Homework 5

1. **Fourier Transformation Matrix.** (20 points) We shall provide an alternate mechanism to construct the Fourier transformation matrix. Recall that, for functions $\{0,1\}^n \to \mathbb{R}$, we defined the basis functions as follows. For all $S, x \in \{0,1\}^n$, we defined

$$\chi_S(x) := (-1)^{S_1 \cdot x_1 + S_2 \cdot x_2 + \cdots + S_n \cdot x_n}$$

Given this definition of the Fourier basis functions, the definition of the Fourier transformation matrix $\mathcal{F}_n \in \frac{1}{N}\{+1,-1\}^{N \times N}$, where $N = 2^n$, is as follows. We shall use row indices $i \in \{0,1,\ldots,N-1\}$ and $j \in \{0,1,\ldots,N-1\}$ and define

$$(\mathcal{F})_{i,j} := \frac{1}{N}\chi_j(i)$$

Now, we begin the new definition using *matrix tensor product*. Let $A \in \mathbb{R}^{a \times b}$ and $B \in \mathbb{R}^{a' \times b'}$ be two matrices. We define the *block matrix* $C = A \otimes B$ as follows. For $i \in \{1,\ldots,a\}$ and $b \in \{1,\ldots,b\}$

$$C_{i,j} := a_{i,j} B$$

Base case. Define

$$\mathcal{G}_1 := \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Recursive construction. Define, for $n > 1$, $\mathcal{G}_n := \mathcal{G}_1 \otimes \mathcal{G}_{n-1}$.

Prove, by induction, that $\mathcal{F}_n = \mathcal{G}_n$.

Solution.

2. **Smoothed Function Property.** (20 points) Let $f\colon \{0,1\}^n \to \mathbb{R}$ be a function. Let $L_p(f)$ be the norm defined as follows

$$L_p(f) := \left(\frac{1}{N} \sum_{x \in \{0,1\}^n} \left| f(x) \right|^p \right)^{1/p}$$

For any $\rho \in [0,1]$, prove that $L_p(T_\rho(f)) \leqslant L_p(f)$. Equality holds if and only if f is a constant function, or $\rho = 1$.

Solution.

3. **Most Random functions are Small Biased.** (20 points) Let $f\colon \{0,1\}^n \to \{+1,-1\}$ be a boolean function. Suppose we consider a *random* boolean function such that, for every $x \in \{0,1\}^n$, we assign $f(x)$ independently and uniformly at random from the set $\{+1,-1\}$. Recall that a function f is small biased if $\left|\mathsf{bias}_f(S)\right| \leqslant \varepsilon$ for all $0 \neq S \in \{0,1\}^n$.

   Formally state and prove a concentration result that proves: "a random boolean function is small-biased with very high probability."

   Solution.

4. **Differential Operator.** (20 points) We shall consider functions $\{0,1\}^n \to \mathbb{R}$. Let us introduce a notation. Given $x \in \{0,1\}^n$, we represent $x|_{i=1}$ as the bit-string identical to x except that its i-th coordinate is fixed to 1. Similarly, $x|_{i=0}$ is the bit-string that is identical to x except that its i-th coordinate is fixed to 0.

   Let $D_i(f)$ be the function $\{0,1\}^n \to \mathbb{R}$ defined as follows

$$D_i(f)(x) = f(x|_{i=1}) - f(x|_{i=0})$$

   Express $\widehat{D_i(f)}$ as a function of $\widehat{f}$.

   Solution.

5. **Flats are Small-biased Distribution.** (20 points) We shall consider function $\mathbb{Z}_p \to \mathbb{C}$ in this problem. Define $\omega = \exp(2\pi\imath/p)$. Recall that we defined, for $S \in \mathbb{Z}_p$, as follows

$$\mathsf{bias}_f(S) = \sum_{x \in \mathbb{Z}_p} f(x)\omega^{S \cdot x}$$

Let $\mathbb{X}$ be a uniform distribution over the set $\{0, 1, \ldots, t-1\}$, for some integer $t < p$. Prove that

$$\mathsf{bias}_{\mathbb{X}}(1) \leqslant \frac{\mathrm{sinc}(\pi t/p)}{\mathrm{sinc}(\pi/p)},$$

where $\mathrm{sinc}(x) := \sin(x)/x$

Solution.

Collaborators :