Homework 2

Collaborators :

1. **Sum of an Interesting Random Variable.** (20 points) Let $\mathbb{X}$ be the random variable over the set of all natural numbers $\{1, 2, 3, \ldots\}$ such that, for any natural number i, we have

$$\mathbb{P}[\mathbb{X} = i] = 3^{-i}.$$

Let $\mathbb{S}_n = \mathbb{X}^{(1)} + \mathbb{X}^{(2)} + \cdots + \mathbb{X}^{(n)}$, where $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \ldots, \mathbb{X}^{(n)}$ are independent and identical to $\mathbb{X}$.

- (5 points) What is $\mathbb{E}[\mathbb{S}_n]$?
- (15 points) Upper-bound the following probability

$$\mathbb{P}\left[\mathbb{S}_n - \mathbb{E}[\mathbb{S}_n] \geqslant E\right]$$

Solution.

2. **Coin-tossing: Word Problem.** (20 points) Suppose you have access to a coin that outputs heads with probability 1/2 and outputs tails with probability 1/2. Let $\mathbb{S}_n$ represent the *number of coin tosses needed* to see exactly n heads.

   - (5 points) What is $\mathbb{E}\left[\mathbb{S}_n\right]$?
   - (15 points) Upper-bound the following probability

$$\mathbb{E}\left[\mathbb{S}_n - \mathbb{E}\left[\mathbb{S}_n\right] \geqslant E\right]$$

   Solution.

3. **Sum of Poisson.** (25 points) Let $\mathbb{Y}$ be the random variable over sample space $\{0, 1, 2, \dots\}$ such that $\Pr[\mathbb{Y} = k] = \frac{e^{-\mu}\mu^k}{k!}$, for all $k \in \{0, 1, 2, \dots\}$. This distribution is the *Poisson distribution* with parameter μ.

   - (3 points) Prove that the mean of the "Poisson distribution with parameter μ" is equal to μ.
   - (7 points) Prove that if $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are independent Poisson distributions with parameters μ_1 and μ_2 respectively, then the random variable $\mathbb{Y}_1 + \mathbb{Y}_2$ is also a Poisson distribution with parameter $(\mu_1 + \mu_2)$.
   - (15 points) Let $\mathbb{X}$ be the Poisson distribution with mean m/n. Let $\mathbb{S}_n := \mathbb{X}^{(1)} + \mathbb{X}^{(2)} + \dots + \mathbb{X}^{(n)}$, where $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \dots, \mathbb{X}^{(n)}$ are all independent and identical to $\mathbb{X}$. Upper-bound the following probability
   $$\mathbb{P}\left[\mathbb{S}_n - \mathbb{E}\left[\mathbb{S}_n\right] \geqslant E\right]$$

   Solution.

4. **Another proof for Chernoff bound** (15 points) Consider the following simple type of Chernoff Bound:

   Suppose $\mathbb{S}_n = \sum_{i=1}^{n} \mathbb{X}^{(i)}$ where $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \ldots, \mathbb{X}^{(n)}$ are i.i.d Bernoulli random variables such that, $\mathbb{X} = \mathrm{Bern}\,(p)$. Then, for any $\varepsilon > 0$, the following Chernoff bound states:

   $$\Pr[\mathbb{S}_n \geqslant n(p+\varepsilon)] \leqslant \exp\left(-n\mathrm{D}_{\mathrm{KL}}\left(p+\varepsilon, p\right)\right).$$

   To prove the inequality above, we define i.i.d Bernoulli random variables $\mathbb{X}'^{(1)}, \mathbb{X}'^{(2)}, \ldots, \mathbb{X}'^{(n)}$ such that $\mathbb{X}' = \mathrm{Bern}\,(p+\varepsilon)$. Define $\mathbb{S}'_n := \sum_{i=1}^{n} \mathbb{X}'^{(i)}$.

   - (3 points) Define $h_k := \frac{\Pr[\mathbb{S}'_n = k]}{\Pr[\mathbb{S}_n = k]}$ and obtain a simplified expression for h_k.
   - (7 points) For any $k \geqslant n(p+\varepsilon)$, prove that $h_k \geqslant \exp\left(n\mathrm{D}_{\mathrm{KL}}\left(p+\varepsilon, p\right)\right)$.
   - (5 points) Use the inequality above to prove the Chernoff bound

   $$\Pr[\mathbb{S}_n \geqslant n(p+\varepsilon)] \leqslant \exp\left(-n\mathrm{D}_{\mathrm{KL}}\left(p+\varepsilon, p\right)\right).$$

   Solution.

5. **Random Walk in 2-D.** (20 points) Suppose an insect starts at $(0,0)$ at time $t=0$. At time t, its position is described by $(\mathbb{X}(t), \mathbb{Y}(t))$. At the next time step $t+1$, the insect uniformly at random moves to (a) $(\mathbb{X}(t)+3, \mathbb{Y}(t))$, (b) $(\mathbb{X}(t)-3, \mathbb{Y}(t))$, (c) $(\mathbb{X}(t), \mathbb{Y}(t)+3)$, or (d) $(\mathbb{X}(t), \mathbb{Y}(t)-3)$.

   State (5 points) and prove (15 points) a theorem that bounds how far from the origin the insect is at time $t=n$.

   Solution.

6. **Negatively Correlated Random Variables.** (20 points) Suppose $\mathbb{X}\colon \Omega \to \mathbb{Z}$ is a discrete random variable. Let $f\colon \mathbb{Z} \to \mathbb{Z}$ and $g\colon \mathbb{Z} \to \mathbb{Z}$ are two increasing and decreasing functions, respectively. Define random variables $\mathbb{R} := f(\mathbb{X})$ and $\mathbb{S} := g(\mathbb{X})$ and assume that $\mathbb{E}[\mathbb{R}^2] < \infty$ and $\mathbb{E}[\mathbb{S}^2] < \infty$. Prove that $\mathbb{R}$ and $\mathbb{S}$ are negatively correlated, i.e., $\mathbb{E}[\mathbb{R} \cdot \mathbb{S}] \leqslant \mathbb{E}[\mathbb{R}] \cdot \mathbb{E}[\mathbb{S}]$.

   Solution.

7. **Chernoff bound for negatively correlated Bernoulli random variables.** (Extra credit: 15 points)

Consider *negatively correlated* random variables $(X_1, X_2, \dots, X_n)$, such that $X_i \in \{0,1\}$, for all $i \in \{1,2,\dots,n\}$. Define $p_i = \mathbb{E}[X_i]$, for all $i \in \{1,2,\dots,n\}$, and $p = (p_1 + p_2 + \cdots + p_n)/n$. Prove that

$$\Pr\left[\sum_{i=1}^{n} X_i \geqslant (p+\varepsilon)n\right] \leqslant \exp\left(-n \cdot \mathrm{D_{KL}}\left(p+\varepsilon, p\right)\right).$$

Useful facts.

- Binary random variables: Consider an arbitrary random variable $X \in \{0,1\}$. Note that the random variable X^k is identical to the random variable X, for all $k \in \{1,2,\dots\}$.
- Negative correlation: For any $I \subseteq \{1,2,\dots,n\}$, the negative correlation of $(X_1, X_2, \dots, X_n)$ implies that

$$\mathbb{E}\left[\prod_{i \in I} X_i\right] \leqslant \prod_{i \in I} \mathbb{E}\left[X_i\right].$$

- Moment generating function: Note that

$$\exp\left(h \sum_{i=1}^{n} X_i\right) = \sum_{k \geqslant 0} \frac{h^k}{k!} \cdot \left(\sum_{i=1}^{n} X_i\right)^k$$

Solution.