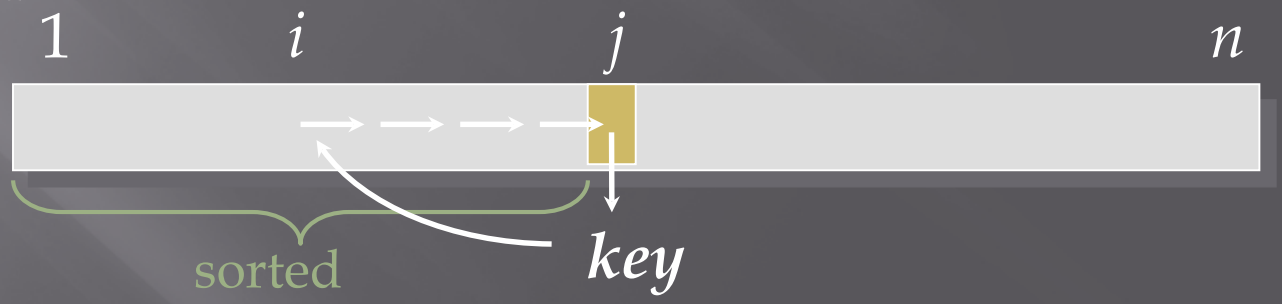
CS 381 – FALL 2019

Week 1.3, Friday, August 23

Insertion Sort

- create the sorted sequence in an incremental way
- start with a sorted sequence of length 1 and insert one more element in each iteration

INSERTION-SORT (A, n)
for $j \leftarrow 2$ to n do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] = key$



Clicker Question

Which of the following claims about insertion sort are true?

Claim 1: On input 1,2,3,4,…,n the algorithm will finish after $0(n)$ steps

Claim 2: On input n,n-1,…,1 the algorithm will finish after $0(n^2)$ steps

Claim 3: On input 1,…,n/2, n,n-1,…n/2+1, the algorithm will finish after $0(n\sqrt{n})$ steps

A. All of the above. **B.** None of the above. **C.** Claim 1 only. **D.** Claim 3 only. **E.** Claims 1 and 2 only.

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Which of the following claims about insertion sort are true?

Claim 1: On input 1,2,3,4,…,n the algorithm will finish after $O(n)$ steps

Claim 2: On input n,n-1,…,1 the algorithm will finish after $O(n^2)$ steps

~~**Claim 3:** On input 1,…,n/2, n,n-1,…n/2+1, the algorithm will finish after $O(n\sqrt{n})$ steps~~

A. All of the above. **B.** None of the above. **C.** Claim 1 only. **D.** Claim 3 only. **E.** Claims 1 and 2 only.

What is the running time of Insertion Sort?

Number of times the while-loop is executed depends on the input

- increasingly sorted input is fast; decreasing is slow.
- Worst case? $\sum_{j=2}^{n} j < n^2$
- Average case?

What all do we count/have to count when analyzing time?

- In (internal) sorting algorithm we generally count the number of comparison

Asymptotic Performance

Pseudo code has two nested loops

- while loop moves left from j to 1
- total time won't be more than quadratic.

Note: A doubly nested loop does not necessarily result in quadratic time

Worst case: $T(n) = O(n^2)$

- Work is bounded by summing the first n-1 integers which is equal to n(n-1)/2
- Time is proportional to n^2
- Also, $T(n) = \Theta(n^2)$

Insertion Sort: Correctness

INSERTION-SORT (A, n)
for $j \leftarrow 2$ to n do
Pre-Condition: A[1]≤ A[2] … ≤ $A[j-1]$
$key \leftarrow A[j]$
$i \leftarrow j - 1$
while $i > 0$ and $A[i] > key$ do
$A[i+1] \leftarrow A[i]$
$i \leftarrow i - 1$
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Post-Condition: A[1]≤ A[2] … ≤ A[j]

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Post-Condition: A[1]≤ A[2] … ≤ A[j]

Post-Condition when j=n → entire array A is sorted.

Insertion Sort: Correctness

Pre-Condition: A[1]≤ A[2] ... ≤ A[j − 1]

key ←*A*[*j*]

i ←*j* –1

Define A_{orig}[1,…,j-1] :=A[1,…,j-1]

while *i* > 0 and *A*[*i*] > *key* do

A[*i*+1] ←*A*[*i*]

i ←*i* –1

Invariant: A[1,…,i-1]= A_{orig}[1,…,i-1] (untouched)

A[i+2,…,j] = A_{orig}[i+1,…,j-1] (shift once*)

key < A[i+2]

A[*i*+1] = *key*

Post-Condition: A[1]≤ A[2] ... ≤ A[j]

Proofs

Let T(n) be a claim in which n is a positive integer.
Prove claim T(n) correct.

Common proof methods

- Direct proof
- Indirect proof
  - By contraposition, by contradiction
- Mathematical induction
  - weak and strong induction
  - Invariants

What do we need to prove in 381?

- Claim of a run time
  Could be a recurrence for a recursive solution, a sum for an iterative solution, an amortized analysis, etc.
- Correctness of your algorithm approach often an inductive argument (even for iterative solutions) or a proof by contradiction
- NP-completeness of a problem
- Lower bound of a problem

Weak Induction

- Basis: T(1) holds (basis is often 1 or 0, or a larger value)
- Induction hypothesis: for every n>1, assume T(n-1) holds
- Using the induction hypothesis, show that T(n) holds.
- It follows that T(n) holds for all n and the claim is proven.

Claim: $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$

Basis for n=1: 2=0+2 true

Assume claim holds for n-1: $\sum_{i=1}^{n-1} \boldsymbol{i2^i} = \boldsymbol{(n-2)2^n + 2}$

Prove the claim for n:

$\sum_{i=1}^{n} \boldsymbol{i2^i} = \boldsymbol{n2^n} + \sum_{i=1}^{n-1} \boldsymbol{i2^i}$ (Split the sum)

$= \boldsymbol{n2^n} + \boldsymbol{(n-2)2^n + 2}$ (Inductive Hyp)

$= \boldsymbol{(2n-2)2^n + 2}$ (algebra)

$= \boldsymbol{2(n-1)2^n + 2}$ (algebra)

$= \boldsymbol{(n-1)2^{n+1} + 2}$ (algebra)

QED

Strong Induction

- Basis: T(1) holds
- Induction hypothesis: For every n>1, assume the claim holds for k = 1, 2, 3, …, n-1
- Using the induction hypothesis, show that T(n) holds.
- It follows that T(n) holds for all n and the claim is proven.

Strong Induction Example

- Claim: Prove that every binary tree with n nodes has n-1 edges
- Let T(n) be the statement that any binary tree with n nodes has n-1 edges
- Base Case: n=1 (check)
- Inductive Hypothesis: For all j < n the statement T(j) holds
- Inductive Step: Prove that T(n) holds

Strong Induction Example

- Let T be a tree with n > 1 nodes and let u be the root of the tree.
- Case 1: u has 1 child v
  - Let T_v be tree rooted at v.
  - Since, T_v has n-1 nodes, by IH T_v has n-2 edges.
  - Total edges: 1+n-2=n-1
- Case 2: u has 2 children w and v
  - Let T_w (resp. T_v) be the corresponding trees with n_w (resp. n_v) nodes with $n_w + n_v = n - 1$.
  - By (Strong) IH T_w (resp. T_v) has $n_w - 1$ edges (resp. $n_v - 1$ edges).
  - Total Edges: $2 + n_w - 1 + n_v - 1 = n_w + n_v = n - 1$

QED

Other Relevant/Common Sums

- $\sum_{i=1}^{n} i$
- $\sum_{i=1}^{n} i^2$
- $\sum_{i=1}^{n} i^k$ k constant
- $\sum_{i=1}^{n} 2^i$
- $\sum_{k=1}^{n} x^k$

…

Exact bounds and asymptotic bounds
CLRS, 182 Text, TCS Cheat Sheet, etc.

Claim: n! > 2^n for n>3

Base case n=4

- $4! = 24 > 2^4 = 16$

Induction hypothesis:

For any $n = k > 3$ it holds that $k! > 2^k$

Show the claim for k+1:

$$(k+1)! = (k+1) \cdot k! > (k+1) \cdot 2^k \quad \text{(by IH)}$$
$$> 2 \cdot 2^k \quad \text{(since } k > 3\text{)}$$
$$\geq 2^{k+1}$$

Analysis of Algorithms

Worst case analysis

- in an asymptotic sense, the maximum time the algorithm takes on any input of size n.

Average case analysis

- expected time; often meaningful
- may need assumptions on the statistical distribution of input data

Best case analysis

- does not mean much; generally easy to determine

In some case, the three bounds are identical

- means performance does not depend on the value of the data
- For some algorithms, average case performance is only known experimentally.

What do we count?

- Time and space
  - time in terms of number of basic operations on basic data types
- Ignore machine dependent factors, but remain realistic
- Random Access Model (RAM)
  - no concurrency
  - count instructions (arithmetic operation, comparison, data movement)
  - each instruction takes constant time
  - realistic assumption on the size of the numbers (to represent n, it takes log n bits)

Asymptotic notation: Big-O

$O(g(n)) = \{ f(n) \mid$ there exist positive constants c and n_0 such that $0 \le f(n) \le c\, g(n)$ or all $n \ge n_0\}$

We write $f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$ such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

$4n + 23\log n - 28 = O(n)$

- Drops low-order terms
- Ignores leading constants
- May not hold for small values of n

$f(n) = O(g(n))$ if there exist constants $c > 0$, $n_0 > 0$

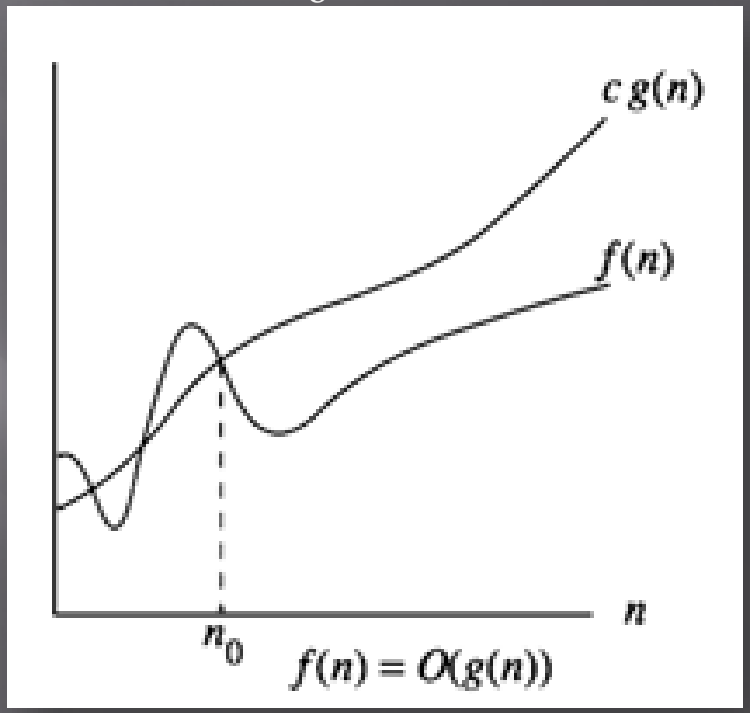
such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0$.

$f(n) = 3n^2 - 4n + 512$

$\le 3n^2 + 512$

$\le 4n^2$ for $n \ge 23$

- $f(n) = O(n^2)$
- $f(n) = O(n^3)$ also holds
- $f(n) = O(n)$ is false



CLRS text Figure 3.1

Which statements are true?

$3n^3 + 90n^2 - 5n = O(n^3)$

$3n^3 + 90n^2 - 5n = O(2^n)$

$3n^3 + 90n^2 - 5n = O(n^2)$

$5 \log n = O(n)$

$\sqrt{n} = O(\log n^8)$

$n \log n = O(n)$

$4n = O(n \log n)$

$n / \log n = O(\sqrt{n})$

$3n^3 + 90n^2 - 5n = O(n^3)$	true
$3n^3 + 90n^2 - 5n = O(2^n)$	true
$3n^3 + 90n^2 - 5n = O(n^2)$	false
$5 \log n = O(n)$	true
$\sqrt{n} = O(\log n^8)$	false
$n \log n = O(n)$	false
$4n = O(n \log n)$	true
$n / \log n = O(\sqrt{n})$	false

Consider two running times: $4n\log n$ and $8nn^{1/8}$

Which relationships hold?

1. $4n\log n = O(8nn^{1/8})$
2. $8nn^{1/8} = O(4n\log n)$
3. $4n\log n = \Theta(8nn^{1/8})$
4. $8nn^{1/8} = \Theta(4n\log n)$

A. None
B. 1
C. 2
D. 1 and 3
E. 4

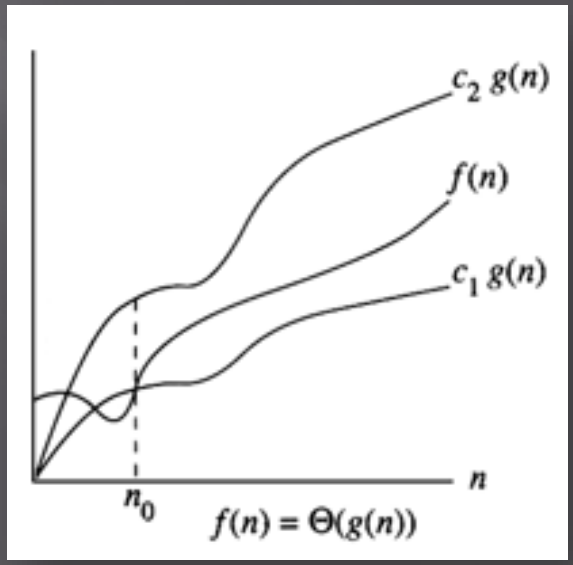
Asymptotic Bounds

$O(g(n)) = \{ f(n) \mid$ there exist positive constants c and n_0 such that $0 \leq f(n) \leq c\, g(n)$ for all $n \geq n_0\}$

O captures upper bounds

$\Theta(g(n)) = \{ f(n) \mid$ there exist positive constants c_1, c_2, and n_0 such that $0 \leq c_1\, g(n) \leq f(n) \leq c_2\, g(n)$ for all $n \geq n_0\}$

Θ captures upper and lower bounds



Examples

- $3n^3 + 90n^2 – 5n$ is $O(n^3)$ and $\Theta(n^3)$ is true
- $3n^3 + 90n^2 – 5n$ is $O(2^n)$ true, but $\Theta(2^n)$ false
- 5 log n is O(n) true, but $\Theta(n)$ false
- 4n = O(n log n) is true, but $\Theta(n \log n)$ false

Asymptotic Bounds

$O(g(n)) = \{ f(n) \mid$ there exist positive constants c and n_0 such that $0 \le f(n) \le c\, g(n)$ for all $n \ge n_0\}$

O captures upper bounds

$\Theta(g(n)) = \{ f(n) \mid$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0\}$

Θ captures upper and lower bounds

$\Omega(g(n)) = \{ f(n) \mid$ there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$

Ω captures lower bounds

$4n \log n = \Omega(n)$

Note

- We will generally assume that n is "nice"
  - E.g., power of 2
  - We are not implementing the algorithms and only need to consider crucial the boundary/special cases
- When asked to design an efficient algorithm
  - sometimes you will be given a target asymptotic bound
  - other times you need to find the "best" one
- You can use known data structures
  - State how they are implemented and give time bounds of operations

How many times is F called?

Assume n is a power of 4 ($n=4^k$)

```
while n > 1 do
    for i = 1 to n do
        F(i,n)
    n = n/4
```

$O(n \log n)$ $\Theta(n \log n)$

$O(n^2)$ $\Theta(n^2)$

$O(n)$ $\Theta(n)$

$O(\log n)$ $\Theta(\log n)$

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$O(n^2)$ $\Theta(n^2)$

$O(n)$ $\Theta(n)$

$O(\log n)$ $\Theta(\log n)$

Useful Fact: Geometric Series

Fact: Suppose $0 < x < 1$ then $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$

Proof: $\sum_{i=0}^{\infty} x^i = \frac{1-x}{1-x} \sum_{i=0}^{\infty} x^i$

$$= \frac{1}{1-x}\left(\sum_{i=0}^{\infty} x^i - \sum_{i=0}^{\infty} x^{i+1}\right)$$

$$= \frac{1}{1-x}\left(\sum_{i=0}^{\infty} x^i - \sum_{i=1}^{\infty} x^i\right) = \frac{1}{1-x}$$

Example: $x < \frac{1}{4}$ we have $\sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^{-i} = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$

How many times is F called?

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```
while n > 1 do
    for i = 1 to n do
        F(i,n)
    n = n/4
```

O(n log n) ~~Θ(n log n)~~

O(n^2) ~~Θ(n^2)~~

O(n) Θ(n)

~~O(log n)~~ ~~Θ(log n)~~