Homework 2

Collaborators:

1. **Sum of an Interesting Random Variable.** (20 points) Let $X$ be the random variable over the set of all natural numbers $\{1, 2, 3, \ldots \}$ such that, for any natural number $i$, we have
   \[ P[X = i] = 3^{-i}. \]

   Let $S_n = X^{(1)} + X^{(2)} + \cdots + X^{(n)}$, where $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ are independent and identical to $X$.

   • (5 points) What is $E[S_n]$?
   • (15 points) Upper-bound the following probability
     \[ P[S_n - E[S_n] \geq E] \]

   **Solution.**
2. **Coin-tossing: Word Problem.** (20 points) Suppose you have access to a coin that outputs heads with probability $1/2$ and outputs tails with probability $1/2$. Let $S_n$ represent the number of coin tosses needed to see exactly $n$ heads.

- (5 points) What is $E[S_n]$?
- (15 points) Upper-bound the following probability

$$E[S_n - E[S_n] \geq E]$$

**Solution.**
3. **Sum of Poisson.** (25 points) Let \(Y\) be the random variable over sample space \(\{0, 1, 2, \ldots\}\) such that \(\Pr[Y = k] = \frac{e^{-\mu} \mu^k}{k!}\), for all \(k \in \{0, 1, 2, \ldots\}\). This distribution is the *Poisson distribution* with parameter \(\mu\).

- (3 points) Prove that the mean of the “Poisson distribution with parameter \(\mu\)” is equal to \(\mu\).
- (7 points) Prove that if \(Y_1\) and \(Y_2\) are independent Poisson distributions with parameters \(\mu_1\) and \(\mu_2\) respectively, then the random variable \(Y_1 + Y_2\) is also a Poisson distribution with parameter \((\mu_1 + \mu_2)\).
- (15 points) Let \(X\) be the Poisson distribution with mean \(m/n\). Let \(S_n := X^{(1)} + X^{(2)} + \cdots + X^{(n)}\), where \(X^{(1)}, X^{(2)}, \ldots, X^{(n)}\) are all independent and identical to \(X\). Upper-bound the following probability

\[
\Pr[S_n - \mathbb{E}[S_n] \geq E]
\]

**Solution.**
4. **Another proof for Chernoff bound** (15 points) Consider the following simple type of Chernoff Bound:

Suppose $S_n = \sum_{i=1}^{n} X^{(i)}$ where $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ are i.i.d Bernoulli random variables such that, $X = \text{Bern}(p)$. Then, for any $\varepsilon > 0$, the following Chernoff bound states:

$$\Pr[S_n \geq n(p + \varepsilon)] \leq \exp\left(-nD_{\text{KL}}(p + \varepsilon, p)\right).$$

To prove the inequality above, we define i.i.d Bernoulli random variables $X'^{(1)}, X'^{(2)}, \ldots, X'^{(n)}$ such that $X' = \text{Bern}(p + \varepsilon)$. Define $S'_n := \sum_{i=1}^{n} X'^{(i)}$.

- (3 points) Define $h_k := \frac{\Pr[S'_{n} = k]}{\Pr[S_{n} = k]}$ and obtain a simplified expression for $h_k$.
- (7 points) For any $k \geq n(p + \varepsilon)$, prove that $h_k \geq \exp\left(nD_{\text{KL}}(p + \varepsilon, p)\right)$.
- (5 points) Use the inequality above to prove the Chernoff bound

$$\Pr[S_n \geq n(p + \varepsilon)] \leq \exp\left(-nD_{\text{KL}}(p + \varepsilon, p)\right).$$

**Solution.**
5. **Random Walk in 2-D.** (20 points) Suppose an insect starts at (0, 0) at time $t = 0$. At time $t$, its position is described by $(X(t), Y(t))$. At the next time step $t + 1$, the insect uniformly at random moves to (a) $(X(t) + 3, Y(t))$, (b) $(X(t) - 3, Y(t))$, (c) $(X(t), Y(t) + 3)$, or (d) $(X(t), Y(t) - 3)$.

State (5 points) and prove (15 points) a theorem that bounds how far from the origin the insect is at time $t = n$.

**Solution.**
6. **Negatively Correlated Random Variables.** (20 points) Suppose $X: \Omega \rightarrow \mathbb{Z}$ is a discrete random variable. Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ are two increasing and decreasing functions, respectively. Define random variables $R := f(X)$ and $S := g(X)$ and assume that $\mathbb{E}[R^2] < \infty$ and $\mathbb{E}[S^2] < \infty$. Prove that $R$ and $S$ are negatively correlated, i.e., $\mathbb{E}[R \cdot S] \leq \mathbb{E}[R] \cdot \mathbb{E}[S]$.

**Solution.**
7. Chernoff bound for negatively correlated Bernoulli random variables.  
(Extra credit: 15 points)

Consider \textit{negatively correlated} random variables \((X_1, X_2, \ldots, X_n)\), such that \(X_i \in \{0, 1\}\), for all \(i \in \{1, 2, \ldots, n\}\). Define \(p_i = \mathbb{E}[X_i]\), for all \(i \in \{1, 2, \ldots, n\}\), and \(p = (p_1 + p_2 + \cdots + p_n)/n\). Prove that

\[
\Pr \left[ \sum_{i=1}^{n} X_i \geq (p + \varepsilon)n \right] \leq \exp \left( -n \cdot D_{\text{KL}}(p + \varepsilon, p) \right).
\]

Useful facts.

- Binary random variables: Consider an arbitrary random variable \(X \in \{0, 1\}\). Note that the random variable \(X^k\) is identical to the random variable \(X\), for all \(k \in \{1, 2, \ldots\}\).

- Negative correlation: For any \(I \subseteq \{1, 2, \ldots, n\}\), the negative correlation of \((X_1, X_2, \ldots, X_n)\) implies that

\[
\mathbb{E} \left[ \prod_{i \in I} X_i \right] \leq \prod_{i \in I} \mathbb{E} [X_i].
\]

- Moment generating function: Note that

\[
\exp \left( h \sum_{i=1}^{n} X_i \right) = \sum_{k \geq 0} \frac{h^k}{k!} \cdot \left( \sum_{i=1}^{n} X_i \right)^k.
\]

Solution.