CS 381 – FALL 2019

Week 11.2, Wednesday, Oct 30

Midterm 2 Tonght! October 30 (8-9:30PM) MTHW 210 and BRNG 2280

Friday PSO → BRNG **2280 (Exam Capacity 62)**

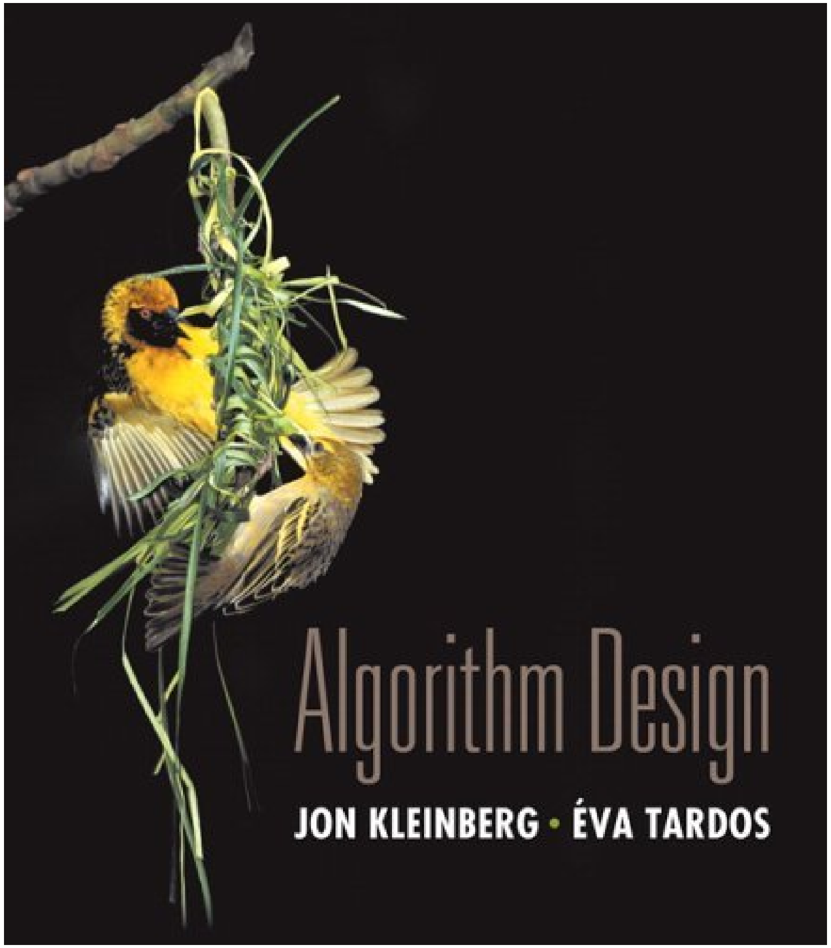
All Others → **MTHW 210 (Exam Capacity 111)**

Practice Midterm 2 & Solutions on Piazza

No PSOs this week (Due to Midterm)

Midterm 2

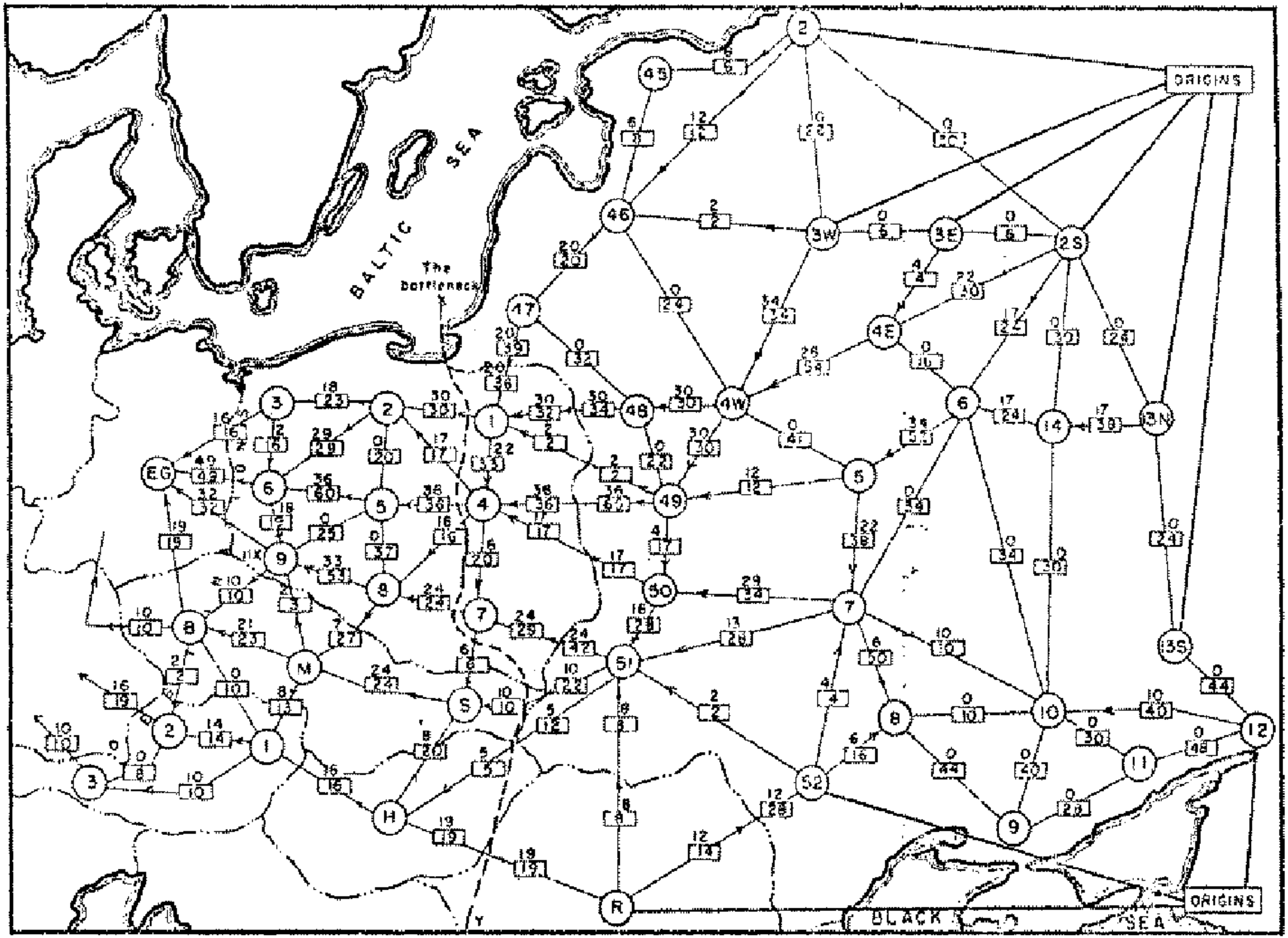
- Friday PSO → BRNG **2280 (Exam Capacity 62)**
- All Others → **MTHW 210 (Exam Capacity 111)**
- Focus: Dynamic Programing and Graph Algorithms
  - Lectures 15 to 28
  - No Network Flow (today's lecture)
- Same Rules as Midterm 1
  - Allowed to prepare 1 page of handwritten notes
  - No calculators, phones, smartwatches etc…
  - Make sure your writing implement shows up clearly when scanned!
    - Number 2 pencils work



Network Flow

Slides by Kevin Wayne.

Soviet Rail Network, 1955



Reference: *On the history of the transportation and maximum flow problems.*
Alexander Schrijver in Math Programming, 91: 3, 2002.

Maximum Flow and Minimum Cut

Max flow and min cut.

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

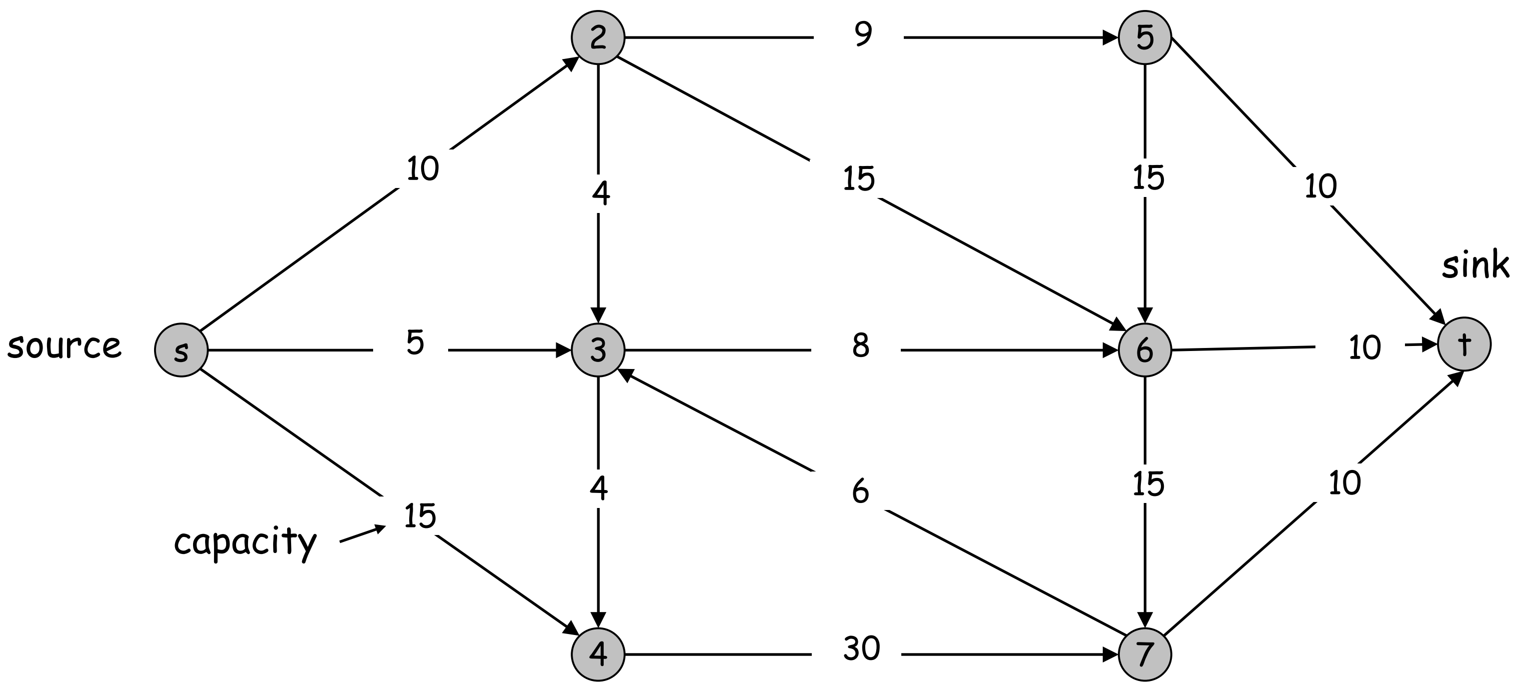
Nontrivial applications / reductions.

- Data mining.
- Open-pit mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Network reliability.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Many many more …

Minimum Cut Problem

Flow network.

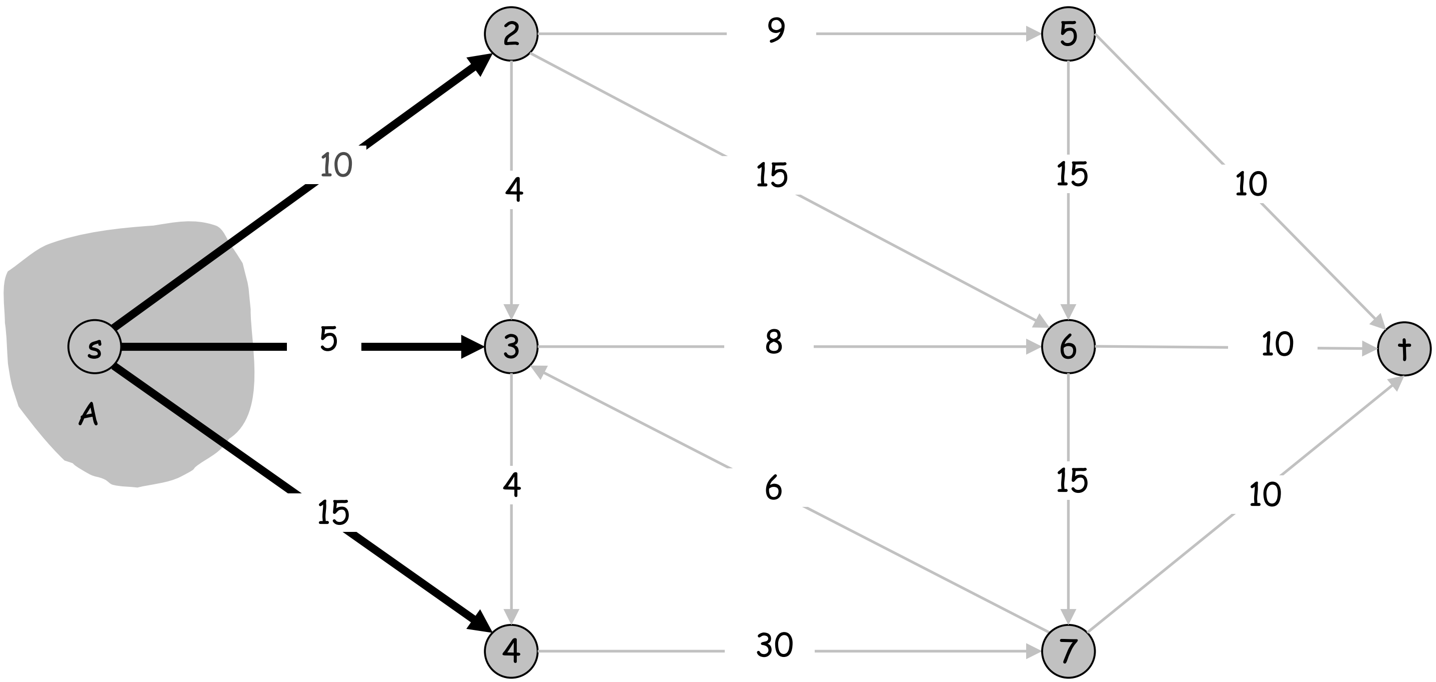
- Abstraction for material flowing through the edges.
- G = (V, E) = directed graph, no parallel edges.
- Two distinguished nodes: s = source, t = sink.
- c(e) = capacity of edge e.



Cuts

Def. An s-t cut is a partition (A, B) of V with s $\in$ A and t $\in$ B.

Def. The capacity of a cut (A, B) is: $cap(A, B) \;=\; \sum_{e \text{ out of } A} c(e)$



Capacity = 10 + 5 + 15
= 30

Cuts

Def. An s-t cut is a partition (A, B) of V with s ∈ A and t ∈ B.

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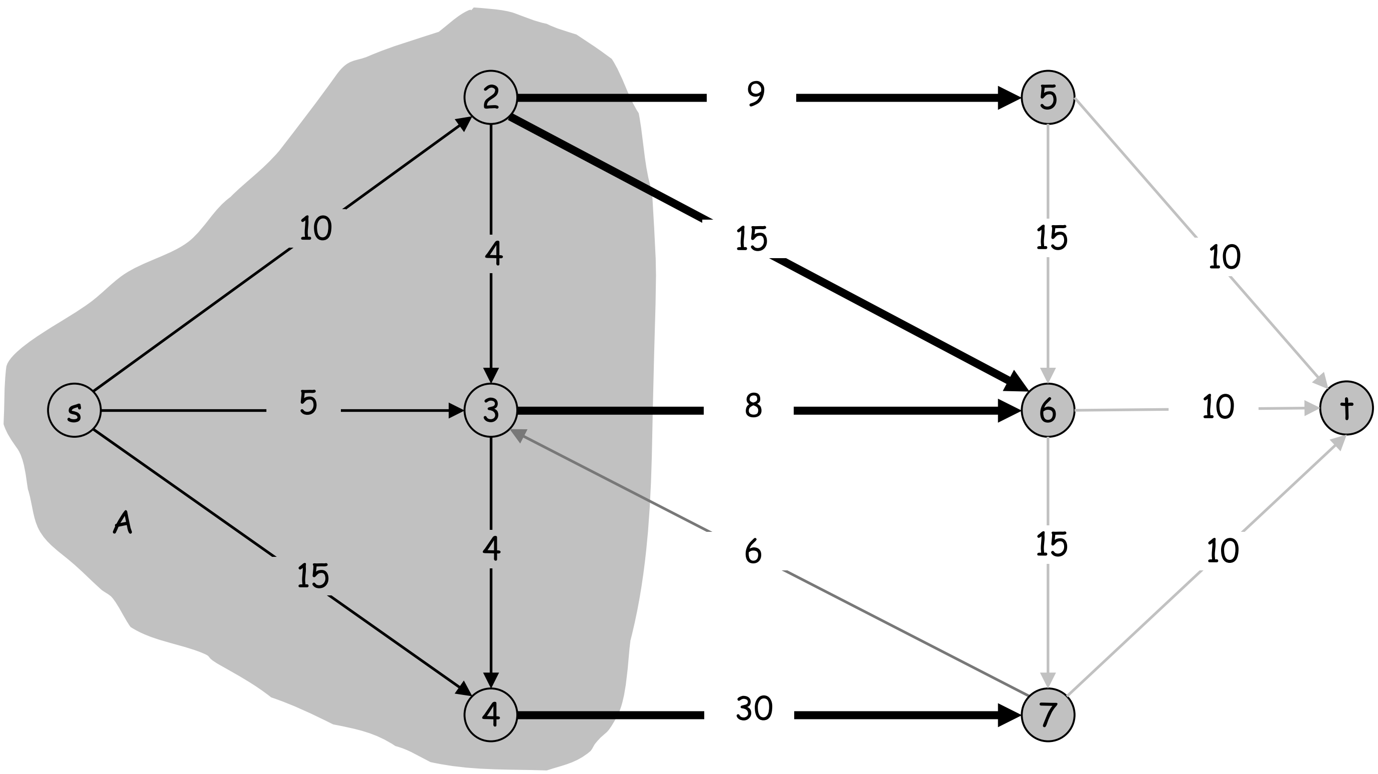
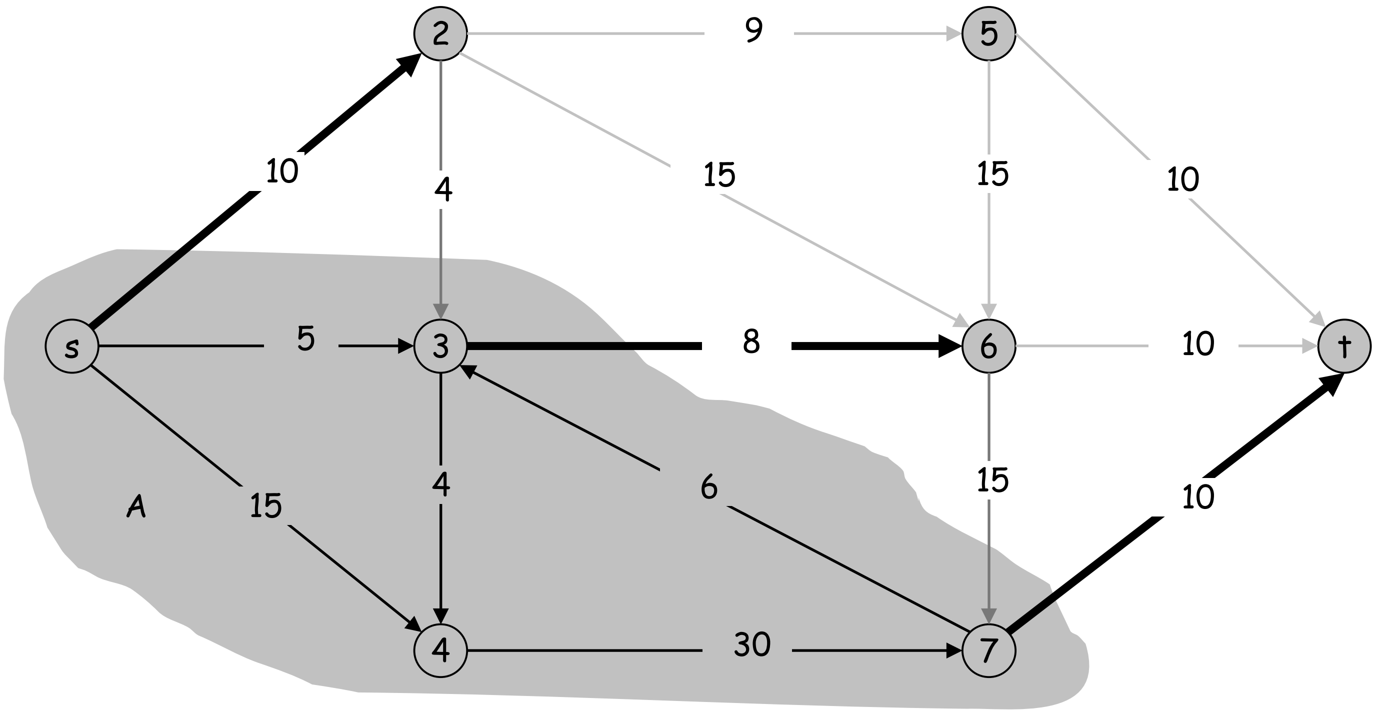
$$cap(A, B) = \sum_{e \text{ out of } A} c(e)$$



Capacity = 9 + 15 + 8 + 30
= 62

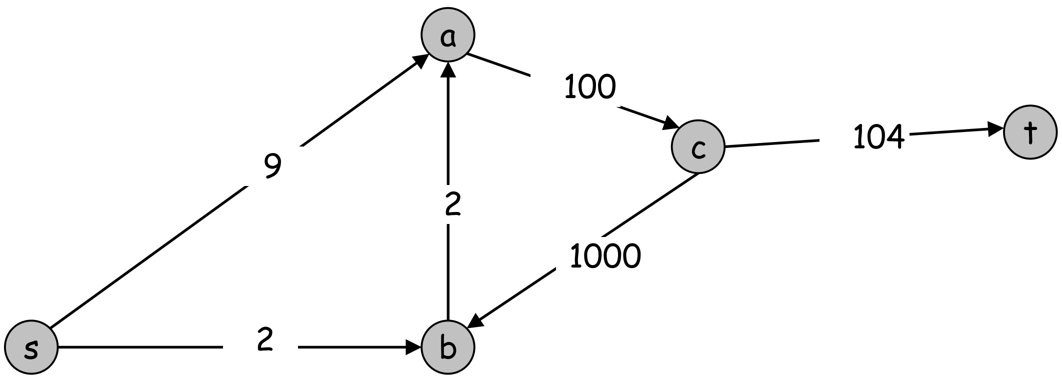
Minimum Cut Problem

Min s-t cut problem. Find an s-t cut of minimum capacity.



Capacity = 10 + 8 + 10
= 28

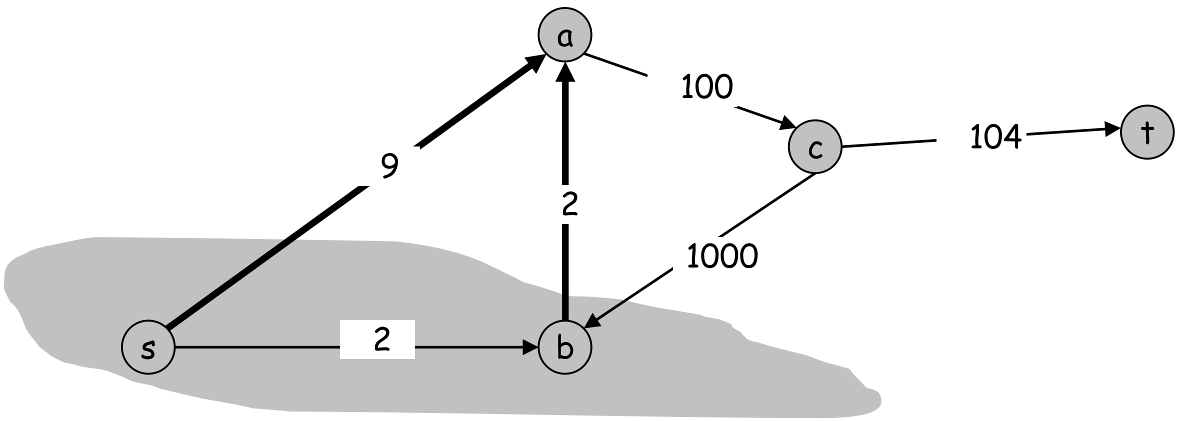
Clicker Question



Find the minimum capacity s-t cut (*A*, B=V*A*)

Choice A: *A*={s,b}, B={a,c,t}
Choice B: *A*={s,a}, B={b,c, t}
Choice C: *A*={s,a,b}, B={c, t}
Choice D: *A*={s,a,b,c,t}, B={}
Choice E: *A*={a,b}, B={s,c,t}

Clicker Question



Find the minimum capacity s-t cut (A, B=V\A)

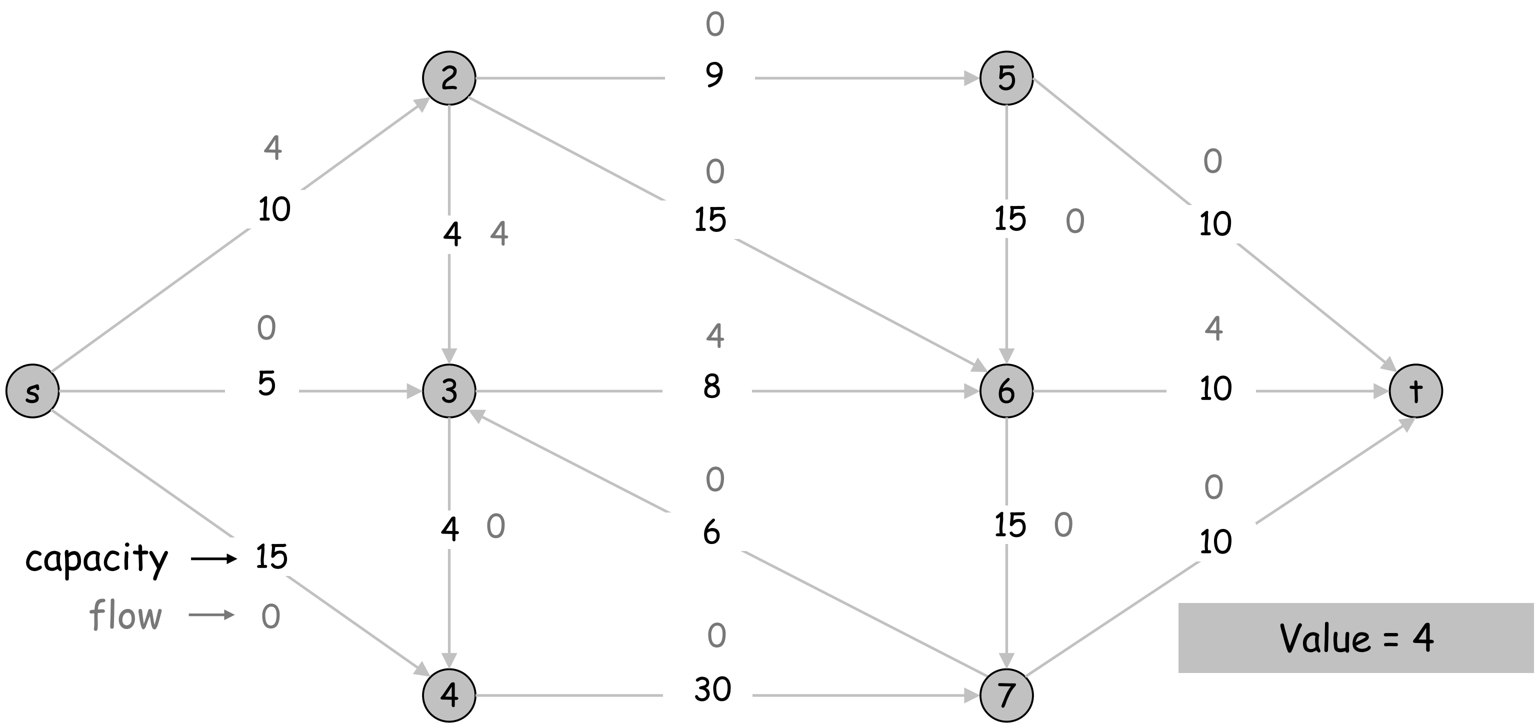
Choice A: A={s,b}, B={a,c,t}
Choice B: A={s,a}, B={b,c, t}
Choice C: A={s,a,b}, B={c, t}
Choice D: A={s,a,b,c,t}, B={} (Not a s-t cut)
Choice E: A={a,b}, B={s,c,t} (Not a s-t cut)

Flows

Def. An s-t flow is a function that satisfies:

- For each $e \in E$: $0 \leq f(e) \leq c(e)$ [capacity]
- For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ [conservation]

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



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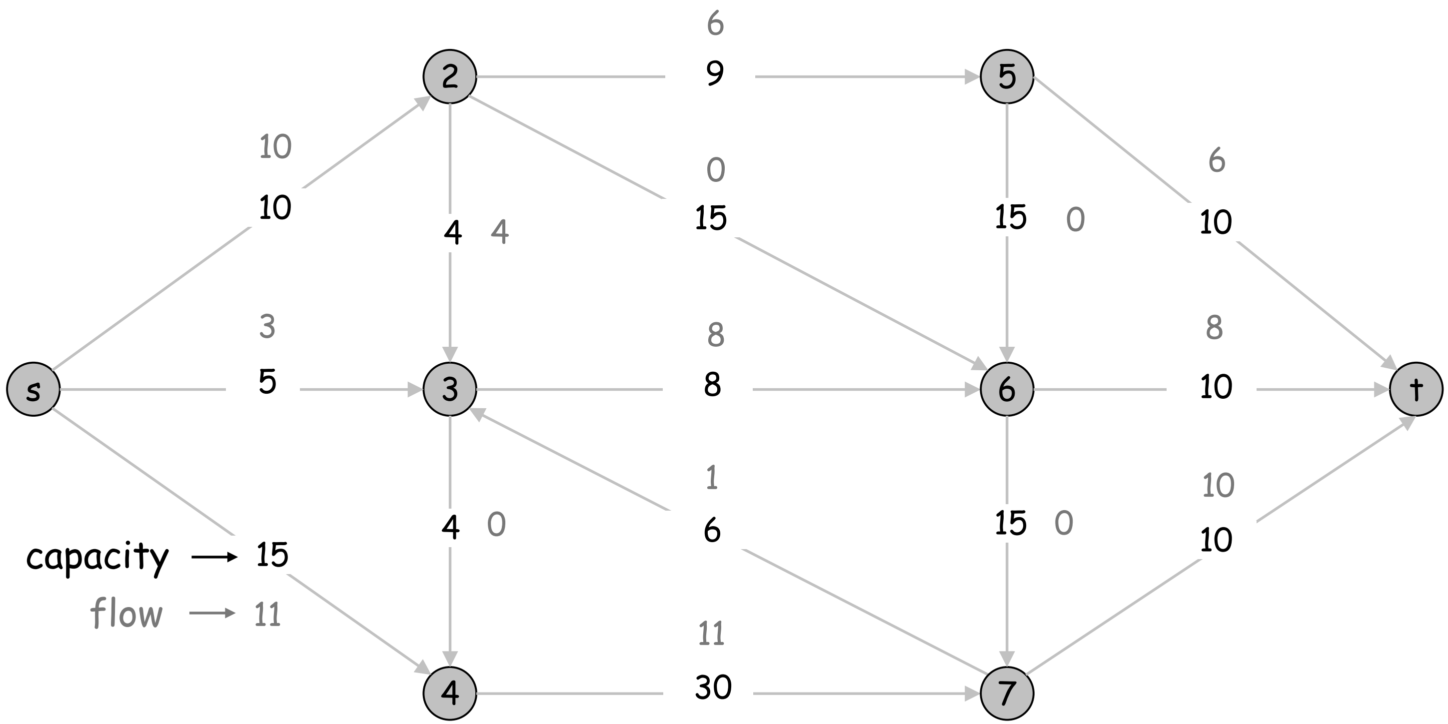
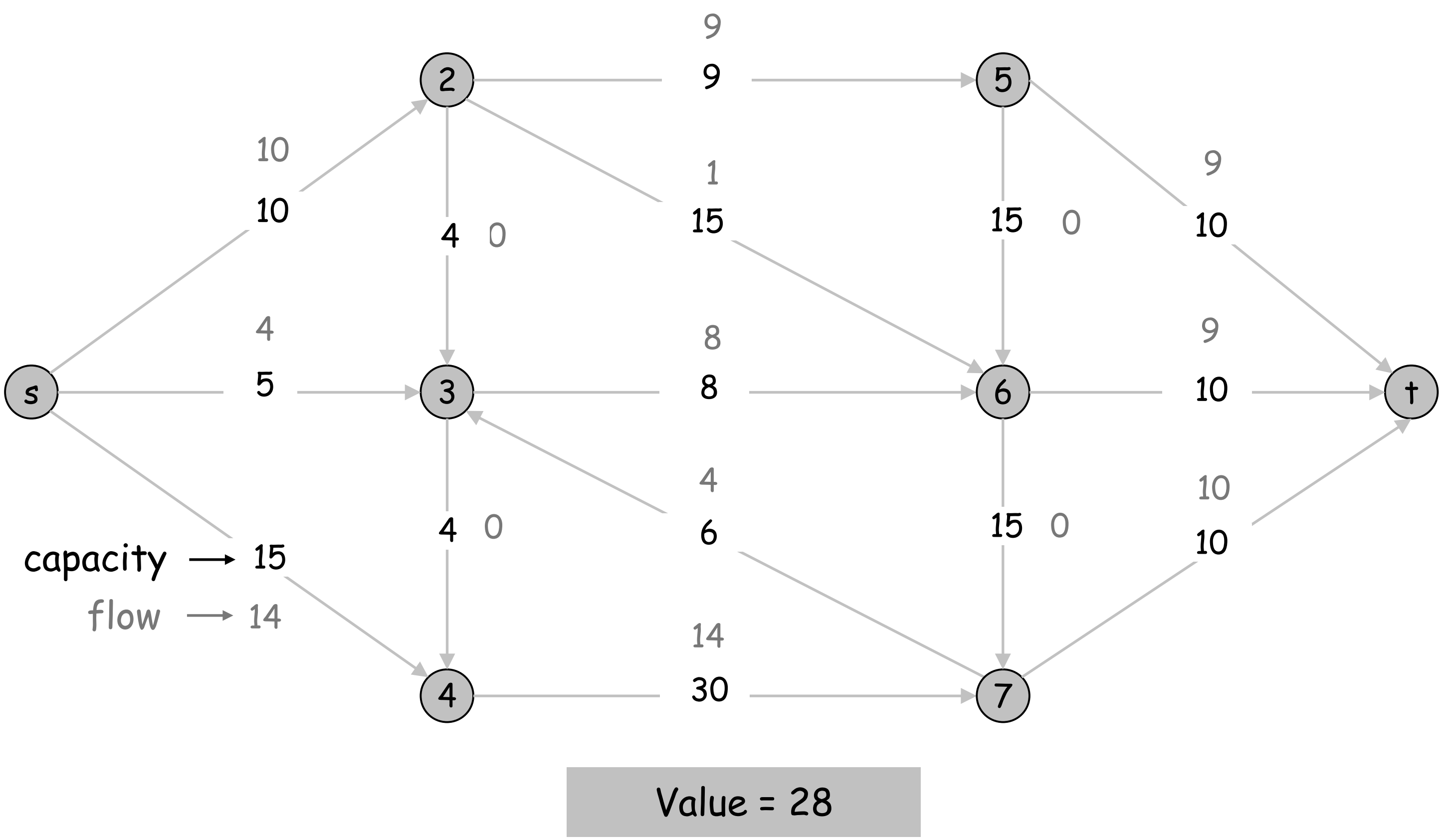


Value = 24

Maximum Flow Problem

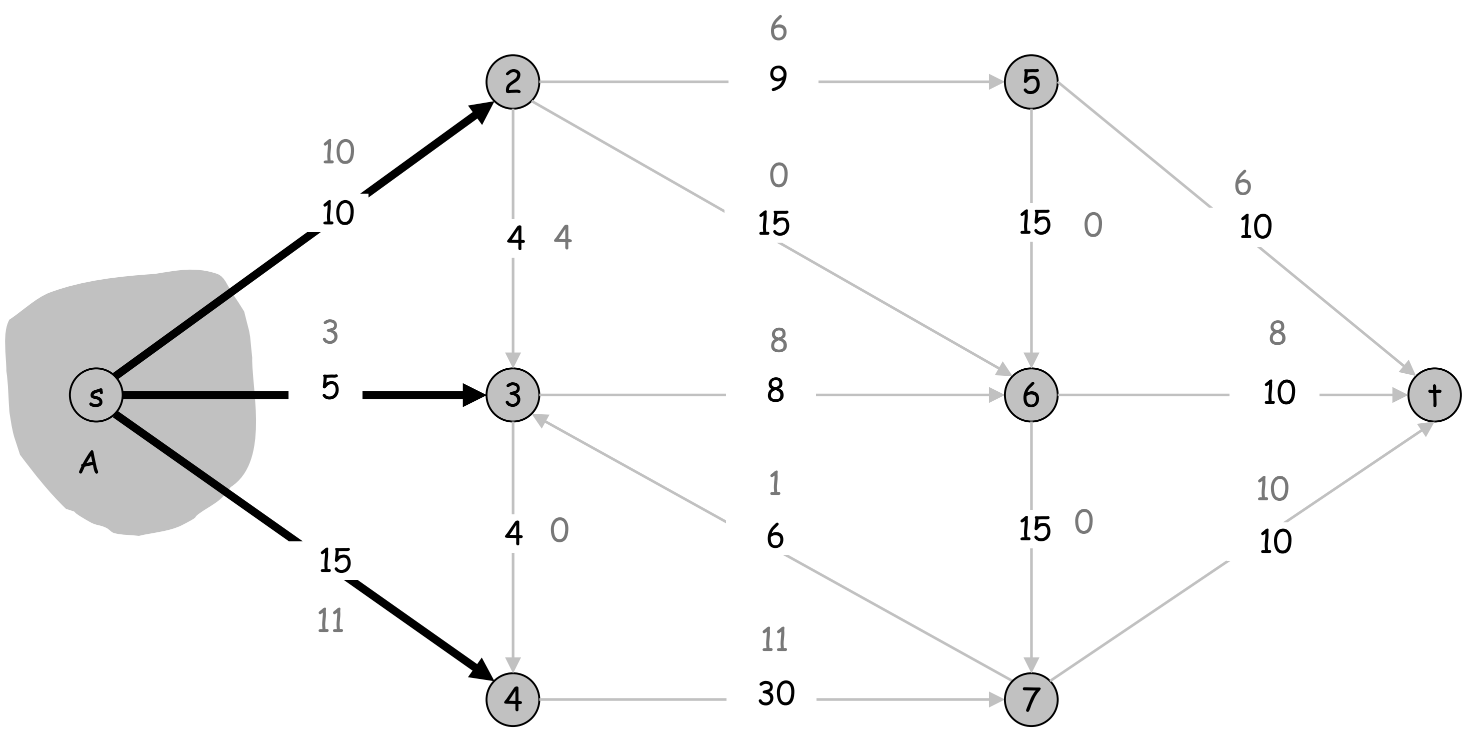
Max flow problem. Find s-t flow of maximum value.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to } A} f(e) \; = \; v(f)$$



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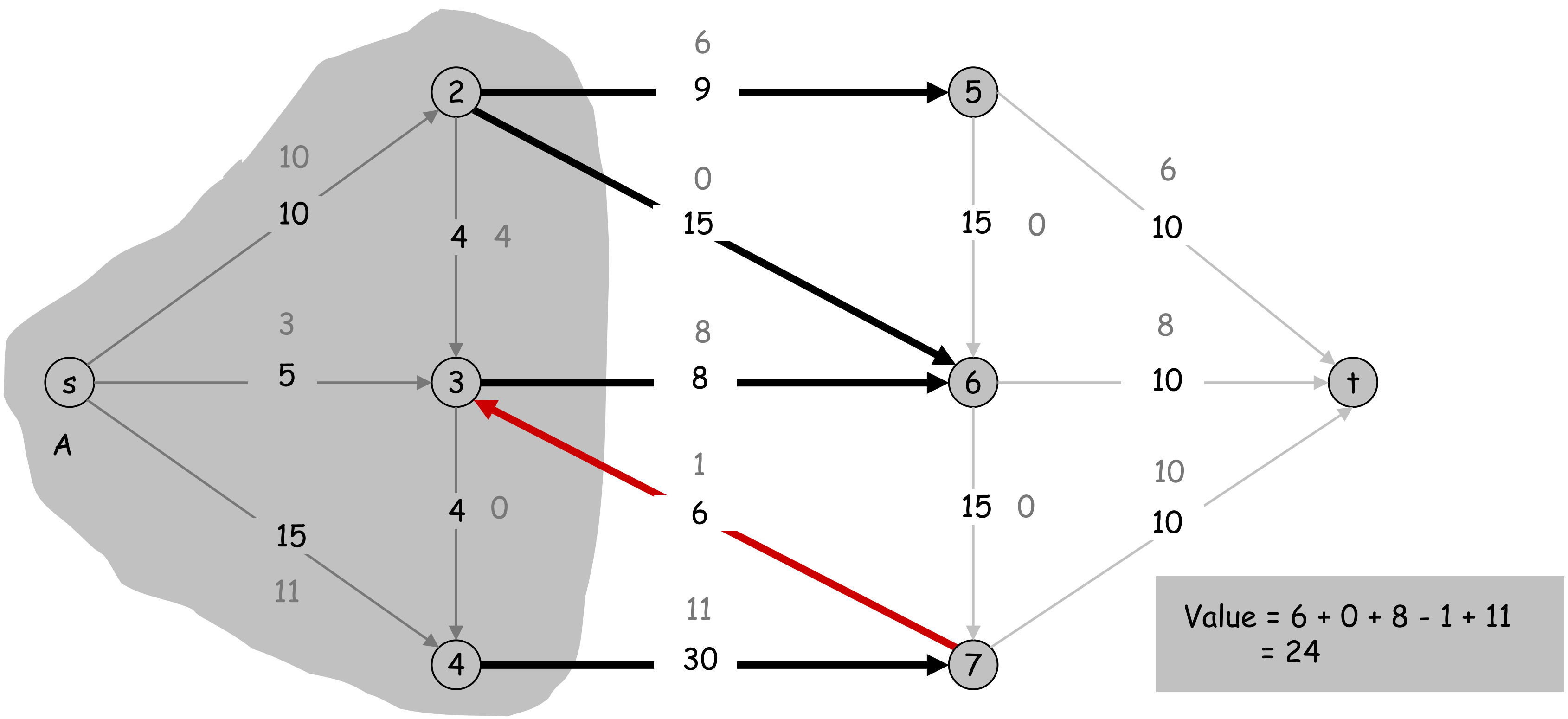

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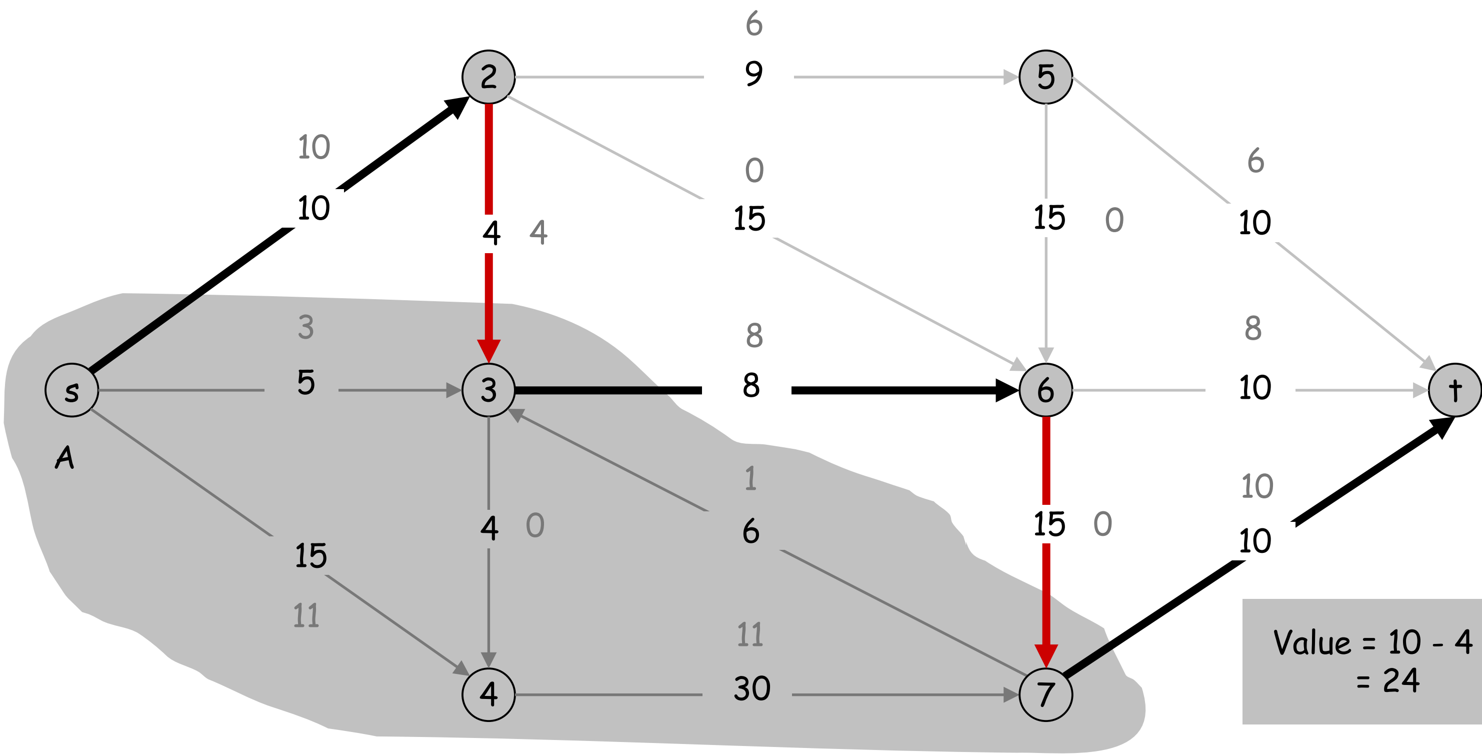


Value = 10 - 4 + 8 - 0 + 10
= 24

Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf. $v(f) = \sum_{\text{e out of s}} f(e) + 0$

$$= \sum_{\text{e out of s}} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{\text{e out of v}} f(e) - \sum_{\text{e in to v}} f(e) \right)$$

by flow conservation, all terms except v = s are 0

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{\text{e out of s}} f(e) + 0$$

$$= \sum_{\text{e out of s}} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{\text{e out of v}} f(e) - \sum_{\text{e in to v}} f(e) \right)$$

$$= \sum_{v \in A} \left(\sum_{\text{e out of v}} f(e) - \sum_{\text{e in to v}} f(e) \right)$$

↑ Flow into s is 0

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{\text{e out of s}} f(e) + 0$$

$$= \sum_{\text{e out of s}} f(e) + \sum_{v \in A \setminus \{s\}} \left(\sum_{\text{e out of v}} f(e) - \sum_{\text{e in to v}} f(e) \right)$$

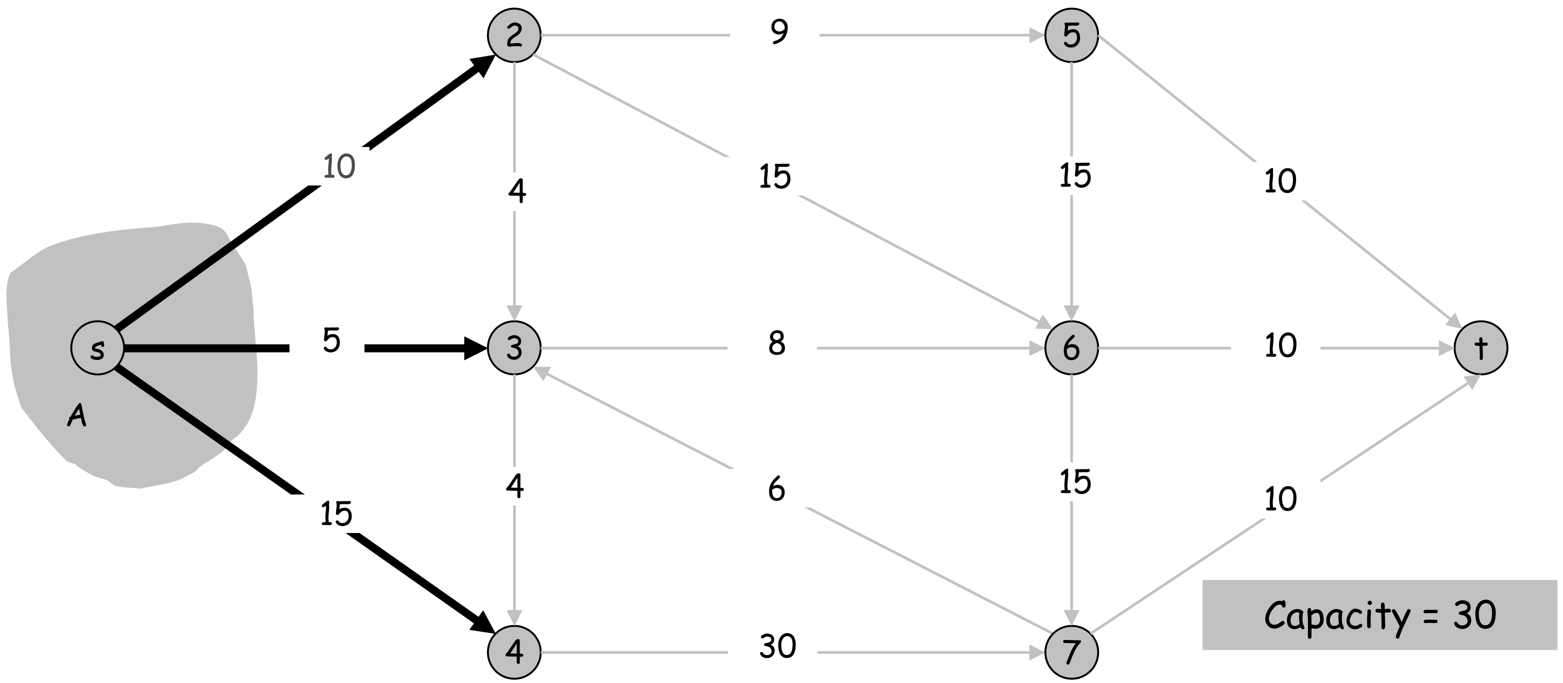
$$= \sum_{v \in A} \left(\sum_{\text{e out of v}} f(e) - \sum_{\text{e in to v}} f(e) \right)$$

$$= \sum_{\text{e out of A}} f(e) - \sum_{\text{e in to A}} f(e)$$

If e=(u,v) with u and v in A then f(e) was added & subtracted in prior sum

Weak duality. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 $\Rightarrow$ Flow value $\leq$ 30



Weak duality. Let f be any flow. Then, for any s-t cut (A, B) we have v(f) $\leq$ cap(A, B).

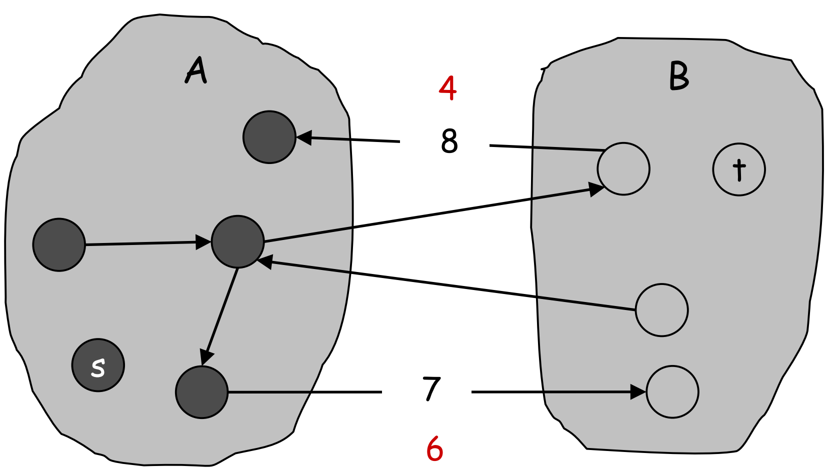
Pf.

$$v(f) = \sum_{\text{e out of A}} f(e) - \sum_{\text{e in to A}} f(e)$$

By Flow value lemma.

$$\leq \sum_{\text{e out of A}} f(e) \leq \sum_{\text{e out of A}} c(e)$$

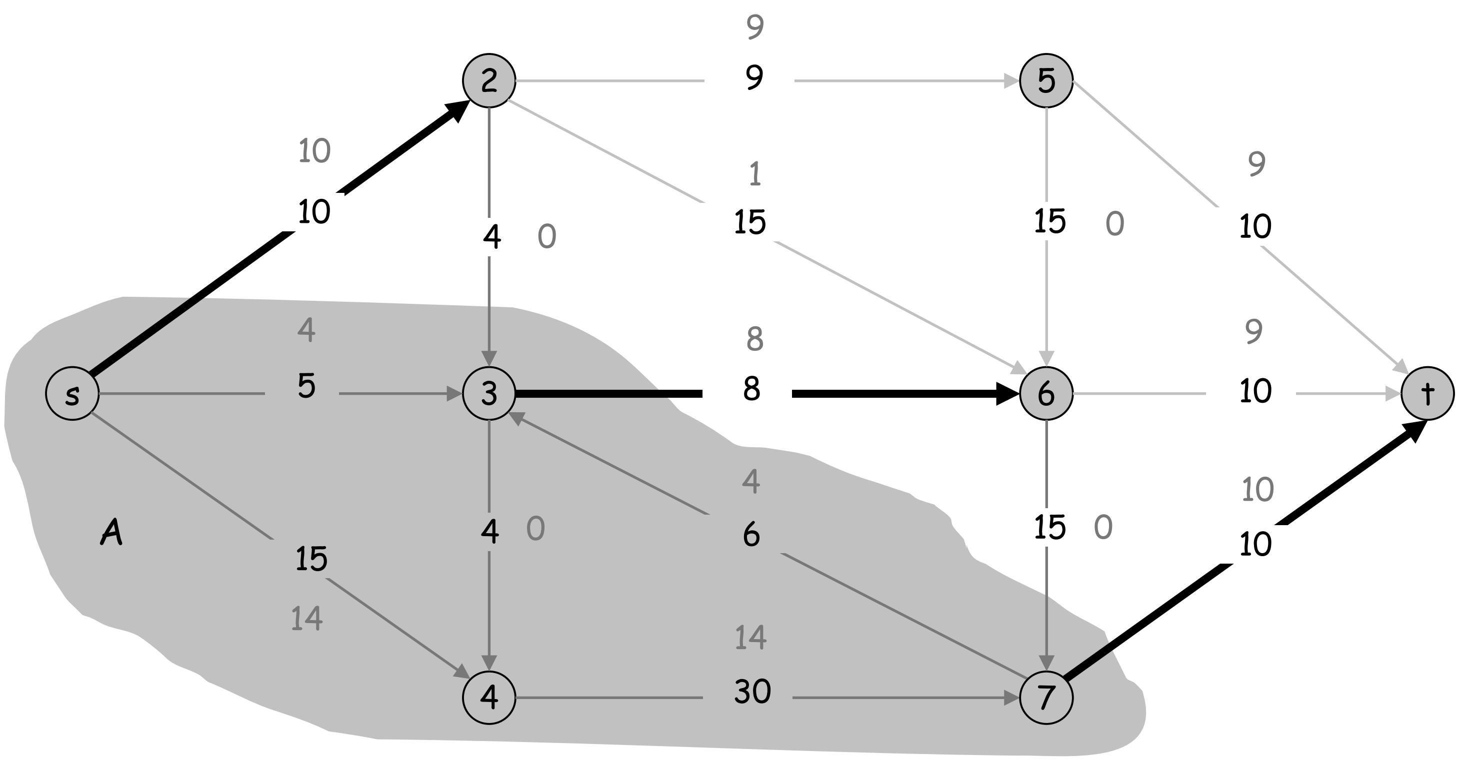
$$= \text{cap}(A, B) \qquad \blacksquare$$



Certificate of Optimality

Corollary. Let f be any flow, and let (A, B) be any cut.
If v(f) = cap(A, B), then f is a max flow and (A, B) is a min cut.

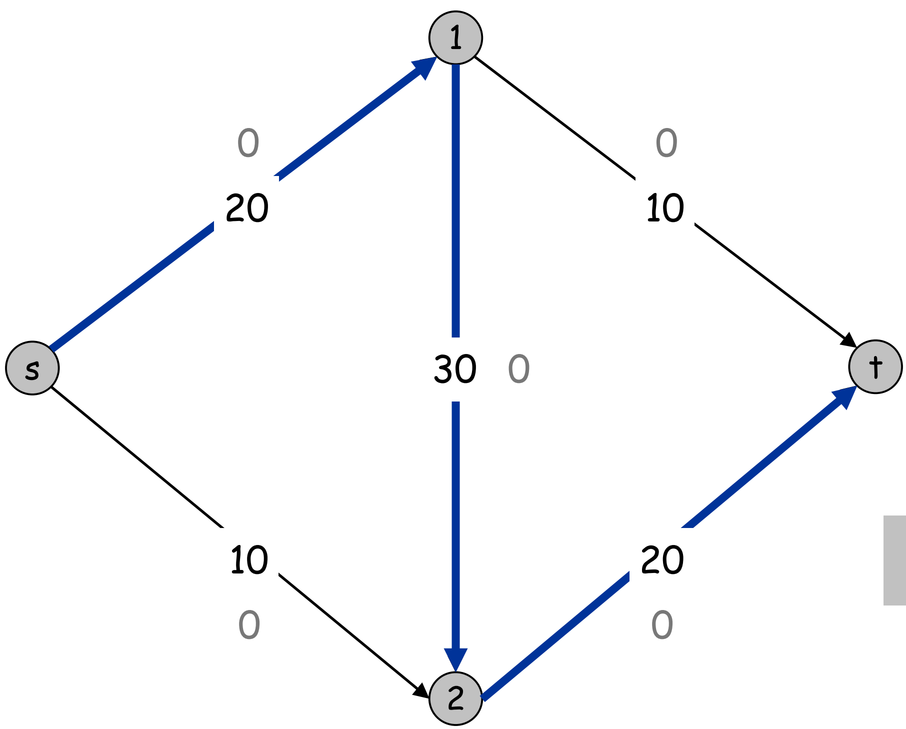
Value of flow = 28
Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28



Towards a Max Flow Algorithm

Greedy algorithm.

- Start with $f(e) = 0$ for all edge $e \in E$.
- Find an s-t path P where each edge has $f(e) < c(e)$.
- Augment flow along path P.
- Repeat until you get stuck.



Flow value = 0

Towards a Max Flow Algorithm

Greedy algorithm.

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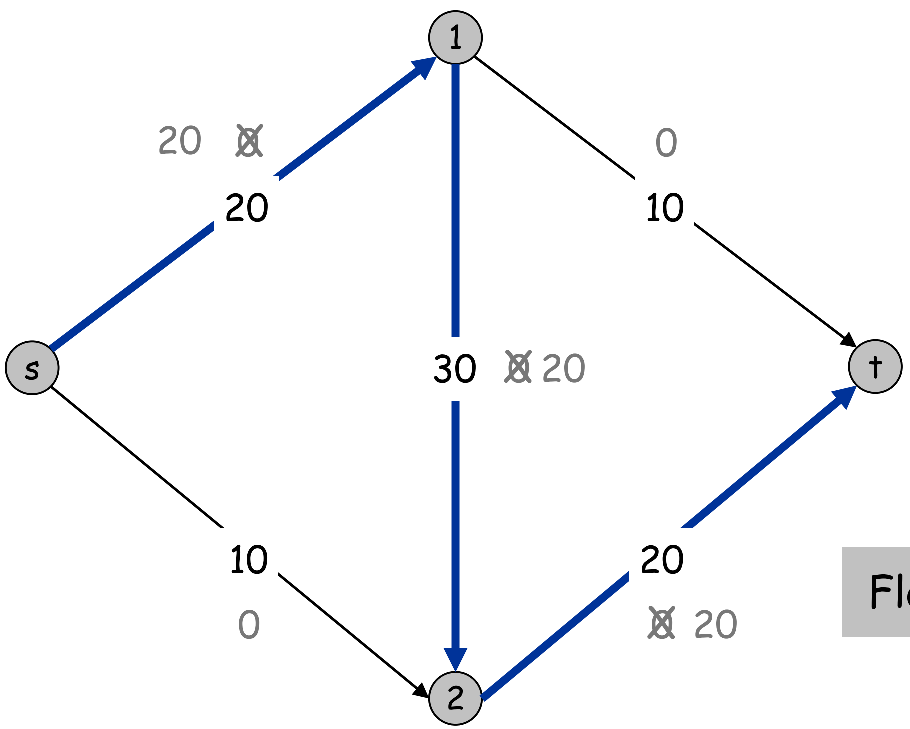

Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.

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- Find an s-t path P where each edge has f(e) < c(e).
- Augment flow along path P.
- Repeat until you get stuck.

↖ locally optimality ⇏ global optimality