

## Homework 1

**Collaborators :**

# Practice Questions

These are practice questions. They will **NOT** be graded. We have also provided the final answers. However, it is up to you to understand how or why the given solution is correct.

You do not need to submit these on Gradescope. However, you may find it easier to just include them in the pdf. In that case, please do not mark these questions on Gradescope.

**1. Basic Probabilities (0 points)**

- (a) **(0 points)** Consider 7 balls: 3 yellow, 2 red, and 2 green. I pick one ball uniformly at random. What is the probability that I pick a red ball?

**Solution.**

$$\frac{2}{7}$$

- (b) **(0 points)** Consider 7 balls: 3 yellow, 2 red, and 2 green. I pick one of them uniformly at random. What is the probability that it is NOT a red ball?

**Solution.**

$$\frac{5}{7}$$

**2. Conditional Probabilities and Bayes' Rule (0 points)**

$$\Pr[X = x] = 0.5$$

$$\Pr[Y = y] = 0.3$$

$$\Pr[X = x, Y = y] = 0.15$$

What is  $\Pr[X = x|Y = y]$ ?

**Solution.**

$$\frac{1}{2}$$

3. **Conditional Probabilities and Bayes' Rule (0 points)** Write  $\Pr[X = x|Y = y]$  in terms of  $\Pr[Y = y|X = x]$ . That is,  $\Pr[X = x|Y = y] = \lambda \cdot \Pr[Y = y|X = x]$  where  $\lambda$  is some expression.

**Solution.**

$$\Pr[X = x|Y = y] = \frac{\Pr[X = x]}{\Pr[Y = y]} \cdot \Pr[Y = y|X = x]$$

or

$$\lambda = \frac{\Pr[X = x]}{\Pr[Y = y]}$$

4. **Understanding Joint Distribution (0 points)** Twelve balls will be tossed into six bins numbered  $\{1, 2, 3, 4, 5, 6\}$ . Each ball is thrown into a bin uniformly and independently into the bins. For  $i \in \{1, 2, 3, 4, 5, 6\}$ , let  $X_i$  represent the *number* of balls that fall into bin  $i$ .

- (a) **(0 points)** Find  $\mathbb{P}[X_1 = 12]$ .

**Solution.**

$$\frac{1}{6^{12}} = \frac{1}{2176782336}$$

- (b) **(0 points)** Find the (marginal) distribution of  $X_5$ .

**Solution.**

The marginal distribution of random variable  $X_5$  is binomial with parameters  $n = 12$ , and  $p = \frac{1}{6}$ :

$$\mathbb{P}[X_5 = k] = \binom{12}{k} \cdot \left(\frac{1}{6}\right)^k \cdot \left(\frac{5}{6}\right)^{12-k}.$$

- (c) **(0 points)** Find  $\mathbb{P}[X_2 = 2 \mid X_1 + X_3 + X_5 = 5]$ .

**Solution.**

$$\frac{224}{729}$$

# Homework Questions

These are homework questions and will be graded. Please make sure to clearly mark each problem on Gradescope.

1. **Basic Probabilities (2 points)** Consider 7 balls: 3 white and 4 black. I pick two balls uniformly at random without replacement. What is the probability that both balls are black?

**Solution.**



**2. Polya's Urn (4+5 points)**

- (a) **(4 points)** Suppose there are  $R > 1$  red balls and  $G > 1$  green balls in a box. We pick a ball from the box uniformly at random. If it is green, we put it back in the box; otherwise, we throw it away. After that, we pick another ball from the box uniformly at random. What is the probability that the first ball we picked is red, conditioned on the fact that the second ball we picked is green?

**Solution.**

- (b) **(5 points)** Suppose there are  $R > 1$  red and  $B > 1$  blue balls in an urn. We sample a ball uniformly at random. We replace the ball in the urn and add one additional ball of the SAME color. After that, we sample a ball and report its color.

What is the distribution of the second ball's color?

**Solution.**

**3. Sending One Bit (5+5+5 points)**

Alice intends to send a bit  $b \in \{0, 1\}$  to Bob. When Alice sends the bit, it goes through a series of  $n$  relays before reaching Bob. Each relay flips the received bit independently with probability  $p$  before forwarding that bit to the next relay.

- (a) **(5 points)** Show that Bob will receive the correct bit with a probability

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \cdot p^{2k} \cdot (1-p)^{n-2k}.$$

Hint: Your solution should work irrespective of whether Alice sends 0 or 1.

**Solution.**

- (b) **(5 points)** Let us consider an alternative way to calculate this probability. We say that the relay has *bias*  $q$  if the probability it flips the bit is  $(1 - q)/2$ . The bias  $q$  is a real number between  $-1$  and  $+1$ . Show that sending a bit through two relays with bias  $q_1$  and  $q_2$  is equivalent to sending a bit through a single relay with bias  $q_1 \cdot q_2$ .

**Solution.**

- (c) **(5 points)** Prove that the probability you receive the correct bit when it passes through  $n$  relays is

$$\frac{1 + (1 - 2p)^n}{2}.$$

**Solution.**

**4. Expected Values (3+5 points)**

(a) **(3 points)** Imagine the following game. A player rolls a fair six-sided die.

If an odd number is rolled, the player loses and has to give the host the roll number times 10 dollars. For example, if 3 is rolled, the player gives the host 30 dollars.

If an even number is rolled, the player wins, and the host gives the player the rolled number times 10 dollars. For example, if 4 is rolled, the player gets 40 dollars.

What is the expected value for the player in this game?

**Solution.**

- (b) **(5 points)** Imagine a bag with 2 white balls and 1 red ball. Every time, you will select one ball from the bag and remove it. If it is not the red ball, you again pick out another ball. What is the expected number of balls you need to take out from the bag to until you have red ball? (Note that the red ball itself is included. If you picked a white ball first, then the red ball, then you have taken 2 balls out of the bag.)

**Solution.**

5. **Mixing Two Boxes (3 points)** Experiment 1: I have two boxes, each with a mix of some red and blue balls. I will select one of the boxes uniformly at random (with a probability of 0.5), then select one of the balls from that box uniformly at random and report its color.

Experiment 2: I pour all the balls from both boxes into one large box. After that, I select one of the balls from this large box uniformly at random and report its color.

Are the two color distributions identical, irrespective of how many red/blue balls were in the two boxes? Please Justify your answer.

**Solution.**

6. **(10 points)** Recall that  $x \bmod p$  means the remainder of  $x$  when divided by  $p$ . For example, “13 divided by 3” is 4 with remainder 1 because  $13 = 4 \cdot 3 + 1$ . We write this as:  $13 \bmod 3 = 1$ .

(a) **(5 points)** Recall the definition of binomial coefficient  $\binom{p}{k} := \frac{p!}{k!(p-k)!}$ . For a prime  $p$ , prove that  $p$  divides  $\binom{p}{k}$ , if  $k \in \{1, 2, \dots, p-1\}$ .

**Solution.**



- (b) **(5 points)** Recall that  $(1+x)^p = \sum_{k=0}^p \binom{p}{k} x^k$ . Prove by induction on  $x$  that, for any  $x \in \{1, 2, \dots, p-1\}$ , we have

$$\overbrace{x \times x \times \dots \times x}^{p\text{-times}} \mod p = x$$

**Solution.**