Homework 2

1. **Sum of an Interesting Random Variable.** (20 points) Let $\mathbb{X}$ be the random variable over the set of all natural numbers $\{1, 2, 3, \ldots\}$ such that, for any natural number i, we have

$$\mathbb{P}[\mathbb{X} = i] = 2^{-i}.$$

Let $\mathbb{S}_n = \mathbb{X}^{(1)} + \mathbb{X}^{(2)} + \cdots + \mathbb{X}^{(n)}$, where $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \ldots, \mathbb{X}^{(n)}$ are independent and identical to $\mathbb{X}$.

- (5 points) What is $\mathbb{E}[\mathbb{S}_n]$?
- (15 points) Upper-bound the following probability

$$\mathbb{P}\left[\mathbb{S}_n - \mathbb{E}[\mathbb{S}_n] \geqslant E\right]$$

Solution.

2. **Coin-tossing: Word Problem.** (20 points) Suppose you have access to a coin that outputs heads with probability 1/2 and outputs tails with probability 1/2. Let $\mathbb{S}_n$ represent the *number of coin tosses needed* to see exactly n heads.

   - (5 points) What is $\mathbb{E}\left[\mathbb{S}_n\right]$?
   - (15 points) Upper-bound the following probability

$$\mathbb{E}\left[\mathbb{S}_n - \mathbb{E}\left[\mathbb{S}_n\right] \geqslant E\right]$$

   Solution.

3. **Sum of Poisson.** (25 points) Let $\mathbb{Y}$ be the random variable over sample space $\{0, 1, 2, \ldots\}$ such that $\Pr[\mathbb{Y} = k] = \frac{e^{-\mu}\mu^k}{k!}$. This is the Poisson distribution with parameter μ.

   - (3 points) Prove that the mean of a Poisson distribution with parameter μ is equal to μ.
   - (7 points) Prove that if $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are independent Poisson distributions with parameters μ_1 and μ_2 respectively, then the random variable $\mathbb{Y}_1 + \mathbb{Y}_2$ is also a Poisson distribution with parameter $\mu_1 + \mu_2$.
   - (15 points) Let $\mathbb{X}$ be the Poisson distribution with mean m/n. Let $\mathbb{S}_n := \mathbb{X}^{(1)} + \mathbb{X}^{(2)} + \cdots + \mathbb{X}^{(n)}$, where $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \ldots, \mathbb{X}^{(n)}$ are all independent and identical to $\mathbb{X}$. Upper-bound the following probability

$$\mathbb{P}\left[\mathbb{S}_n - \mathbb{E}\left[\mathbb{S}_n\right] \geqslant E\right]$$

   Solution.

4. **Empty Bins in the Poisson Model.** (20 points) Let $\mathbb{X}$ represent the Poisson distribution with mean m/n. Let $\mathbb{Y}$ be the indicator variable $\mathbf{1}_{\{\mathbb{X}=0\}}$. That is, $\mathbb{Y}$ is the random variable that is 1 if and only if the random variable $\mathbb{X}$ is 0.

   Let $\mathbb{S}_n = \mathbb{Y}^{(1)} + \mathbb{Y}^{(2)} + \ldots + \mathbb{Y}^{(n)}$, where $\mathbb{Y}^{(1)}, \mathbb{Y}^{(2)}, \ldots, \mathbb{Y}^{(n)}$ are independent and identical to $\mathbb{Y}$.

   - (5 points) What is $\mathbb{E}\left[\mathbb{S}_n\right]$?
   - (15 points) Upper-bound the following probability

$$\mathbb{P}\left[\mathbb{S}_n - \mathbb{E}\left[\mathbb{S}_n\right] \geqslant E\right]$$

   Solution.

5. **Another proof for Chernoff bound** (15 points) Consider the following simple type of Chernoff Bound:

Suppose $\mathbb{S}_n = \sum_{i=1}^{n} \mathbb{X}^{(i)}$ where $\mathbb{X}^{(1)}, \mathbb{X}^{(2)}, \dots, \mathbb{X}^{(n)}$ are i.i.d Bernoulli random variables such that, $\mathbb{X} = \text{Bern}\,(p)$. Then, for any $\varepsilon > 0$, the following Chernoff bound states:

$$\Pr[\mathbb{S}_n \geqslant n(p+\varepsilon)] \leqslant e^{-n\mathrm{D}_{\mathrm{KL}}(p+\varepsilon,p)}$$

To prove the inequality above, we define i.i.d Bernoulli random variables $\mathbb{X}'^{(1)}, \mathbb{X}'^{(2)}, \dots, \mathbb{X}'^{(n)}$ such that $\mathbb{X}' = \text{Bern}\,(p+\varepsilon)$. Define $\mathbb{S}'_n := \sum_{i=1}^{n} \mathbb{X}'^{(i)}$.

- (3 points) Define $h_k := \frac{\Pr[\mathbb{S}'_n = k]}{\Pr[\mathbb{S}_n = k]}$ and obtain a simplified expression for h_k.
- (7 points) For any $k \geqslant n(p+\varepsilon)$, prove that $h_k \geqslant e^{n\mathrm{D}_{\mathrm{KL}}(p+\varepsilon,p)}$.
- (5 points) Use the inequality above to prove the Chernoff bound

$$\Pr[\mathbb{S}_n \geqslant n(p+\varepsilon)] \leqslant e^{-n\mathrm{D}_{\mathrm{KL}}(p+\varepsilon,p)}.$$

Solution.

6. **Random Walk in 2-D.** (20 points) Suppose an insect starts at $(0, 0)$ at time $t = 0$. At time t, its position is described by $(\mathbb{X}(t), \mathbb{Y}(t))$. At the next time step $t + 1$, the insect uniformly at random moves to (a) $(\mathbb{X}(t) + 1, \mathbb{Y}(t))$, $(\mathbb{X}(t) - 1, \mathbb{Y}(t))$, $(\mathbb{X}(t), \mathbb{Y}(t) + 1)$, or $(\mathbb{X}(t), \mathbb{Y}(t) - 1)$.

   State (5 points) and prove (15 points) a theorem that bounds how far from the origin the insect is at time $t = n$.

   Solution.

Collaborators :