Partially Ordered Sets

Background for Program Analysis

Partial ordering

- A *partial ordering* is a relation:
  $$\sqsubseteq : L \times L \rightarrow \{\text{true}, \text{false}\}$$
- that is:
  - ***reflexive***: $\forall l : l \sqsubseteq l$
  - ***transitive***: $\forall l_1, l_2, l_3 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_3 \Rightarrow l_1 \sqsubseteq l_3$
  - ***anti-symmetric***: $\forall l_1, l_2 : l_1 \sqsubseteq l_2 \wedge l_2 \sqsubseteq l_1 \Rightarrow l_1 = l_2$

Partially ordered set

- A *partially ordered set* $(L, \sqsubseteq)$

  is a set L equipped with a partial ordering $\sqsubseteq$
- $l_2 \sqsupseteq l_1 \equiv l_1 \sqsubseteq l_2$
- $l_1 \sqsubset l_2 \equiv l_1 \sqsubseteq l_2 \wedge l_1 \neq l_2$

Bounds

- $Y \subseteq L$ has $l \in L$ as an *upper bound* if

$$\forall l' \in Y : l' \sqsubseteq l$$

  and $l \in L$ as a *lower bound* if

$$\forall l' \in Y : l' \sqsupseteq l$$

Least upper bound

- A *least upper bound* l of Y is an upper bound of Y that satisfies $l \sqsubseteq l_0$ whenever l_0 is another upper bound of Y
- A subset Y need not have a least upper bound but if one exists then it is unique (since $\sqsubseteq$ is anti-symmetric)
- The least upper bound of Y is written $\bigsqcup Y$
- $l_1 \sqcup l_2 \equiv \bigsqcup\{l_1, l_2\}$

Greatest lower bound

- A *greatest lower bound* l of Y is a lower bound of Y that satisfies $l_0 \sqsubseteq l$ whenever l_0 is another lower bound of Y
- A subset Y need not have a greatest lower bound but if one exists then it is unique (since $\sqsubseteq$ is anti-symmetric)
- The greatest lower bound of Y is written $\bigsqcap Y$
- $l_1 \sqcap l_2 \equiv \bigsqcap\{l_1, l_2\}$

Complete lattice

- A *complete lattice* $L = (L, \sqsubseteq) = (L, \sqsubseteq, \bigsqcap, \bigsqcup, \bot, \top)$ is a partially ordered set $(L, \sqsubseteq)$ where all subsets have an lub and a glb
- $\bot = \bigsqcup \emptyset = \bigsqcap L$ is the *least element*
- $\top = \bigsqcap \emptyset = \bigsqcup L$ is the *greatest element*

Example

- For some set S

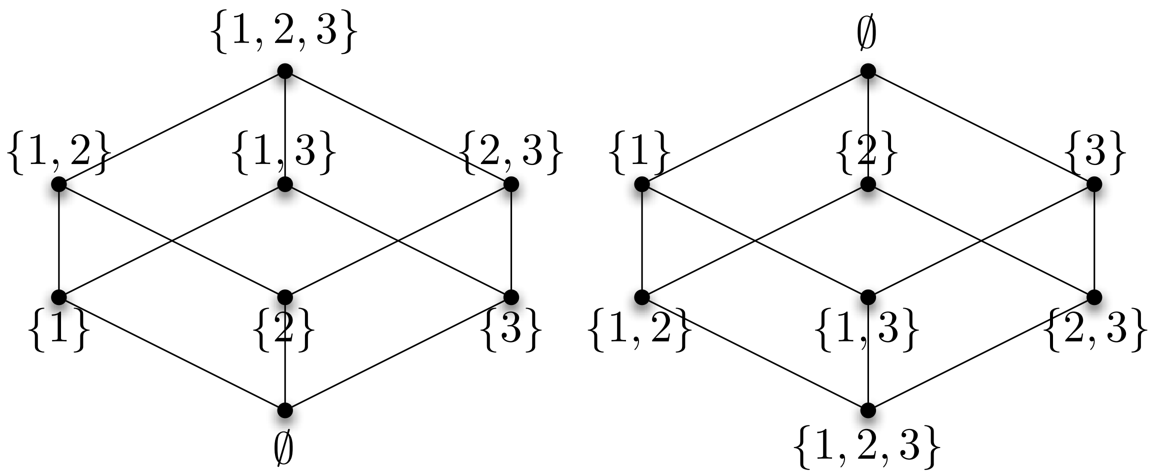
$$L = (\mathcal{P}(S), \subseteq) \quad \Rightarrow \quad \begin{array}{rcl} \sqsubseteq & \equiv & \subseteq \\ \bigsqcup Y & = & \bigcup Y \\ \bigsqcap Y & = & \bigcap Y \\ \bot & = & \emptyset \\ \top & = & S \end{array}$$

$$L = (\mathcal{P}(S), \supseteq) \quad \Rightarrow \quad \begin{array}{rcl} \sqsubseteq & \equiv & \supseteq \\ \bigsqcup Y & = & \bigcap Y \\ \bigsqcap Y & = & \bigcup Y \\ \bot & = & S \\ \top & = & \emptyset \end{array}$$

both are complete lattices

Example 1

$$S = \{1, 2, 3\}$$



$L = (\mathcal{P}(S), \subseteq)$ $\quad$ $L = (\mathcal{P}(S), \supseteq)$

Lemma 2

For a partially ordered set $L = (L, \sqsubseteq)$ the claims:

(i) L is a complete lattice

(ii) every subset of L has a lub

(iii) every subset of L has a glb

are equivalent

Proof

Clearly, (i) implies (ii) and (iii).

To show that (ii) implies (i) let $Y \subseteq L$ and define

$$\sqcap Y = \sqcup \{l \in L \mid \forall l' \in Y : l \sqsubseteq l'\}$$

Now, prove this defines a glb:

RHS set elements are all lbs so the equation defines a lb. Since any lb will be in the set it follows that the equation defines the glb.

Similarly, for $\sqcup Y = \sqcap \{l \in L \mid \forall l' \in Y : l' \sqsubseteq l\}$ ■

Moore family

- A *Moore family* is a subset Y of a complete lattice $L = (L, \sqsubseteq)$ that is closed under glbs:

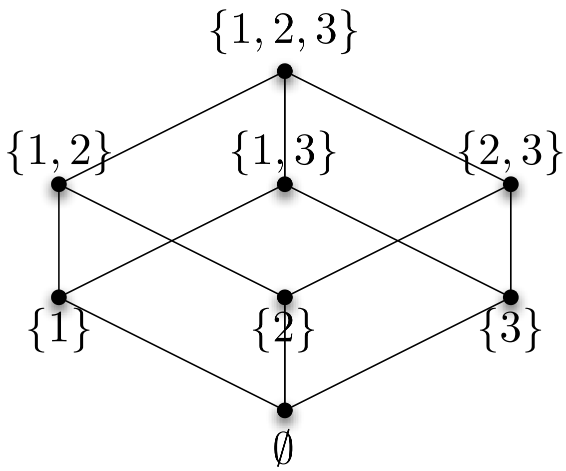
$$\forall Y' \subseteq Y : \sqcap Y' \in Y$$

- Thus, a Moore family always contains a least element, $\sqcap Y$, and a greatest element, $\sqcap \emptyset$, which equals the greatest element, $\top$, from L
- A Moore family is never empty

Example 3

Consider the complete lattice $L = (\mathcal{P}(S), \subseteq)$

$S = \{1, 2, 3\}$



$\{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$

$\{\emptyset, \{1,2,3\}\}$

are both Moore families

$\{\{1\}, \{2\}\}$

$\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

are not Moore families

Properties of functions

- A function $f : L_1 \to L_2$ between posets

  $L_1 = (L_1, \sqsubseteq_1)$ and $L_2 = (L_2, \sqsubseteq_2)$
- is *onto* if $\forall l_2 \in L_2 : \exists l_1 \in L_1 : f(l_1) = l_2$
- is 1-1 if $\forall l, l' \in L_1 : f(l) = f(l') \Rightarrow l = l'$
- is *monotone* if $\forall l, l' \in L_1 : l \sqsubseteq_1 l' \Rightarrow f(l) \sqsubseteq_2 f(l')$
- is *additive* if $\forall l_1, l_2 \in L_1 : f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$
- is *multiplicative* if

  $\forall l_1, l_2 \in L_1 : f(l_1 \sqcap l_2) = f(l_1) \sqcap f(l_2)$

Properties of functions

- The function f is *completely additive* if for all $Y \subseteq L_1$:

$$f(\bigsqcup_1 Y) = \bigsqcup_2 \{f(l') \mid l' \in Y\}$$

whenever $\bigsqcup_1 Y$ exists

- It is *completely multiplicative* if for all $Y \subseteq L_1$:

$$f(\bigsqcap_1 Y) = \bigsqcap_2 \{f(l') \mid l' \in Y\}$$

whenever $\bigsqcap_1 Y$ exists

Properties of functions

- Clearly $\bigsqcup_1 Y$ and $\bigsqcap_1 Y$ exist when L_1 is a complete lattice
- When L_2 is not a complete lattice the above statements also require the appropriate lubs and glbs to exist in L_2
- The function is *affine* if for all $Y \subseteq L_1, Y \neq \emptyset$

$$f(\bigsqcup_1 Y) = \bigsqcup_2 \{f(l') \mid l' \in Y\}$$

whenever $\bigsqcup_1 Y$ exists (and $Y \neq \emptyset$)

Properties of functions

- The function is *affine* if for all $Y \subseteq L_1, Y \neq \emptyset$
  $$f(\bigsqcup_1 Y) = \bigsqcup_2 \{f(l') \mid l' \in Y\}$$
  whenever $\bigsqcup_1 Y$ exists (and $Y \neq \emptyset$)
- The function is *strict* if $f(\bot_1) = \bot_2$
- Note that a function is completely additive iff it is both affine and strict

Lemma 4

If $L = (L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ and $M = (M, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ are complete lattices and M is finite then the three conditions:

(i) $\gamma : M \to L$ is monotone

(ii) $\gamma(\top) = \top$, and

(iii) $\gamma(m_1 \sqcap m_2) = \gamma(m_1) \sqcap \gamma(m_2)$ whenever $m_1 \not\sqsubseteq m_2 \wedge m_2 \not\sqsubseteq m_1$

are jointly equivalent to $\gamma : M \to L$ being completely multiplicative

Proof

First note that if γ is completely multiplicative then all three conditions hold.

For the converse note that by monotonicity of γ we have $\gamma(m_1 \sqcap m_2) = \gamma(m_1) \sqcap \gamma(m_2)$ also when $m_1 \sqsubseteq m_2 \vee m_2 \sqsubseteq m_1$
By induction on the (finite) cardinality of $M' \subseteq M$ we prove that $\gamma(\bigsqcap M') = \bigsqcap\{\gamma(m) \mid m \in M'\}$

If the cardinality of M' is 0 then the equation follows from (ii). If the cardinality of M' is larger than 0 then we write $M' = M'' \cup \{m''\}$ where $m'' \notin M''$ to ensure that the cardinality of M'' is strictly less than that of M'; hence:

$$\begin{array}{rcl} \gamma(\bigsqcap M') & = & \gamma((\bigsqcap M'') \sqcap m'') \\ & = & \gamma(\bigsqcap M'') \sqcap \gamma(m'') \\ & = & (\bigsqcap\{\gamma(m) \mid m \in M''\}) \sqcap \gamma(m'') \\ & = & \{\gamma(m) \mid m \in M'\} \end{array}$$

■

Lemma 5

A function $f : (\mathcal{P}(D), \subseteq) \to (\mathcal{P}(E), \subseteq)$ is *affine* iff there exists a function $\varphi : D \to \mathcal{P}(E)$ and an element $\varphi_\emptyset \in \mathcal{P}(E)$ such that

$$f(Y) = \bigcup\{\varphi(d) \mid d \in Y\} \cup \varphi_\emptyset$$

The function f is completely additive iff additionally $\varphi_\emptyset = \emptyset$

Proof

Suppose f is as displayed and assume $\mathcal{Y} \neq \emptyset$; then

$$\begin{aligned}
\bigcup\{f(Y) \mid Y \in \mathcal{Y}\} &= \bigcup\{\bigcup\{\varphi(d) \mid d \in Y\} \cup \varphi_\emptyset \mid Y \in \mathcal{Y}\} \\
&= \bigcup\{\bigcup\{\varphi(d) \mid d \in Y\} \mid Y \in \mathcal{Y}\} \cup \varphi_\emptyset \\
&= \bigcup\{\varphi(d) \mid d \in \bigcup\mathcal{Y}\} \cup \varphi_\emptyset \\
&= f(\bigcup\mathcal{Y})
\end{aligned}$$

showing that f is affine.

Next, suppose that f is affine and define $\varphi(d) = f(\{d\})$ and $\varphi_\emptyset = f(\emptyset)$. For $Y \in \mathcal{P}(D)$ let $\mathcal{Y} = \{\{d\} \mid d \in Y\} \cup \{\emptyset\}$ and note that $Y = \bigcup \mathcal{Y}$ and $\mathcal{Y} \neq \emptyset$. Then

$$\begin{aligned} f(Y) &= f(\bigcup \mathcal{Y}) \\ &= \bigcup(\{f(\{d\}) \mid d \in Y\} \cup \{f(\emptyset)\}) \\ &= \bigcup(\{\varphi(d) \mid d \in Y\} \cup \{\varphi_\emptyset\}) \\ &= \bigcup\{\varphi(d) \mid d \in Y\} \cup \varphi_\emptyset \end{aligned}$$

so f can be written in the required form. Completely additive follows straightforwardly.

■

Isomorphism

An *isomorphism* from a poset $(L_1, \sqsubseteq_1)$ to a poset $(L_2, \sqsubseteq_2)$ is a *monotone* function $\theta : L_1 \to L_2$ such that there exists a (necessarily unique) monotone function $\theta^{-1} : L_2 \to L_1$ with $\theta \circ \theta^{-1} = id_2$ and $\theta^{-1} \circ \theta = id_1$ (where id_i is the identity function over $L_i, i = 1, 2$.

Construction of Complete Lattices

- Complete lattices can be combined to construct new complete lattices.
- *Cartesian product*
- *Total function space*
- *Monotone function space*

Cartesian Product

Let $L_1 = (L_1, \sqsubseteq_1)$ and $L_2 = (L_2, \sqsubseteq_2)$ be posets.
Define $L = (L, \sqsubseteq) = L_1 \times L_2$ by

$$L = \{(l_1, l_2) \mid l_1 \in L_1 \land l_2 \in L_2\}$$

and $(l_{11}, l_{21}) \sqsubseteq (l_{12}, l_{22})$ iff $l_{11} \sqsubseteq l_{12} \land l_{21} \sqsubseteq l_{22}$

It is then straightforward to verify that L is a poset. If additionally each $L_i = (L_i, \sqsubseteq_i, \sqcup_i, \sqcap_i, \bot_i, \top_i)$ is a complete lattice then so is $L = (L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ and furthermore

$$\bigsqcup Y = (\bigsqcup{}_1\{l_1 \mid \exists l_2 : (l_1, l_2) \in Y\}, \bigsqcup{}_2\{l_2 \mid \exists l_1 : (l_1, l_2) \in Y\})$$

and $\bot = (\bot_1, \bot_2)$ and similarly for $\sqcap Y$ and $\top$.

Total function space

Let $L_1 = (L_1, \sqsubseteq_1)$ be a poset and let S be a set.
Define $L = (L, \sqsubseteq) = S \rightarrow L_1$ by

and
$$L = \{f : S \rightarrow L_1 \mid f \text{ is a total function}\}$$
$$f \sqsubseteq f' \text{ iff } \forall s \in S : f(s) \sqsubseteq_1 f'(s)$$

It is then straightforward to verify that L is a poset. If additionally $L_1 = (L_1, \sqsubseteq_1, \sqcup_1, \sqcap_1, \bot_1, \top_1)$ is a complete lattice then so is $L = (L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ and furthermore $\bigsqcup Y = \lambda s. \bigsqcup_1 \{f(s) \mid f \in Y\}$ and $\bot = \lambda s. \bot_1$ and similarly for $\sqcap Y$ and $\top$.

Monotone function space

Let $L_1 = (L_1, \sqsubseteq_1)$ and $L_2 = (L_2, \sqsubseteq_2)$ be posets.
Define $L = (L, \sqsubseteq) = L_1 \rightarrow L_2$ by

and
$$L = \{f : L_1 \rightarrow L_2 \mid f \text{ is a monotone function}\}$$
$$f \sqsubseteq f' \text{ iff } \forall l_1 \in L_1 : f(l_1) \sqsubseteq_2 f'(l_1)$$

It is then straightforward to verify that L is a poset. If additionally each $L_i = (L_i, \sqsubseteq_i, \sqcup_i, \sqcap_i, \bot_i, \top_i)$ is a complete lattice then so is $L = (L, \sqsubseteq, \sqcup, \sqcap, \bot, \top)$ and furthermore

$$\bigsqcup Y = \lambda l_1. \bigsqcup_2 \{f(l_1) \mid f \in Y\}$$

and $\bot = \lambda l_1. \bot_2$ and similarly for $\sqcap Y$ and $\top$.

Chains

A subset $Y \subseteq L$ of a poset $L = (L, \sqsubseteq)$ is a *chain* if

$$\forall l_1, l_2 \in Y : (l_1 \sqsubseteq l_2) \vee (l_2 \sqsubseteq l_1)$$

Thus a chain is a (possibly empty) subset of L that is totally ordered. It is a *finite chain* if it is a finite subset of L.

A sequence $(l_n)_n = (l_n)_{n\in\mathbb{N}}$ of elements in L is an *ascending chain* if $n \leq m \Rightarrow l_n \sqsubseteq l_m$

Writing $(l_n)_n$ also for $\{l_n \mid n \in \mathbb{N}\}$ it is clear that an ascending chain is also a chain.

Similarly, a sequence $(l_n)_n$ is a *descending chain* if

$$n \leq m \Rightarrow l_n \sqsupseteq l_m$$

and clearly a descending chain is also a chain.

A sequence $(l_n)_n$ *eventually stabilizes* iff

$$\exists n_0 \in \mathbb{N} : \forall n \in \mathbb{N} : n \geq n_0 \Rightarrow l_n = l_0$$

For $(l_n)_n$ we write $\bigsqcup_n l_n$ for $\bigsqcup\{l_n \mid n \in \mathbb{N}\}$

and similarly we write $\bigsqcap_n l_n$ for $\bigsqcap\{l_n \mid n \in \mathbb{N}\}$

Ascending Chain and Descending Chain Conditions

- A poset $L = (L, \sqsubseteq)$ has finite height iff all chains are finite.
- It has finite height at most h if all chains contain at most h+1 elements; it has finite height h if additionally there is a chain with h+1 elements.
- The poset satisfies the *Ascending Chain Condition* iff all ascending chains eventually stabilize. Similarly, it satisfies the *Descending Chain Condition* iff all descending chains eventually stabilize.

Lemma 6

A poset $L = (L, \sqsubseteq)$ has finite height iff it satisfies both the Ascending and Descending Chain Conditions.

Proof

First assume that L has finite height. If $(l_n)_n$ is an ascending chain then it must be a finite chain and hence eventually stabilize; thus L satisfies the Ascending Chain Condition. Similarly for the Descending Chain Condition.

Next assume that L satisfies both the Ascending and Descending Chain Conditions, and consider a chain $Y \subseteq L$. We prove that Y is a finite chain. This is obvious if Y is empty so assume it is not. Then also $(Y, \sqsubseteq)$ is a non-empty poset satisfying the Ascending and Descending Chain Conditions.

As an auxiliary result we now show that

each non-empty $Y' \subseteq Y$ contains a least element

Construct a descending chain $(l'_n)_n$ in Y' as follows: first let l'_0 be an arbitrary element of Y'. For the inductive step let $l'_{n+1} = l'_n$ if l'_n is the least element of Y'; otherwise we can find $l'_{n+1} \in Y'$ such that $l'_{n+1} \sqsubseteq l'_n \wedge l'_{n+1} \neq l'_n$. Clearly $(l'_n)_n$ is a descending chain in Y; since Y satisfies the Descending Chain Condition the chain will eventually stabilize:
i.e. $\exists n'_0 : \forall n \geq n'_0 : l'_n = l'_{n'_0}$ and the construction is such that $l'_{n'_0}$ is the least element of Y'.

Back to the main proof: construct an ascending chain $(l_n)_n$ in Y. Using the side result each l_n is chosen as the least element of the set $Y \setminus \{l_0, \ldots, l_{n-1}\}$ as long as the latter set is non-empty, and this yields $l_{n-1} \sqsubseteq l_n \wedge l_{n-1} \neq l_n$; when $Y \setminus \{l_0, \ldots, l_{n-1}\}$ is empty set $l_n = l_{n-1}$, and since $Y \neq \emptyset$ we know that $n > 0$. Thus we have an ascending chain in Y and using the Ascending Chain Condition we have $\exists n_0 : \forall n \geq n_0 : l_n = l_{n_0}$. But this means that $Y \setminus \{l_0, \ldots, l_{n_0}\} = \emptyset$ since this is the only way that we can achieve $l_{n_0+1} = l_{n_0}$. It follows that Y is finite.

$\blacksquare$

Example 7

- 0
- –1
- –2

  ⋮
- –∞

Ascending Chain but does not have finite height

- ∞

  ⋮
- 2
- 1
- 0

Descending Chain but does not have finite height

Preservation

- If L_1 and L_2 each satisfy one of the conditions finite height, ascending chain, descending chain then $L_1 \times L_2$ also satisfies that condition.
- If S is finite then $S \rightarrow L$ preserves the conditions of L.
- $L_1 \rightarrow L_2$ does not in general preserve the conditions.

Lemma 8

For a poset $L = (L, \sqsubseteq)$ the conditions

(i) L is a complete lattice satisfying the Ascending Chain Condition, and

(ii) L has a least element, $\perp$, and binary lubs and satisfies the Ascending Chain Condition

are equivalent.

Proof

It is immediate that (i) implies (ii) so we prove that (ii) implies (i). Using Lemma 2 it suffices to prove that all subsets Y of L have a lub $\bigsqcup Y$. If Y is empty clearly $\bigsqcup Y = \perp$. If Y is finite and non-empty then we can write $Y = \{y_1, \ldots, y_n\}$ for $n \geq 1$ and it follows that $\bigsqcup Y = (\ldots(y_1 \sqcup y_2) \sqcup \ldots) \sqcup y_n$.

If Y is infinite then construct $(l_n)_n$ of elements of L: let l_0 be an arbitrary element y_0 of Y and given l_n take $l_{n+1} = l_n$ in the case $\forall y \in Y : y \sqsubseteq l_n$ and take $l_{n+1} = l_n \sqcup y_{n+1}$ in the case where some $y_{n+1} \in Y$ satisfies $y_{n+1} \not\sqsubseteq l_n$. Clearly this sequence is an ascending chain. Since L satisfies the Ascending Chain Condition it follows that the chain eventually stabilizes:
i.e., $\exists n : l_n = l_n + 1 = \ldots$
This means that $\forall y \in Y : y \sqsubseteq l_n$ because if $y \not\sqsubseteq l_n$ then $l_n \neq l_n \sqcup y$ for a contradiction. So we have constructed an upper bound for Y. Since it is the lub for $\{y_0, \ldots, y_n\} \subseteq Y$ it is also the lub for Y. ■

Lemma 9

For a complete lattice $L = (L, \sqsubseteq)$ satisfying the Ascending Chain Condition and a total function $f : L \to L$, the conditions

(i) f is additive
i.e. $\forall l_1, l_2 : f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$, and

(ii) f is affine
i.e. $\forall Y \subseteq L, Y \neq \emptyset : f(\bigsqcup Y) = \bigsqcup\{f(l) \mid l \in Y\}$

are equivalent and in both cases f is a monotone function.

Proof

It is immediate that (ii) implies (i): take $Y = \{l_1, l_2\}$
Also, (i) implies that f is monotone since $l_1 \sqsubseteq l_2$
is equivalent to $l_1 \sqcup l_2 = l_2$.
Next, suppose that f satisfies (i) and prove (ii).
If Y is finite write $Y = \{y_1, \ldots, y_n\}$ for $n \geq 1$ and

$$\begin{array}{rcl} f(\bigsqcup Y) = f(y_1 \sqcup \ldots \sqcup y_n) & = & f(y_1) \sqcup \ldots \sqcup f(y_n) \\ & \sqsubseteq & \bigsqcup \{f(l) \mid l \in Y\} \end{array}$$

If Y is infinite then the construction of the proof of Lemma 8 gives $\bigsqcup Y = l_n$ and $l_n = y_n \sqcup \ldots \sqcup y_0$ for some $y_i \in Y$ and $0 \leq i \leq n$. Then

$$\begin{array}{rcl} f(\bigsqcup Y) = f(l_n) & = & f(y_n \sqcup \ldots \sqcup y_0) \\ & = & f(y_n) \sqcup \ldots \sqcup f(y_0) \\ & \sqsubseteq & \bigsqcup \{f(l) \mid l \in Y\} \end{array}$$

Furthermore

$$f(\bigsqcup Y) \sqsupseteq \bigsqcup \{f(l) \mid l \in Y\}$$

follows from the monotonicity of f .
This completes the proof.

■

Fixed points

- Consider a monotone function $f : L \to L$ on a complete lattice $L = (L, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$. A *fixed point* of f is an element $l \in L : f(l) = l$. Write $Fix(f) = \{l \mid f(l) = l\}$ for the set of fixed points.
- f is *reductive at* l iff $f(l) \sqsubseteq l$. Write $Red(f) = \{l \mid f(l) \sqsubseteq l\}$ for the set of elements on which f is reductive, and say that f itself is *reductive* if $Red(f) = L$.

- Similarly, f is *extensive at* l iff $f(l) \sqsupseteq l$. Write $Ext(f) = \{l \mid f(l) \sqsupseteq l\}$ for the set of elements on which f is extensive, and say that f itself is *extensive* if $Ext(f) = L$.
- Since L is a complete lattice it is always the case that $Fix(f)$ will have a glb in L:
  $$lfp(f) = \bigsqcap Fix(f)$$
  Similarly, $Fix(f)$ will have a lub in L:
  $$gfp(f) = \bigsqcup Fix(f)$$
- *Tarski's Fixed Point Theorem* establishes that $lfp(f)$ is the *least fixed point* of f and that $gfp(f)$ is the *greatest fixed point* of f.

Proposition 10

Tarski's Fixed Point Theorem

Let $L = (L, \sqsubseteq, \sqcup, \sqcap, \perp, \top)$ be a complete lattice.
If $f : L \to L$ is a monotone function then $lfp(f)$ and $gfp(f)$ satisfy:

$$\begin{aligned} lfp(f) &= \bigsqcap Red(f) \in Fix(f) \\ gfp(f) &= \bigsqcup Ext(f) \in Fix(f) \end{aligned}$$

Proof

For $lfp(f)$ define $l_0 = \bigsqcap Red(f)$. First show that $f(l_0) \sqsubseteq l_0$ so that $l_0 \in Red(f)$. Since $l_0 \sqsubseteq l \ \forall l \in Red(f)$ and f is monotone we have

$$f(l_0) \sqsubseteq f(l) \sqsubseteq l, \forall l \in Red$$

and hence $f(l_0) \sqsubseteq l_0$.
To prove $l_0 \sqsubseteq f(l_0)$ observe that $f(f(l_0)) \sqsubseteq f(l_0)$ showing that $f(l_0) \in Red(f)$ and hence $l_0 \sqsubseteq f(l_0)$ by definition of l_0. Together this shows that l_0 is a fixed point of f and so $l_0 \in Fix(f)$. To see that l_0 is least in $Fix(f)$ note that $Fix(f) \subseteq Red(f)$ so $lfp(f) = l_0$. Similarly for $gfp(f)$. ■

Iteration

Iterating to the lfp by taking the lub of the sequence $(f^n(\bot))_n$ implies need for continuity of f (i.e. $f(\bigsqcup_n l_n) = \bigsqcup_n (f(l_n))$ for all ascending chains $(l_n)_n$), and similarly for the glb. One can show that

$$\begin{aligned} \bot \sqsubseteq f_n(\bot) \sqsubseteq \bigsqcup_n f_n(\bot) &\sqsubseteq lfp(f) \\ &\sqsubseteq gfp(f) \sqsubseteq \bigsqcap_n f^n(\top) \sqsubseteq f^n(\top) \sqsubseteq \top \end{aligned}$$

However, if L satisfies the Ascending Chain Condition then $\exists n : f^n(\bot) = f^{n+1}(\bot)$ and hence $lfp = f^n(\bot)$. Similarly for $gfp(f)$.