Lecture 19: Discrete Fourier Analysis on the Boolean Hypercube (Introduction)
Functions with domain \( \{0, 1\}^n \) and range \( \mathbb{R} \)

Let \( f : \{0, 1\}^n \rightarrow \mathbb{R} \)

We shall always use \( N = 2^n \)

Any \( n \)-bit binary string shall be canonically interpreted as an integer in the range \( \{0, 1, \ldots, N - 1\} \)

For any function \( f : \{0, 1\}^n \rightarrow \mathbb{R} \) we shall associate the following unique vector in \( \mathbb{R}^N \)

\[
(f(0), f(1), \ldots, f(N - 1))
\]
For \( i \in \{0, 1, \ldots, N - 1\} \), we define the function \( \delta_i : \{0, 1\}^n \rightarrow \mathbb{R} \) as follows:

\[
\delta_i(x) = \begin{cases} 
1, & \text{if } x = i \\
0, & \text{otherwise}
\end{cases}
\]

Note that the functions \( \{\delta_0, \delta_1, \ldots, \delta_{N-1}\} \) form a basis for \( \mathbb{R}^N \).

Any function \( f \) can be expressed as a linear combination of these basis functions as follows:

\[
f = f(0)\delta_0 + f(1)\delta_1 + \cdots + f(N - 1)\delta_{N-1}
\]

Our goal is to study the function \( f \) in a new basis, namely, the “Fourier Basis,” which shall be introduced next. We emphasize that this basis need not be unique.
For $S = (S_1, S_2, \ldots, S_n) \in \{0, 1\}^n$, we define the following function

$$\chi_S(x) := (-1)^{\sum_{i=1}^{n} S_i \cdot x_i}$$

Several introductory materials on Fourier analysis interpret $S$ as a subset of $\{1, 2, \ldots, n\}$. Although, the definition presented here is equivalent to this interpretation, I personally prefer this notation because it generalizes to other domains.
An Example

- Suppose $n = 3$ and we are working with functions $f : \{0, 1\}^n \rightarrow \mathbb{R}$
- Note that there are 8 different Fourier basis functions

\[
\begin{align*}
\chi_{000}(x) &= (-1)^0 = 1 \\
\chi_{100}(x) &= (-1)^{x_1} \\
\chi_{010}(x) &= (-1)^{x_2} \\
\chi_{110}(x) &= (-1)^{x_1 + x_2} \\
\chi_{001}(x) &= (-1)^{x_3} \\
\chi_{101}(x) &= (-1)^{x_1 + x_3} \\
\chi_{011}(x) &= (-1)^{x_2 + x_3} \\
\chi_{111}(x) &= (-1)^{x_1 + x_2 + x_3}
\end{align*}
\]
(All non-trivial) Basis Functions are balanced

**Lemma**

\[ \sum_{x \in \{0,1\}^n} \chi_R(x) = \begin{cases} N, & \text{if } R = 0 \\ 0, & \text{otherwise} \end{cases} \]

**Proof:**

- Suppose \( R = 0 \), then we have

\[ \sum_{x \in \{0,1\}^n} \chi_R(x) = \sum_{x \in \{0,1\}^n} 1 = N \]
(All non-trivial) Basis Functions are balanced

Suppose $R \neq 0$. Let \( \{i_1, i_2, \ldots, i_r\} \) be the set of indices \( \{i: R_i = 1\} \)

\[
\sum_{x \in \{0,1\}^n} \chi_R(x) = \sum_{x \in \{0,1\}^n} (-1)^{R_1x_1 + \cdots + R_nx_n}
\]
\[
= \sum_{x \in \{0,1\}^n} (-1)^{R_{i_1}x_{i_1} + \cdots + R_{i_r}x_{i_r}}
\]
\[
= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2}x_{i_2} + \cdots + R_{i_r}x_{i_r}} \sum_{x_{i_1} \in \{0,1\}} (-1)^{x_{i_1}}
\]
\[
= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2}x_{i_2} + \cdots + R_{i_r}x_{i_r}} \left((-1)^0 + (-1)^1\right)
\]
\[
= \sum_{x_{-i_1} \in \{0,1\}^{n-1}} (-1)^{R_{i_2}x_{i_2} + \cdots + R_{i_r}x_{i_r}} \cdot 0 = 0
\]
Definition (Inner Product)

The inner-product of two functions $f, g : \{0, 1\}^n \rightarrow \mathbb{R}$ is defined as follows

$$\langle f, g \rangle := \frac{1}{N} \sum_{x \in \{0,1\}^n} f(x) \cdot g(x)$$
Lemma

\[ \langle \chi_S, \chi_T \rangle = \begin{cases} 1, & \text{if } S = T \\ 0, & \text{otherwise} \end{cases} \]

Proof:

\[ \langle \chi_S, \chi_T \rangle = \frac{1}{N} \sum_{x \in \{0,1\}^n} \chi_S(x) \chi_T(x) \]

\[ = \frac{1}{N} \sum_{x \in \{0,1\}^n} (-1)^{(S_1+T_1)x_1 + \ldots + (S_n+T_n)x_n} \]

Note that if \( S_i = T_i \) then \((-1)^{(S_i+T_i)x_i} = 1\); otherwise \((-1)^{(S_i+T_i)x_i} = (-1)^{x_i}\)
Define $R$ such that $R_i = 1$ if $S_i \neq T_i$; otherwise $R_i = 0$

Then, the right-hand side expression becomes

$$\langle \chi_S, \chi_T \rangle = \frac{1}{N} \sum_{x \in \{0,1\}^n} (-1)^{R_1x_1 + \cdots + R_nx_n}$$

$$= \frac{1}{N} \sum_{x \in \{0,1\}^n} \chi_R(x)$$

$$= \begin{cases} \frac{1}{N} \cdot N, & \text{if } R = 0 \\ \frac{1}{N} \cdot 0, & \text{otherwise} \end{cases}$$

Note that $R = 0$ if and only if $S = T$. This observation completes the proof
Our objective is to study a function $f : \{0, 1\}^n \to \mathbb{R}$.

Every function $f$ is equivalently represented as the vector $(f(0), f(1), \ldots, f(N - 1)) \in \mathbb{R}^N$, where $N = 2^n$.

For $S = S_1 S_2 \ldots S_n \in \{0, 1\}^n$, define the following function

$$\chi_S(x) := (-1)^{S_1 x_1 + S_2 x_2 + \ldots + S_n x_n},$$

where $x = x_1 x_2 \ldots x_n \in \{0, 1\}^n$.

We defined an inner product of functions

$$\langle f, g \rangle := \frac{1}{N} \sum_{x \in \{0,1\}^n} f(x) \cdot g(x).$$

We showed that $\left\{ \chi_S : S \in \{0, 1\}^N \right\}$ is an orthonormal basis. That is,

$$\langle \chi_S, \chi_T \rangle = \begin{cases} 0, & \text{if } S \neq T \\ 1, & \text{if } S = T \end{cases}$$
Fourier Coefficients

- Since \( \{ \chi_S : S \in \{0, 1\}^n \} \) is an orthonormal basis, we can express any \( f \) as follows

\[
f = \hat{f}(0) \cdot \chi_0 + \hat{f}(1) \cdot \chi_1 + \cdots + \hat{f}(N - 1) \cdot \chi_{N-1},
\]

where \( \hat{f}(S) \in \mathbb{R} \) and \( S \in \{0, 1\}^n \)

- We interpret \( \left( \hat{f}(0), \hat{f}(1), \ldots, \hat{f}(N - 1) \right) \) as a function \( \hat{f} \)
Fourier Transformation

- Fourier Transformation is a basis change that maps the function $f$ to the function $\hat{f}$.
- We shall represent it as $f \overset{\mathcal{F}}{\mapsto} \hat{f}$, where $\mathcal{F}$ is the Fourier Transformation.
Note that we have the following property. For any $S \in \{0, 1\}^n$, we have $\langle f, \chi_S \rangle = \hat{f}(S)$. So, we get

$$ (f(0)f(1) \cdots f(N - 1)) \cdot \frac{1}{N} (\chi_S(0)\chi_S(1) \cdots \chi_S(N - 1))^\top = \hat{f}(S) $$

Define the matrix

$$\mathcal{F} = \frac{1}{N} \begin{bmatrix}
\chi_0(0) & \chi_1(0) & \cdots & \chi_{N-1}(0) \\
\chi_0(1) & \chi_1(1) & \cdots & \chi_{N-1}(1) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_0(N - 1) & \chi_1(N - 1) & \cdots & \chi_{N-1}(N - 1)
\end{bmatrix}$$

From the property mentioned above, note that we have the identity

$$ f \cdot \mathcal{F} = \hat{f} $$

Fourier Analysis
Claim

For two functions $f, g : \{0, 1\}^n \to \mathbb{R}$, we have

$$(f + g) = \hat{f} + \hat{g}$$

Proof.

$$(f + g) = (f + g)\mathcal{F} = f\mathcal{F} + g\mathcal{F} = \hat{f} + \hat{g}$$
Claim

For a function \( f : \{0, 1\}^n \to \mathbb{R} \) and \( c \in \mathbb{R} \), we have

\[
\hat{(cf)} = c\hat{f}
\]

Proof.

\[
\hat{(cf)} = (cf)\mathcal{F} = c(f\mathcal{F}) = c\hat{f}
\]
Theorem

Let \( f : \{0, 1\}^n \to \mathbb{R} \). Then, we have

\[
\widehat{(f)} = \frac{1}{N} \cdot f
\]

Proof.

- We shall prove that \( \mathcal{F} \cdot \mathcal{F} = \frac{1}{N} I_{N \times N} \). This result shall directly imply that \( \widehat{(f)} = (f \mathcal{F}) \mathcal{F} = f \left( \frac{1}{N} I_{N \times N} \right) = \frac{1}{N} \cdot f \)
- Let us compute the element \((\mathcal{F} \cdot \mathcal{F})_{i,j}\). This element is the product of the \(i\)-th row of \( \mathcal{F} \) and the \(j\)-th column of \( \mathcal{F} \)
- The \(j\)-th column of \( \mathcal{F} \) is \( \left( \frac{1}{N} \chi_j \right)^T \)
- The \(i\)-th row of \( \mathcal{F} \) is \( (\chi_0(i) \chi_1(i) \cdots \chi_{N-1}(i)) \)
- Note that \( \chi_S(x) = \chi_x(S) \), i.e., the matrix \( \mathcal{F} \) is symmetric
So, the $i$-th row of $\mathcal{F}$ is $\frac{1}{N} \chi_i$

Therefore, we have $(\mathcal{F}\mathcal{F})_{i,j} = \frac{1}{N^2} \cdot \chi_i \cdot \chi_j^T = \frac{1}{N} \langle \chi_i, \chi_j \rangle$. The orthonormality of the Fourier basis completes the proof
Theorem (Plancherel)

Suppose \( f, g : \{0, 1\}^n \rightarrow \mathbb{R} \). Then, the following holds

\[
\langle f, g \rangle = \sum_{S \in \{0, 1\}^n} \hat{f}(S)\hat{g}(S)
\]
Proof.

\[ \langle f, g \rangle = \left\langle \sum_{S \in \{0,1\}^n} \widehat{f}(S) \chi_S, \sum_{T \in \{0,1\}^n} \widehat{g}(T) \chi_T \right\rangle \]

\[ = \sum_{S \in \{0,1\}^n} \widehat{f}(S) \left\langle \chi_S, \sum_{T \in \{0,1\}^n} \widehat{g}(T) \right\rangle \]

\[ = \sum_{S \in \{0,1\}^n} \widehat{f}(S) \sum_{T \in \{0,1\}^n} \widehat{g}(T) \langle \chi_S, \chi_T \rangle \]

\[ = \sum_{S \in \{0,1\}^n} \widehat{f}(S) \cdot \widehat{g}(S) \]
Note that, if $f, g : \{0, 1\}^n \to \{+1, -1\}$ and we have $\langle f, g \rangle = 1 - \varepsilon$, then $f$ and $g$ disagree at $\varepsilon N$ inputs. Intuitively, if $|\langle f, g \rangle|$ is close to 1, the functions are highly correlated. On the other hand, if $|\langle f, g \rangle|$ is close to 0 then the functions are independent.
Theorem (Parseval’s Identity)

Suppose $f : \{0, 1\}^n \to \mathbb{R}$. Then

$$\langle f, f \rangle = \sum_{S \in \{0, 1\}^n} \hat{f}(S)^2$$

Substitute $f = g$ in Plancherel’s theorem.
Corollary

If \( f : \{0, 1\}^n \rightarrow \{+1, -1\} \), then \( \sum_{S \in \{0, 1\}^n} \hat{f}(S)^2 = 1 \)

Follows from the fact that \( \langle f, f \rangle = 1 \) and the Parseval’s identity