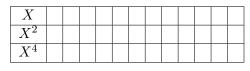
Homework 6

- 1. **RSA Assumption (5+12+5).** Consider RSA encryption scheme with parameters $N = 35 = 5 \times 7$.
 - (a) Compute $\varphi(N)$ and write down the set \mathbb{Z}_N^* . Solution.

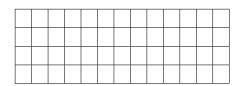
(b) Use repeated squaring and complete the rows X, X^2, X^4 for all $X \in \mathbb{Z}_N^*$ as you have seen in the class (slides), that is, fill in the following table by adding as many columns as needed.





(c) Find the row X^7 and show that X^7 is a bijection from \mathbb{Z}_N^* to \mathbb{Z}_N^* . Solution.

X						
X^2						
X^4						
X^7						



2. Answer the following questions (7+7+7+7 points):

(a) (7 points) By hand, compute the three least significant (decimal) digits of 25114997⁹³⁰¹⁴⁰³. Explain your logic.

(b) (7 points) Is the following RSA signature scheme valid? (Justify your answer)

$$(r||m) = 21, \sigma = 226, N = 339, e = 23$$

Here, m denotes the message, r denotes the randomness used to sign m, and σ denotes the signature. Moreover, (r||m) denotes the concatenation of r and m. The signature algorithm Sign(m) returns $(r||m)^d \mod N$ where d is the inverse of e modulo $\varphi(N)$. The verification algorithm $Ver(m,\sigma)$ returns $((r||m) == \sigma^e \mod N)$.

(c) (7 points) Remember that in RSA encryption and signature schemes, $N = p \times q$ where p and q are two large primes. Show that in the RSA scheme (with public parameters N and e), if you know N and $\varphi(N)$, then you can efficiently factorize N, i.e., you can recover p and q.

Solution.

(d) (7 points) Consider an encryption scheme where $Enc(m) := m^e \mod N$ where e is a positive integer relatively prime to $\varphi(N)$ and $Dec(c) := c^d \mod N$ where d is the inverse of e modulo $\varphi(N)$. Show that in this encryption scheme, if you know the encryption of m_1 and the encryption of m_2 , then you can find the encryption of $(m_1 \times m_2)^6$.

- (e) (7 points) Suppose $n=76499=227\cdot 337,$ where 227 and 337 are primes. Let $e_1=6039$ and $e_2=9031.$
 - i. (2 points) Only one of the two exponents e_1, e_2 is a valid RSA encryption key, which one? Solution.

ii. (3 points) For the valid encryption key, compute the corresponding decryption key d.

Solution.

iii. (2 points) Decrypt the cipher text c = 33. Solution.

3. Euler Phi Function (30 points)

(a) (10 points) Let $N=p_1^{e_1}\cdot p_2^{e_2}\cdots p_t^{e_t}$ represent the unique prime factorization of a natural number N, where $p_1< p_2<\cdots< p_t$ are prime numbers and e_1,e_2,\ldots,e_t are natural numbers. Let $\mathbb{Z}_N^*=\left\{x\colon 0\leqslant x< N-1,\gcd(x,N)=1\right\}$ and $\varphi(N)=\left|\mathbb{Z}_N^*\right|$. Using the inclusion exclusion principle, prove that

$$\varphi(N) = N \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_t}\right).$$

(b) (5 points) For any $x \in \mathbb{Z}_N^*$, prove that

$$x^{\varphi(N)} = 1 \mod N.$$

Hint: Consider the subgroup generated by x and its order. Solution.

(c) Replacing $\varphi(N)$ with $\frac{\varphi(N)}{2}$ in RSA. (15 points)

In RSA, we pick the exponent e and the decryption key d from the set $\mathbb{Z}_{\varphi(N)}^*$. This problem shall show that we can choose $e, d \in \mathbb{Z}_{\varphi(N)/2}^*$ instead.

Let p, q be two distinct odd primes and define N = pq.

i. (2 points) For any $e \in \mathbb{Z}_{\varphi(N)/2}^*$, prove that $x^e \colon \mathbb{Z}_N^* \to \mathbb{Z}_N^*$ is a bijection. Solution.

ii. (7 points) Consider any $x \in \mathbb{Z}_N^*$. Prove that $x^{\frac{\varphi(N)}{2}} = 1 \mod p$ and $x^{\frac{\varphi(N)}{2}} = 1 \mod q$.

iii. (3 points) Consider any $x \in \mathbb{Z}_N^*$. Prove that $x^{\frac{\varphi(N)}{2}} = 1 \mod N$. Solution.

iv. (3 points) Suppose e,d are integers that $e\cdot d=1 \mod \frac{\varphi(N)}{2}$. Show that $(x^e)^d=x \mod N$, for any $x\in \mathbb{Z}_N^*$. Solution.

4. Understanding hardness of the Discrete Logarithm Problem. (15 points) Suppose (G, \circ) is a group of order N generated by $g \in G$. Suppose there is an algorithm A_{DL} that, when given input $X \in G$, it outputs $x \in \{0, 1, ..., N-1\}$ such that $g^x = X$ with probability p_X .

Think of it this way: The algorithm \mathcal{A}_{DL} solves the discrete logarithm problem; however, for different inputs $X \in G$, its success probability p_X may be different.

Let $p = \frac{(\sum_{X \in G} p_X)}{N}$ represent the average success probability of \mathcal{A}_{DL} solving the discrete logarithm problem when X is chosen uniformly at random from G.

Construct a new algorithm \mathcal{B} that takes $any \ X \in G$ as input and outputs $x \in \{0, 1, \ldots, N-1\}$ (by making one call to the algorithm \mathcal{A}_{DL}) such that $g^x = X$ with probability p. This new algorithm that you construct shall solve the discrete logarithm problem for $every \ X \in G$ with the same probability p.

(Remark: Intuitively, this result shows that solving the discrete logarithm problem for any $X \in G$ is no harder than solving the discrete logarithm problem for a random $X \in G$.)

5. Concatenating a random bit string before a message. (15 points)

Let $m \in \{0,1\}^a$ be an arbitrary message. Define the set

$$S_m = \left\{ (r||m) \colon r \in \{0,1\}^b \right\}.$$

Let p be an odd prime. Recall that in the RSA encryption algorithm, we encrypted a message y chosen uniformly at random from this set S_m .

Prove the following

$$\Pr_{y \overset{\$}{\leftarrow} S_m}[p \text{ divides } y] \leqslant 2^{-b} \cdot \left\lceil 2^b/p \right\rceil.$$

(Remark: This bound is tight as well. There exists m such that equality is achieved in the probability expression above. Intuitively, this result shows that the message y will be relatively prime to p with probability (roughly) (1-1/p).

6. Properties of x^e when e is relatively prime to $\varphi(N)$ (20 points)

In this problem, we will partially prove a result from the class that was left unproven. Suppose N=pq, where p and q are distinct prime numbers. Let e be a natural number that is relatively prime to $\varphi(N)=(p-1)(q-1)$. In the lectures, we claimed (without proof) that the function $x^e\colon \mathbb{Z}_N^*\to \mathbb{Z}_N^*$ is a bijection. The following problem is key to proving this result.

Let N = pq, where p and q are distinct prime numbers. Let e be a natural number relatively prime to (p-1)(q-1). Consider $x, y \in \mathbb{Z}_N^*$. If $x^e = y^e \mod N$, then prove that x = y.

Hint: You might find the following facts useful.

- Every $\alpha \in \mathbb{Z}_N$ can be uniquely written as (α_p, α_q) such that $\alpha = \alpha_p \mod p$ and $\alpha = \alpha_q \mod q$, using the Chinese Remainder theorem. We will write this observation succinctly as $\alpha = (\alpha_p, \alpha_q) \mod (p, q)$.
- For $\alpha, \beta \in \mathbb{Z}_N$, and $e \in \mathbb{N}$ we have $\alpha^e = \beta \mod N$ if and only if $\alpha_p^e = \beta_p \mod p$ and $\alpha_q^e = \beta_q \mod q$. We will write this succinctly as $\alpha^e = (\alpha_p^e, \alpha_q^e) \mod (p, q)$.
- From the Extended GCD algorithm, if u and v are relatively prime then, there exists integers $a, b \in \mathbb{Z}$ such that au + bv = 1.
- Fermat's little theorem states that $x^{p-1} = 1 \mod p$ if x is a natural number that is relatively prime to the prime p.

7. Challenging: Inverting exponentiation function. (20 points)

Fix N = pq, where p and q are distinct odd primes. Let e be a natural number such that $gcd(e, \varphi(N)) = 1$. Suppose there is an adversary \mathcal{A} running in time T such that

$$\Pr\left[\left[\mathcal{A}([x^e \mod N]) = x\right]\right] = 0.01$$

for x chosen uniformly at random from \mathbb{Z}_N^* . Intuitively, this algorithm successfully finds the e-th root with probability 0.01, for a random x.

For any $\varepsilon \in (0,1)$, construct an adversary $\mathcal{B}_{\varepsilon}$ (which, possibly, makes multiple calls to the adversary \mathcal{A}) such that

$$\Pr [[\mathcal{B}_{\varepsilon}([x^e \mod N]) = x]] = 1 - \varepsilon,$$

for every $x \in \mathbb{Z}_N^*$. The algorithm $\mathcal{B}_{\varepsilon}$ should have a running time polynomial in T, $\log N$, and $\log 1/\varepsilon$.

${\bf Collaborators:}$