Lecture 02: Estimating Summations & Stirling Approximation

Summation I

- Let us try to write a closed form expression for the following summation

$$S = \sum_{i=1}^{n} 1$$

- It is trivial to see that $S = n$

Summation II

- Now, let us try to write a closed form expression for the following summation

$$S = \sum_{i=1}^{n} i$$

- We can prove that $S = \frac{n(n+1)}{2}$
  - How do you prove this statement? (Use Induction? Use the formula for the Sum of an Arithmetic Progression?)
- Using Asymptotic Notation, we can say that $S = \frac{n^2}{2} + o(n^2)$

Summation III

- Now, let us try to write a closed form expression for the following summation

$$S = \sum_{i=1}^{n} i^2$$

- We can prove that $S = \frac{n(n+1)(2n+1)}{6}$
  - Why is the expression on the right an integer? (Prove by induction that 6 divides $n(n+1)(2n+1)$ for all positive integer n)
  - How do you prove this statement? (Use Induction?)
- Using Asymptotic Notation, we can say that $S = \frac{n^3}{3} + o(n^3)$

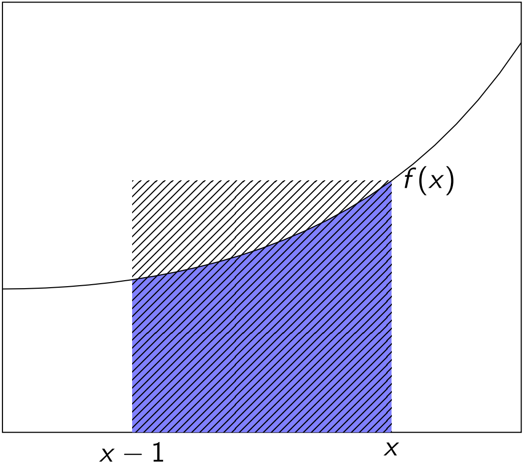
Summation IV

- Do we see a pattern here?
- Conjecture: For $k \geqslant 1$, we have $\sum_{i=1}^{n} i^{k-1} = \frac{n^k}{k} + o(n^k)$.
  - How do we prove this statement?

Estimating Summations by Integration I

- Let f be an increasing function
- For example, $f(x) = x^{k-1}$ is an increasing function for $k > 1$ and $x \geqslant 0$

Estimating Summations by Integration II



Estimating Summations by Integration III

- Observation: "Blue area under the curve" is smaller than the "Shaded area of the rectangle"
  - Blue area under the curve is:

$$\int_{x-1}^{x} f(t)dt$$

  - Shaded area of the rectangle is:

$$f(x)$$

  - So, we have the inequality:

$$\int_{x-1}^{x} f(t)\,\mathrm{d}t \leqslant f(x)$$

  - Summing both side from $x = 1$ to $x = n$, we get

$$\sum_{x=1}^{n} \int_{x-1}^{x} f(t)\,\mathrm{d}t \leqslant \sum_{x=1}^{n} f(x)$$

Estimating Summations by Integration IV

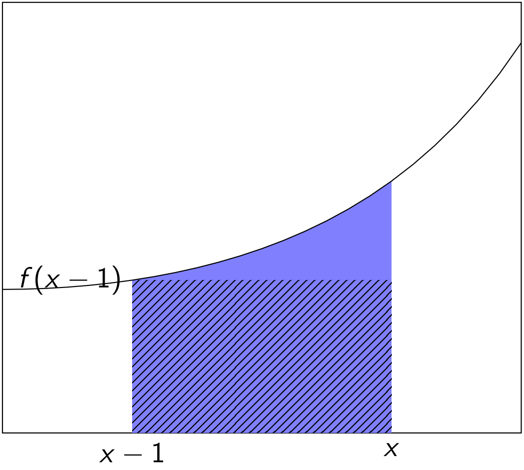
- The left-hand side of the inequality is

$$\int_0^1 f(t)\,\mathrm{d}t + \int_1^2 f(t)\,\mathrm{d}t + \cdots + \int_{n-1}^n f(t)\,\mathrm{d}t = \int_0^n f(t)\,\mathrm{d}t$$

- So, for an increasing f, we have the following lower bound.

$$\int_0^n f(t)\,\mathrm{d}t \leqslant \sum_{x=1}^{n} f(x) \qquad (1)$$

- Now, we will upper bound the summation expression. Consider the figure below



Estimating Summations by Integration VI

- Observation: "Blue area under the curve" is greater than the "Shaded area of the rectangle"
- So, we have the inequality:

$$\int_{x-1}^{x} f(t)\,\mathrm{d}t \geqslant f(x-1)$$

- Now we sum the above inequality from $x = 2$ to $x = n + 1$
- We get

$$\int_{1}^{2} f(t)\,\mathrm{d}t + \int_{2}^{3} f(t)\,\mathrm{d}t + \cdots + \int_{n}^{n+1} f(t)\,\mathrm{d}t \geqslant f(1) + f(2) + \cdots + f(n)$$

- So, for an increasing f, we get the following upper bound

$$\int_{1}^{n+1} f(t)\,\mathrm{d}t \geqslant \sum_{x=1}^{n} f(x) \qquad (2)$$

Summary: Estimation of Summation using Integration

Theorem

For an increasing function f, we have

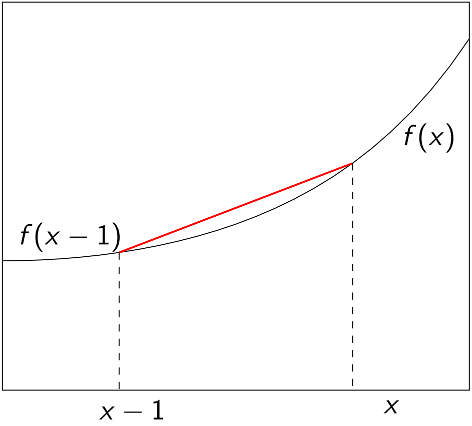
$$\int_0^n f(t)\,\mathrm{d}t \leqslant \sum_{x=1}^{n} f(x) \leqslant \int_1^{n+1} f(t)\,\mathrm{d}t$$

Exercise:

- Use this theorem to prove that $\sum_{i=1}^{n} i^{k-1} = \frac{n^k}{k} + o(n^k)$, for $k \geqslant 1$
- Consider the function $f(x) = 1/x$ to find upper and lower bounds for the sum $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ using the approach used to prove Theorem 1

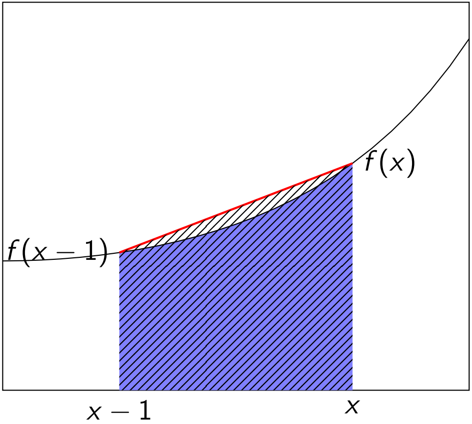
Property of Concave Upwards Function I

- Consider the coordinates $(x-1, f(x-1))$ and $(x, f(x))$
- For a concave upwards function, the secant between the two coordinates is always (on or) above the part of the curve f between the two coordinates



Property of Concave Upwards Function II

- So, the shaded area of the trapezium is greater than the blue area under the curve



Property of Concave Upwards Function III

- So, we get

$$\frac{f(x-1)+f(x)}{2} \geqslant \int_{x-1}^{x} f(t)\,\mathrm{d}t$$

- Now, use this new observation to obtain a better lower bound for the sum $\sum_{x=1}^{n} f(x)$
- Think: Can you get even tighter bounds?
- Additional Reading: Read on the "trapezoidal rule"

Stirling Approximation I

- Let us try to estimate $n!$
- Note that $\ln n! = \sum_{i=1}^{n} \ln i$. So, we can apply the first part of the lecture on estimating summations by integrals to obtain an estimation
- Note that $\int \ln t \, \mathrm{d}t = t \ln t - t + \text{const.}$. Furthermore, $\ln t$ is an increasing function. So, we obtain the following result

$$n \ln n - (n-1) = \ln 1 + \int_{t=1}^{n} \ln t \, \mathrm{d}t \leqslant \sum_{i=1}^{n} \ln i \leqslant \int_{t=1}^{n+1} \ln t \, \mathrm{d}t = (n+1)\ln(n+1) - n$$

- Exponentiating both sides, we obtain the bound

$$\frac{n^n}{\mathrm{e}^{n-1}} \leqslant n! \leqslant \frac{(n+1)^{n+1}}{\mathrm{e}^n}$$

Stirling Approximation II

- This estimate is a good starting point that might suffice for some applications. Next, we shall learn about a very tight estimate of $n!$ known as the Stirling's approximation

Theorem (Stirling Approximation)

$$\sqrt{2\pi n}\cdot\frac{n^n}{\mathrm{e}^n}\exp\left(\frac{1}{12n+1}\right)\leqslant n!\leqslant\sqrt{2\pi n}\cdot\frac{n^n}{\mathrm{e}^n}\exp\left(\frac{1}{12n}\right)$$

- We can use this expression to obtain very tight estimates of n-choose-k. Consider the following upper-bound

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{k!(n-k)!}\\ &\leqslant \frac{\sqrt{2\pi n}\cdot\frac{n^n}{\mathrm{e}^n}\exp\left(\frac{1}{12n}\right)}{\sqrt{2\pi k}\cdot\frac{k^k}{\mathrm{e}^k}\exp\left(\frac{1}{12k+1}\right)\sqrt{2\pi(n-k)}\cdot\frac{(n-k)^{(n-k)}}{\mathrm{e}^{(n-k)}}\exp\left(\frac{1}{12(n-k)+1}\right)}\\ &= \sqrt{\frac{n}{2\pi k(n-k)}}\frac{n^n}{k^k(n-k)^{(n-k)}}\exp\left(\frac{1}{12n}-\frac{1}{12k+1}-\frac{1}{12(n-k)+1}\right)\end{aligned}$$

Stirling Approximation III

Let us use $p = k/n$ and $q = 1 - p$ to simplify this upper-bound

$$\binom{n}{k} \leqslant \frac{1}{\sqrt{2\pi npq}} \left(p^p q^q\right)^{-n} \exp\left(\frac{1}{12n} - \frac{1}{12k+1} - \frac{1}{12(n-k)+1}\right)$$

- Similarly, we can obtain the following lower-bound

$$\binom{n}{k} \geqslant \frac{1}{\sqrt{2\pi npq}} \left(p^p q^q\right)^{-n} \exp\left(\frac{1}{12n+1} - \frac{1}{12k} - \frac{1}{12(n-k)}\right)$$

Stirling Approximation IV

- Although these bounds are very tight, they are not very easy to use. The above bounds can be used to obtain the following theorem (this proof is left as an exercise).

Theorem (Binomial Coefficient Estimate)

Let $0 \leqslant k \leqslant n$ and $p = k/n$ and $q = 1 - p$. Then, the following bound holds

$$\frac{1}{\sqrt{8npq}} \left(p^p q^q\right)^{-n} \leqslant \binom{n}{k} \leqslant \frac{1}{\sqrt{2\pi npq}} \left(p^p q^q\right)^{-n}$$