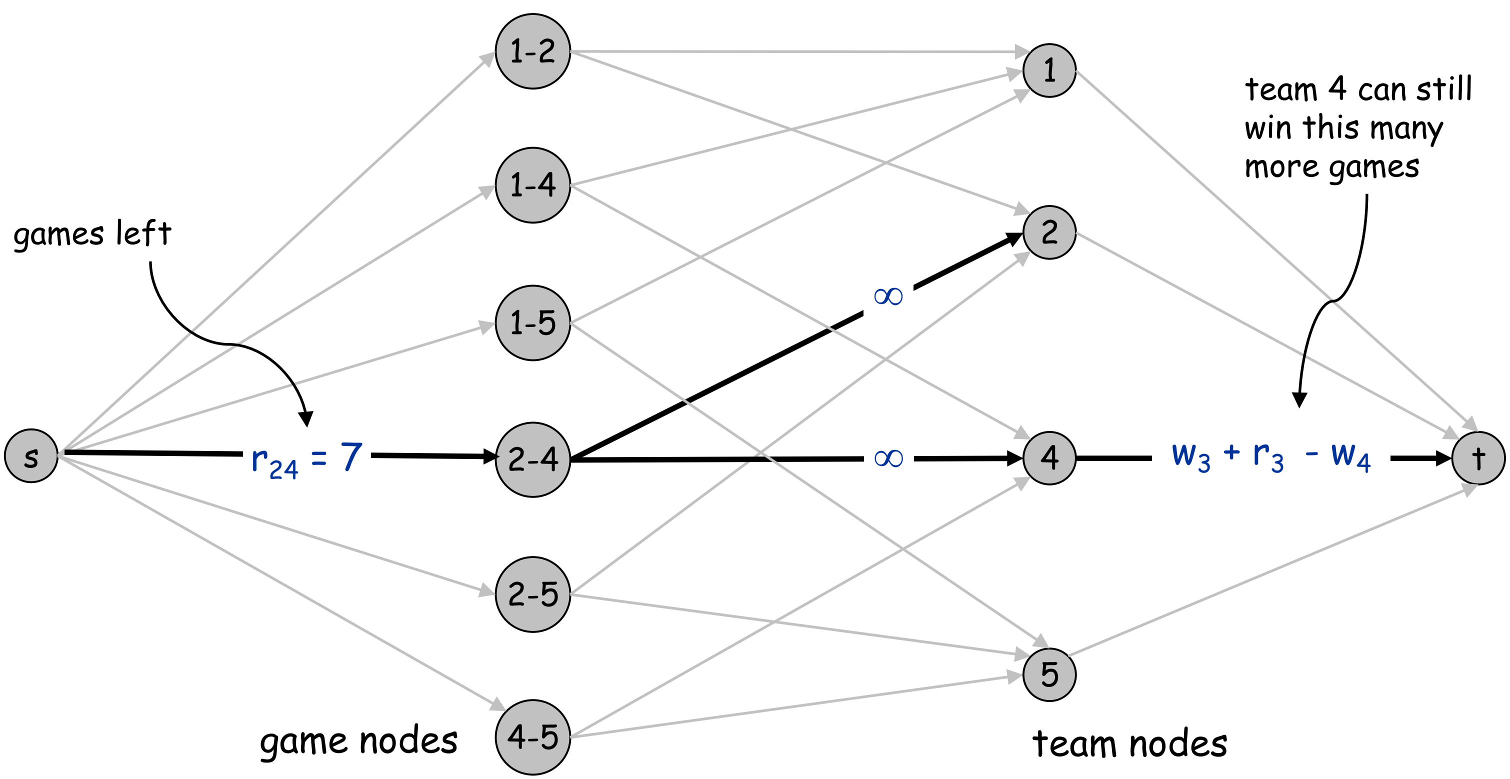
- Assume team 3 wins all remaining games $\Rightarrow$ $w_3 + r_3$ wins.
- Divvy remaining games so that all teams have $\leq$ $w_3 + r_3$ wins.



# Baseball Elimination: Max Flow Formulation

Theorem. Team 3 is not eliminated iff max flow saturates all edges leaving source.

- Integrality theorem $\Rightarrow$ each remaining game between x and y added to number of wins for team x or team y.
- Capacity on (x, t) edges ensure no team wins too many games.



# Clicker Question

Baseball elimination problem.

- Set of teams $S$.
- Distinguished team $s \in S$.
- Team $x$ has won $w_x$ games already.
- Teams $x$ and $y$ play each other $r_{xy}$ additional times.
- Is there any outcome of the remaining games in which team $s$ finishes with the most (or tied for the most) wins?

Which of the following claims about the flow network for the baseball elimination problem are true?

A. The flow network has $O(|S|)$ nodes.

B. The maximum flow is at most $|S|$

C. The flow network does not include a node for the distinguished team $s$.

D. The capacity of the edge $(z,t)$ for team $z$ is $w_s - w_z$ to ensure that team $z$ does not win more games than $s$.

E. All of the claims are true

# Clicker Question

Baseball elimination problem.

- Set of teams S.
- Distinguished team $s \in S$.
- Team x has won $w_x$ games already.
- Teams x and y play each other $r_{xy}$ additional times.
- Is there any outcome of the remaining games in which team s finishes with the most (or tied for the most) wins?

Which of the following claims about the flow network for the baseball elimination problem are true?

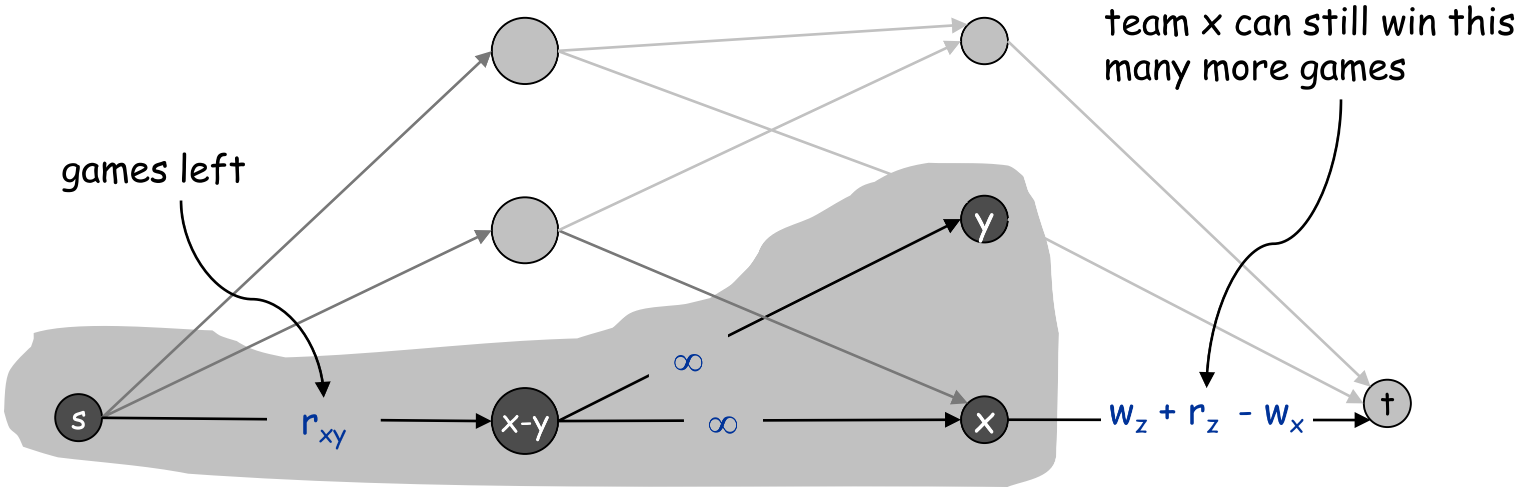
A. The flow network has $O(|S|)$ nodes. $O(|S|^2)$ pairs of teams
B. The maximum flow is at most $|S|$ Teams can play multiple games
C. The flow network does not include a node for the distinguished team s.
D. The capacity of the edge (z,t) for team z is $r_s + w_s - w_z$ to ensure that team z does not win more games than s. (Missing $r_s$)
E. All of the claims are true

# Certificate of Elimination

**Minimum Cut:** (A,B)

**Theorem:** Suppose that the max-flow does not saturate edges exiting source (team s is eliminated). If $T \subseteq S$ denotes teams on A side of the cut then

$$\frac{\overbrace{\sum_{\{x,y\}\subseteq T} r_{xy}}^{\text{some team in } T \text{ must win each of these games}} + \overbrace{\sum_{x\in T} w_x}^{\#\text{wins by } T}}{|T|} > w_s + r_s$$



# 7.11 Project Selection

# Project Selection

Projects with prerequisites.

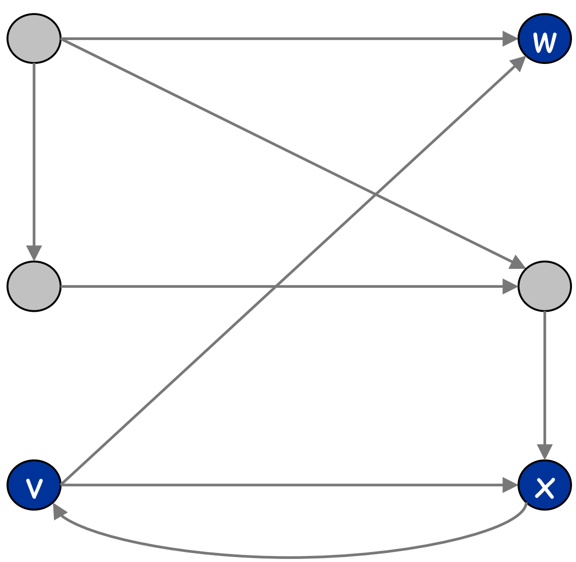
can be positive or negative

- Set P of possible projects. Project v has associated revenue $p_v$.
  - some projects generate money: create interactive e-commerce interface, redesign web page
  - others cost money: upgrade computers, get site license
- Set of prerequisites E. If $(v, w) \in E$, can't do project v and unless also do project w.
- A subset of projects $A \subseteq P$ is feasible if the prerequisite of every project in A also belongs to A.

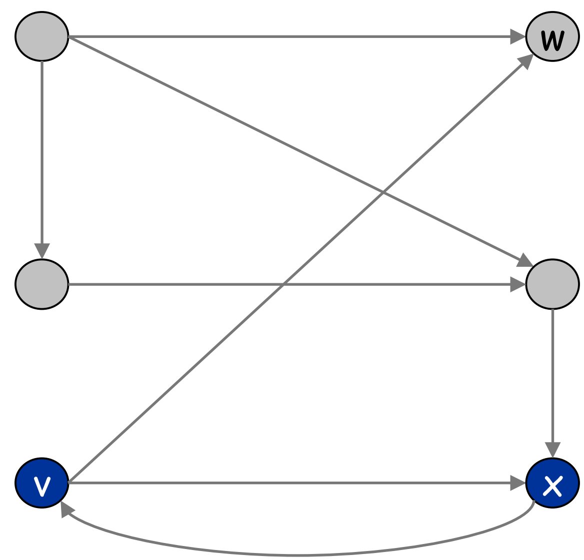
Project selection. Choose a feasible subset of projects to maximize revenue.

# Project Selection: Prerequisite Graph

Prerequisite graph.
- Include an edge from v to w if can't do v without also doing w.
- {v, w, x} is feasible subset of projects.
- {v, x} is infeasible subset of projects.




feasible

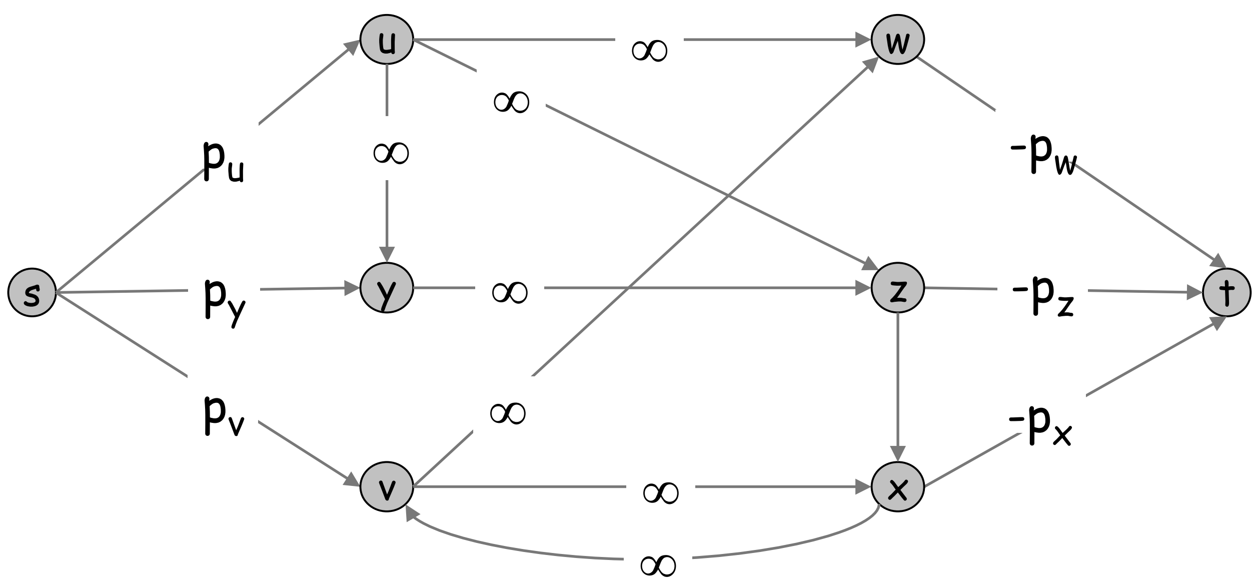



infeasible

# Project Selection: Min Cut Formulation

Min cut formulation.

- Assign capacity ∞ to all prerequisite edge.
- Add edge (s, v) with capacity $p_v$ if $p_v > 0$.
- Add edge (v, t) with capacity $-p_v$ if $p_v < 0$.
- For notational convenience, define $p_s = p_t = 0$.



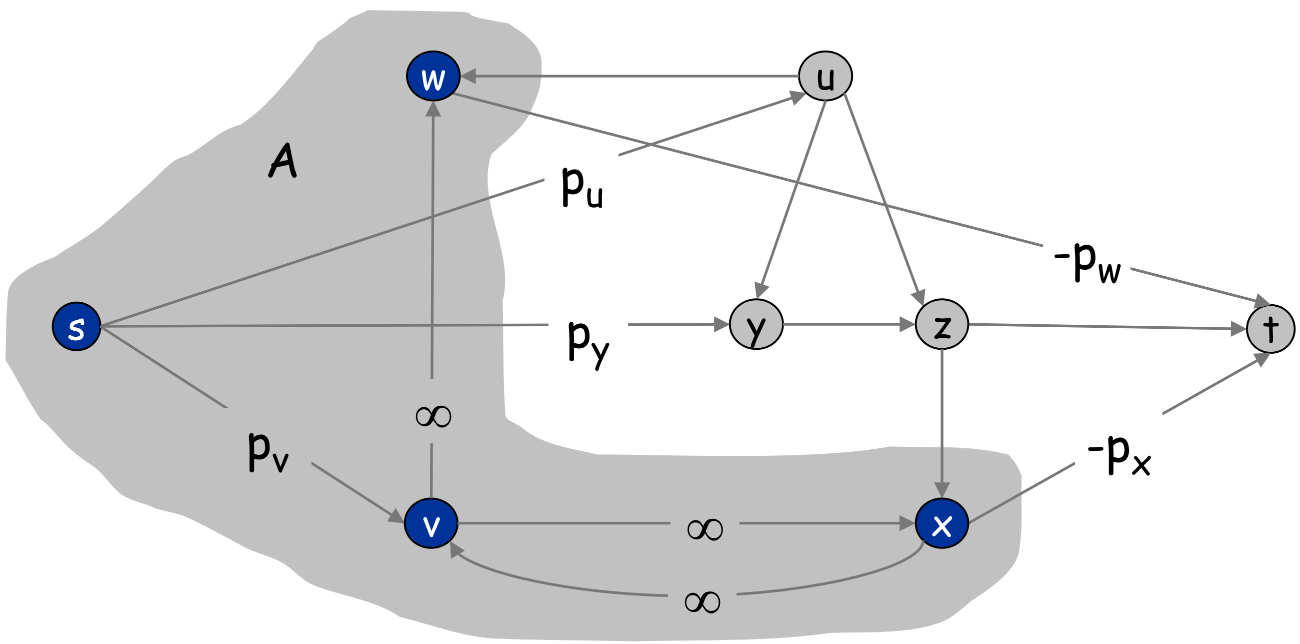
# Project Selection: Min Cut Formulation

Claim. (A, B) is min cut iff A – { s } is optimal set of projects.

- Infinite capacity edges ensure A – { s } is feasible.
- Max revenue because:

$$cap(A,\ B) = \sum_{v \in B:\ p_v > 0} p_v + \sum_{v \in A:\ p_v < 0} (-p_v)$$

$$= \underbrace{\sum_{v:\ p_v > 0} p_v}_{\text{constant}} - \sum_{v \in A} p_v$$



# Algorithm Design Patterns and Anti-Patterns

| Algorithm design patterns. | Ex. |
| --- | --- |
| ▪ Greedy. | $O(n \log n)$ interval scheduling. |
| ▪ Divide-and-conquer. | $O(n \log n)$ Closest Pair of Points. |
| ▪ Dynamic programming. | $O(n^2)$ edit distance. |
| ▪ Duality. | $O(n^3)$ bipartite matching. |
| ▪ Reductions. | Circulation via Network Flow<br>Bipartite Matching via Network Flow<br>Baseball elimination |
| ▪ Local search. | |
| ▪ Randomization. | |

| Algorithm design anti-patterns. | |
| --- | --- |
| ▪ NP-completeness. | $O(n^k)$ algorithm unlikely. |
| ▪ PSPACE-completeness. | $O(n^k)$ certification algorithm unlikely. |
| ▪ Undecidability. | No algorithm possible. |