

Lecture 23: RSA Encryption

Recall: RSA Assumption

- We pick two primes uniformly and independently at random $p, q \xleftarrow{s} P_n$
- We define $N = p \cdot q$
- We shall work over the group (\mathbb{Z}_N^*, \times) , where \mathbb{Z}_N^* is the set of all natural numbers $< N$ that are relatively prime to N , and \times is integer multiplication mod N
- We pick $y \xleftarrow{s} \mathbb{Z}_N^*$
- Let $\varphi(N)$ represent the size of the set \mathbb{Z}_N^* , which is $(p - 1)(q - 1)$
- We pick any $e \in \mathbb{Z}_{\varphi(N)}^*$, that is, e is a natural number $< \varphi(N)$ and is relatively prime to $\varphi(N)$
- We give (n, N, e, y) to the adversary \mathcal{A} as ask her to find the e -th root of y , i.e., find x such that $x^e = y$

RSA Assumption. For any computationally bounded adversary, the above-mentioned problem is hard to solve

Recall: Properties

- The function $x^e: \mathbb{Z}_N^* \rightarrow \mathbb{Z}_N^*$ is a bijection for all e such that $\gcd(e, \varphi(N)) = 1$
- Given (n, N, e, y) , where $y \stackrel{\$}{\leftarrow} \mathbb{Z}_N^*$, it is difficult for any computationally bounded adversary to compute the e -th root of y , i.e., the element $y^{1/e}$
- But given d such that $e \cdot d = 1 \pmod{\varphi(N)}$, it is easy to compute $y^{1/e}$, because $y^d = y^{1/e}$

Now, think about how we can design a key-agreement scheme using these properties. Once the key agreement protocol is ready, we can create a public-key encryption scheme with a one-time pad.

Key-Agreement

First, Alice and Bob establish a key that is hidden from the adversary

Alice

Bob

$$p, q \xleftarrow{\$} P_n$$

$$N = p \cdot q$$

$$r \xleftarrow{\$} \mathbb{Z}_N^* \xleftarrow{\text{pk} = (n, N, e)} \text{Pick any } e \in \mathbb{Z}_{\varphi(N)}^*$$

$$y = r^e \xrightarrow{y} \tilde{r} = y^d$$

Note that $r = \tilde{r}$ and is hidden from an adversary based on the RSA assumption

Public-key Encryption after the Key-Agreement Protocol

Using this key, Alice sends the encryption of $m \in \mathbb{Z}_N^*$ using the one-time pad encryption scheme.

$$\begin{array}{ccc} \underline{\text{Alice}} & & \underline{\text{Bob}} \\ c = m \cdot r & \xrightarrow{c} & \tilde{m} = c \cdot \text{inv}(\tilde{r}) \end{array}$$

Since we always have $r = \tilde{r}$, this encryption scheme always decrypts correctly. Note that $\text{inv}(\tilde{r})$ can be computed only by knowing $\varphi(N)$.

Putting the two together: RSA Encryption (First Attempt) I

Alice

Bob

$$p, q \xleftarrow{\$} P_n$$

$$N = p \cdot q$$

$$r \xleftarrow{\$} \mathbb{Z}_N^* \longleftarrow \text{pk} = (n, N, e) \quad \text{Pick any } e \in \mathbb{Z}_{\varphi(N)}^*$$

$$y = r^e$$

$$c = m \cdot r \longrightarrow (y, c) \longrightarrow \tilde{r} = y^d$$

$$\tilde{m} = c \cdot \text{inv}(\tilde{r})$$

Putting the two together: RSA Encryption (First Attempt) II

We emphasize that this encryption scheme work only for $m \in \mathbb{Z}_N^*$. In particular, this works for all messages m that have a binary representation of length less than n -bits because p and q are n -bit primes.

HOWEVER, THIS SCHEME IS INSECURE

- Let us start with a simpler problem.

Suppose I pick an integer x and give $y = x^3$ to you. Can you efficiently find the x ?

- Running for for loop with $i \in \{0, \dots, y\}$ and testing whether $i^3 = y$ or not is an inefficient solution
- However, binary search on the domain $\{0, \dots, y\}$ is an efficient algorithm
- Then why does the RSA assumption that says “computing the e -th root is difficult if $\varphi(N)$ is unknown” hold? Answer: Because we are working over \mathbb{Z}_N^* and not \mathbb{Z} ! “Wrapping around” due to the modulus operation while cubing kills the binary search approach.
- However, if x is such that $x^e < N$ then the modulus operation does not take effect. So, if $x < N^{1/e}$ then we can find the e -th root of y !

- Now, let us try to attack the “first attempt” algorithm
- Recall that we have $c = m \cdot r$ and $y = r^e$. So, we have $c^e = m^e \cdot r^e$. Now, note that $c^e \cdot \text{inv}(y) = m^e \cdot r^e \cdot y^{-1} = m^e$.
- So, the adversary can compute $c^e \cdot \text{inv}(y)$ to obtain m^e . If $m < N^{1/e}$, then the adversary can use binary search to recover m .
- There is another problem! If Alice is encrypting and sending multiple messages $\{m_1, m_2, \dots\}$, then the eavesdropper can recover $\{m_1^e, m_2^e, \dots\}$. So, she can find which of these $\{m_1^e, m_2^e, \dots\}$ are identical. In turn, she can find out the messages in $\{m_1, m_2, \dots\}$ that are identical (because $x^e: \mathbb{Z}_N^* \rightarrow \mathbb{Z}_N^*$ is a bijection).
- How do we fix these attacks?

RSA Encryption

- Our idea is to pad the message m with some randomness s . The new message $s||m$, with high probability, satisfies $(s||m)^e > N$ (that is, it wraps around)
- How does it satisfy the second attack mentioned above (Think: Birthday bound)
- Let us write down the new encryption scheme for $m \in \{0, 1\}^{n/2}$

$\text{Enc}_{n,N,e}(m)$:

- 1 Pick $r \xleftarrow{\$} \mathbb{Z}_N^*$
- 2 Pick $s \xleftarrow{\$} \{0, 1\}^{n/2}$
- 3 Compute $y = r^e$, and $c = (s||m) \cdot r$
- 4 Return (y, c)

Final Optimized RSA Encryption

- Note that masking with r is not helping at all! Let us call $s||m$ as the payload. An adversary can obtain the “ e -th power of the payload” by computing $c^e \cdot y^{-1}$
- So, we can use the following optimized encryption algorithm instead

$\text{Enc}_{n,N,e}(m)$:

- 1 Pick $s \xleftarrow{\$} \{0, 1\}^{n/2}$
- 2 Return $c = (s||m)^e$

Looking Ahead: Implementing RSA

Let us summarize all the algorithms that we need to implement the RSA algorithm

- 1 Generating n -bit primes to sample p and q
- 2 Generating e such that e is relatively prime to $\varphi(N)$, where $N = pq$
- 3 Finding the trapdoor d such that $e \cdot d = 1 \pmod{\varphi(N)}$