Stirling Approximation

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1 Stirling's Approximation

Our objective is to prove the following result.

Theorem 1 (Stirling's approximation, Robbins Formula [Rob55])**.**

$$\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \exp\left(\frac{1}{12n+1}\right) \leqslant n! \leqslant \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \exp\left(\frac{1}{12n}\right).$$

Before we proceed with the proof, I want to mention the following stronger conjectured bound that I discovered (numerically); however, I was unable to prove this bound.

$$\sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{\mathrm{e}}\right)^n \leqslant n! \leqslant \sqrt{2\pi\left(n+\frac{1}{6}+\frac{1}{72n}\right)}\left(\frac{n}{\mathrm{e}}\right)^n.$$

Let us now proceed with the proof of Robbins formula.

Proof. Our objective is to prove tight upper and lower bounds of the form

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n.$$

So, we shall obtain bounds of the form

$$\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \cdot g(n) \leqslant n! \leqslant \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n \cdot f(n),$$

where

$$f(n) = \exp\left(\frac{1}{12n}\right)$$
$$g(n) = \exp\left(\frac{1}{12n+1}\right).$$

1. First, the idea is to prove that the following limit exists.

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n} = L.$$

2. Next, we prove that the following sequence is (weakly) increasing.

$$\left\{\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n f(n)}\right\}_{n\in\mathbb{N}}.$$

Since, this sequence also tends to L and is (weakly) increasing, we get that

$$n! \leqslant L\cdot\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n f(n).$$

3. Similarly, if we prove that the following sequence is (weakly) decreasing

$$\left\{ \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n g(n)} \right\}_{n\in\mathbb{N}},$$

then

$$n! \geqslant L \cdot \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n g(n).$$

These high-level technique is used to derive several tight bounds of this form.

Part 1. Take a look at the video. Then, you can take a look at more formal presentations as well.

Part 2. Let us prove the following

$$\begin{aligned}
& \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n f(n)} \leqslant \frac{(n+1)!}{\sqrt{2\pi(n+1)}\left(\frac{n+1}{\mathrm{e}}\right)^{n+1} f(n+1)} \\
\iff \quad & \frac{f(n+1)}{f(n)} \leqslant \mathrm{e}\cdot\left(\frac{n}{n+1}\right)^{n+1/2} \\
\iff \quad & \ln f(n+1) - \ln f(n) \leqslant 1 + (n+1/2)\ln\left(1-\frac{1}{n+1}\right) \\
\iff \quad & -\frac{1}{12n(n+1)} \leqslant 1 + (n+1/2)\ln\left(1-\frac{1}{n+1}\right)
\end{aligned}$$

We substitute $\varepsilon = 1/(n+1)$. Then, we have $(n+1) = 1/\varepsilon$, $n = (1-\varepsilon)/\varepsilon$ and $n+1/2 = (1-\varepsilon/2)/\varepsilon$. Therefore, the inequality is equivalent to

$$-\frac{\varepsilon^2}{12(1-\varepsilon)} \leqslant 1 - (1-\varepsilon/2)\left(\sum_{i\geqslant 0}\frac{\varepsilon^i}{i+1}\right) = -\sum_{i\geqslant 2}\left(\frac{1}{i+1}-\frac{1}{2i}\right)\varepsilon^i.$$

It suffices to prove that, for all $i \geqslant 2$, we have

$$-\frac{1}{12} \leqslant -\left(\frac{1}{i+1}-\frac{1}{2i}\right) \iff \left(\frac{1}{i+1}-\frac{1}{2i}\right) \leqslant \frac{1}{12}.$$

Consider the function $h(x) = \frac{1}{x+1} - \frac{1}{2x}$ over the domain $x \in [2,\infty)$. Our objective is to find its extreme values. Observe that

$$h'(x) = -\frac{1}{(x+1)^2} + \frac{1}{2x^2} = 0.$$

Note that $2x^2 > (x+1)^2 \equiv \sqrt{2} > 1 + \frac{1}{x}$, for all integer $x \geqslant 3$. Therefore, $h(x)$ is decreasing in $[3,\infty)$.

For integer $x \geqslant 2$, the maximum is achieved at $x = 2$ or $x = 3$. Note that $h(2) = h(3) = \frac{1}{12}$. So, for integer x, we have $h(x) \leqslant \frac{1}{12}$. This observation completes the proof.

Part 3. Let us prove the following

$$\begin{aligned}
& \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n g(n)} \geqslant \frac{(n+1)!}{\sqrt{2\pi(n+1)}\left(\frac{n+1}{\mathrm{e}}\right)^{n+1} g(n+1)} \\
\iff \quad & \frac{1}{12n+13} - \frac{1}{12n+1} \geqslant 1 + (n+1/2)\ln\left(1-\frac{1}{n+1}\right)
\end{aligned}$$

We substitute $\varepsilon = 1/(n+1)$. Then, we have $12n+13 = (12+\varepsilon)/\varepsilon$ and $12n+1 = (12-11\varepsilon)/\varepsilon$. Therefore, the LHS becomes

$$\frac{1}{12n+13} - \frac{1}{12n+1} = \frac{\varepsilon}{12+\varepsilon} - \frac{\varepsilon}{12-11\varepsilon} = \frac{\varepsilon/12}{1+\varepsilon/12} - \frac{\varepsilon/12}{1-11\varepsilon/12} = \frac{1}{12}\sum_{j\geqslant 1}\left((-1/12)^j - (11/12)^j\right)\varepsilon^{j+1}.$$

We know from the above derivation that the RHS is

$$-\sum_{i\geqslant 2}\left(\frac{1}{i+1}-\frac{1}{2i}\right)\varepsilon^i.$$

Therefore, for integer $i\geqslant 2$, it suffices to show that

$$\frac{1}{12}\left(\,(-1/12)^{i-1}-(11/12)^{i-1}\,\right)\geqslant -\left(\frac{1}{i+1}-\frac{1}{2i}\right).$$

I leave this as an exercise. □

References

[Rob55] Herbert Robbins. A remark on stirling's formula. *The American mathematical monthly*, 62(1):26–29, 1955. 1