Lecture 27: Hypercontractivity
Today, we shall learn about an advanced tool in Fourier Analysis called Hypercontractivity. We shall see the theorem and a few of its applications. However, we shall not see the proof.
For $p \geq 1$ and any function $f : \{0, 1\}^n \to \mathbb{R}$, we define

$$L_p(f) := \left( \frac{1}{N} \sum_{x \in \{0,1\}^n} |f(x)|^p \right)^{1/p}$$

There are two interesting properties of the $L_p(\cdot)$ norm.

**Lemma (Monotonicity of Norm)**

For $1 \leq p < q$ and $f : \{0, 1\}^n \to \mathbb{R}$ we have

$$L_p(f) \leq L_q(f)$$

Moreover, equality holds if and only if $f$ is a constant function.
Further, we also have the “Contractivity Property.” The smoothed version of the function has a smaller norm than the original function.

**Lemma (Contractivity)**

For \( p \geq 1 \) and \( \rho \in [0, 1] \), we have

\[
L_p(T_\rho(f)) \leq L_p(f)
\]

Equality holds if (and only if) \( \rho = 0 \) or \( f \) is a constant function.
By the “contractivity property” we know that

\[ L_p(T_\rho(f)) \leq L_p(f) \]

By monotonicity of norm, we know that

\[ L_p(T_\rho(f)) \leq L_q(T_\rho(f)), \]

where \( q > p \)

However, how does \( L_p(f) \) compare with \( L_q(T_\rho(f)) \)?

Answer: It depends.

**Hypercontractivity.** Even the \( q \)-th norm of \( T_\rho(f) \) is smaller than the \( p \)-th norm of \( f \) if \( \rho \leq \sqrt{\frac{p-1}{q-1}} \).
Formally, we have the following result

**Theorem (Hypercontractivity)**

Let \( f : \{0, 1\}^n \to \mathbb{R} \) be an arbitrary function. For \( 1 \leq p < q \) and \( \rho \leq \sqrt{\frac{p-1}{q-1}} \) we have

\[
L_q(T_\rho(f)) \leq L_p(f)
\]

**Proof Outline.**

- Prove the statement for \( 1 \leq p < q = 2 \) (The proof of this statement proceeds by induction on \( n \) and the base case of \( n = 1 \) is the toughest case)
- Reduce the proof of the statement for the case \( 2 \leq p < q \) to the case of \( 1 \leq p < q = 2 \) (using Hölder’s inequality)
- Reduce the proof of the statement for the case \( 1 \leq p < 2 < q \) to the two cases above (using the homomorphical property of the noise operator)
Special Case of $q = 2$

- Let us state the hypercontractivity theorem for the special case of $1 \leq p < q = 2$
- Suppose $p = 1 + \delta$, where $\delta \in [0, 1)$
- Suppose $\rho = \sqrt{\frac{p-1}{q-1}} = \delta^{1/2}$
- The hypercontractivity theorem states that

$$L_2(T_\rho(f)) \leq L_p(f)$$

- Squaring both sides and applying Parseval’s identity to the LHS, we get

$$\sum_{S \in \{0,1\}^n} \delta^{|S|} \hat{f}(S)^2 \leq \left( \frac{1}{N} \sum_{x \in \{0,1\}^n} |f(x)|^{1+\delta} \right)^{2/1+\delta}$$
Let $f : \{0, 1\}^n \to \{+1, 0, -1\}$. Recall that we denoted the Boolean functions by functions with range $\{+1, -1\}$. Suppose we want to denote boolean functions only defined on a subset of $\{0, 1\}^n$. In this case, we use functions $\{0, 1\}^n \to \{+1, 0, -1\}$. Wherever the function is not defined, it evaluates to 0; otherwise, it takes value $\in \{+1, -1\}$.

The KKL in “KKL Lemma” stands for “Kahn-Kalai-Linial”

Note that, for a function $f : \{0, 1\}^n \to \{+1, 0, -1\}$, we have

$$L_p(f) = \mathbb{P} \left[ f(x) \neq 0 : x \leftarrow \{0, 1\}^n \right]^{1/p}$$
• From the hypercontractivity theorem, we directly have the KKL Lemma

**Lemma (KKL Lemma)**

*For any function* $f : \{0, 1\}^n \rightarrow \{+1, 0, -1\}$ *and* $\delta \in [0, 1)$ *we have*

$$\sum_{S \subseteq \{0,1\}^n} \delta^{|S|} \hat{f}(S)^2 \leq \Pr[f(x) \neq 0 : x \leftarrow \{0, 1\}^n]^{2/1+\delta}$$

• **Intuition.** Note that the RHS is $\ll \Pr[f(x) \neq 0 : x \leftarrow \{0, 1\}^n]$ because $\delta < 1$, i.e., the ratio of the support of $f$ to the size of the entire sample space $N$. On the other hand, the LHS is dominated by the Fourier coefficients on $S$ that have small support.
So, the inequality states that the total mass of the Fourier coefficients on $S$ that have a small support is $\ll$ the ratio of the support of $f$ to the size of the entire sample space $N$. Effectively, this lemma states that if a boolean function has a small support, then most of its mass of the Fourier coefficients is on the $S$ that has a large support.

In the next lecture, we shall prove a formal result that makes this intuition concrete. We shall show that the uniform distribution on any large subset $A \subseteq \{0, 1\}^n$ fools most large linear tests.